

## On static and spherically symmetric solutions of Starobinsky model

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**Abstract** We investigate the problem of static and spherically symmetric solutions in the Starobinsky gravity model. By extending the Lichnerowicz and Israel theorems, William Nelson has demonstrated that the Schwarzschild solution is the unique static, spherically symmetric and asymptotically flat black hole solution in the Starobinsky model. However, Hong Lü et al. find that there are sign errors in the proof of Nelson. This raises the problem of whether Nelson's proof is correct or not. In order to answer this question, we explore the corresponding solutions by using the Taylor series expansion method. We find that Nelson's conclusion is indeed correct despite the flaw in the proof.

**Key words:** black hole physics: gravitation

### 1 INTRODUCTION

The most recent observations from the *Planck* satellite (Planck Collaboration et al. 2016) are remarkably consistent with the Starobinsky model of gravity  $f(R) = R + sR^2$  (Starobinsky 1980). On the contrary, chaotic inflationary models like large field inflation and natural inflation are disfavored because of their high tensor-to-scalar ratio. On the other hand, compared to the Starobinsky model, other alternatives of  $f(R)$  modified gravity theories are usually plagued by various problems. For example, they generally lead to the Einstein gravity plus a cosmological constant (Thongkool et al. 2009), have a field-matter dominated epoch rather than a standard matter-dominated epoch (Amendola et al. 2007), run into a curvature singularity (Frolov 2008) (see Kobayashi & Maeda (2008) for the existence of a singularity in an asymptotic de Sitter Universe and Upadhye & Hu (2009) for the opposite statement), produce high frequency oscillations and a singularity at finite time in cosmology (Appleby & Battye 2008), or give rise to a fine tuning of the exponent (Faraoni 2011). In view of these points, great enthusiasm for research on Starobinsky gravity has been generated.

We shall deal with the problem of solving Starobinsky gravity. Actually, investigations on the static, spherically symmetric spacetime solutions in other modified gravity theories have also been carried out. In the first place, numerical solutions modeled as neutron stars in modified gravities are studied by the following references (Cooney et al. 2010; Babichev & Langlois 2010; Arapoğlu et al. 2011; Jaime et al. 2011; Santos 2012; Orellana et al. 2013; Alavirad & Weller 2013; Astashenok et al. 2013; Ganguly et al. 2014; Astashenok et al. 2014; Yazadjiev et al. 2014; Capozziello et al. 2016; Astashenok et al. 2017). Secondly, many efforts are devoted to aspects of black hole solutions. Here we are unable to exhaust all the references and we only present a brief review. Using the Taylor expansion method, the near-horizon geometry of black hole spacetime is investigated in Lü et al. (2015a). The Lifshitz black hole in four dimensional R-squared gravity is presented in Cai et al. (2009). Some exact, asymptotically non-flat and static black hole solutions are given in Gao & Shen (2016). Static spherically symmetric solutions of  $f(R)$  gravity have also been studied in both the vacuum case (Multamäki & Vilja 2006) and the perfect fluid model (Multamäki & Vilja 2007). Following a pertur-

bative approach in  $f(R)$  gravity around the Einstein-Hilbert action, de La Cruz-Dombriz et al. (2009) found that only solutions of the Schwarzschild-(Anti) de Sitter type are present up to second order in perturbations. A class of exact static spherically symmetric solutions has been presented in Clifton & Barrow (2005) for the form  $R^{1+\delta}$ . Using the Lagrange multiplier method, Sebastiani & Zerbini (2011) present a Lagrangian derivation of the equation of motion for static spherically symmetric spacetimes in the  $f(R)$  theory of gravity. It is found that the corresponding equations of motion simply have the form of a first order derivative and thus some new solutions are obtained. Shojai & Shojai (2012) construct some new static spherically symmetric interior solutions in  $R^{1+\delta}$  theory. With the method of Noether symmetries, Capozziello et al. (2012) find some new static spherically symmetric solutions in  $f(R)$  theory. Multamäki & Vilja (2006) construct the spherically symmetric solutions of  $f(R)$  gravity using their input function method. Gutiérrez-Piñeres & López-Monsalvo (2013) present a static axially symmetric vacuum solution for  $f(R)$  gravity in Weyl's canonical coordinates. Finally, using the so-called generator method, Amirabi et al. (2016) present some static spherically symmetric solutions in  $f(R)$  theory in  $n$  dimensional spacetimes. For more works on static solutions in  $f(R)$  gravity, we refer to Goswami et al. (2014); Hendi et al. (2012); Bergliaffa & Nunes (2011); Cembranos et al. (2011); Aparicio Resco et al. (2016); Lobo & Oliveira (2009); Caramês & de Mello (2009); Azadi et al. (2008); Capozziello et al. (2008); Kainulainen et al. (2007); Cognola et al. (2015); Nojiri & Odintsov (2013, 2014); Hendi et al. (2014b,a); Babichev & Langlois (2009); Hendi & Momeni (2011); Myrzakulov et al. (2016); Lü et al. (2015b); Chakraborty & SenGupta (2015); Frolov & Shapiro (2009).

In this paper, we shall look for the static, spherically symmetric and asymptotically flat black hole solution in the Starobinsky model of gravity. In fact, Nelson (2010) has shown that, by extending the Lichnerowicz and Israel theorems from Einstein gravity to higher derivative gravities, the Schwarzschild solution is the unique static, spherically symmetric and asymptotically flat black hole solution in the Starobinsky model of gravity. However, Hong Lü et al. (Lü et al. 2015b) found that there are sign errors in the proof of Nelson. This raises the problem of whether Nelson's proof is correct or not. In order to answer this question, we explore the corresponding so-

lutions by using the Taylor series expansion method. We find that Nelson's conclusion is indeed correct despite the flaw in the proof. Compared to Nelson's verification, we can go even further. Using the Taylor series expansion method, we find the conclusion is applicable not only to the Starobinsky model but also to the more generalized modified gravity,  $f(R) = R + s_2R^2 + s_3R^3 + s_4R^4 + O(R^4)$  (in contrast, Nelson's verification is applicable to the theory of  $f(R) = R + sR^2 + \alpha R_{\mu\nu}R^{\mu\nu}$ ). Thus it is an important conclusion. Throughout this paper, we adopt the system of units in which  $G = c = \hbar = 1$  and the metric signature  $(-, +, +, +)$ .

## 2 SOLUTION IN THE EINSTEIN GRAVITY

In this section, as a warm-up, we shall derive the static, spherically symmetric and asymptotically flat spacetime solution in the Einstein gravity by using the Taylor expansion method.

Let us start from the Einstein-Hilbert action

$$S = \int d^4x \sqrt{-g} \frac{R}{16\pi}, \quad (1)$$

where  $R$  is the Ricci scalar. Since we shall look for the vacuum solutions, the contribution of matters is absent in the action.

The metric for static and spherically symmetric spacetime in the isotropic coordinate system can be written as

$$ds^2 = -A(r)^2 dt^2 + B(r)^2 (dr^2 + r^2 d\Omega^2), \quad (2)$$

where  $d\Omega^2 = d\theta^2 + \sin^2\theta d\phi^2$ . Then we obtain the Einstein equations

$$2BB''r - B'^2r + 4BB' = 0, \quad (3)$$

$$2BB'A'r + B'^2Ar + 2B^2A' + 2BB'A = 0, \quad (4)$$

$$A''B^2r + BAB''r - B'^2Ar + B^2A' + BB'A = 0. \quad (5)$$

The system of equations can be solved and the solution is

$$A = \frac{1 - \frac{M}{2r}}{1 + \frac{M}{2r}}, \quad B = \left(1 + \frac{M}{2r}\right)^2, \quad (6)$$

which is the Schwarzschild solution.  $M$  is the mass of the source. Different from the Schwarzschild coordinate, the metric components in isotropic coordinates are convergent on the event horizon. So, we shall carry out the

calculation in the isotropic coordinates. We expand  $A(r)$  and  $B(r)$  in the form of a Taylor series

$$A(r) = 1 + \frac{a_1 M}{r} + \frac{a_2 M^2}{r^2} + \frac{a_3 M^3}{r^3} + \frac{a_4 M^4}{r^4} + \frac{a_5 M^5}{r^5} + \frac{a_6 M^6}{r^6} + O\left(\frac{1}{r^6}\right), \quad (7)$$

$$B(r) = 1 + \frac{b_1 M}{r} + \frac{b_2 M^2}{r^2} + \frac{b_3 M^3}{r^3} + \frac{b_4 M^4}{r^4} + \frac{b_5 M^5}{r^5} + \frac{b_6 M^6}{r^6} + O\left(\frac{1}{r^6}\right), \quad (8)$$

where  $a_i$  and  $b_i$  are dimensionless constants to be determined. For sufficiently large  $r$ , we have  $A = 1$  and  $B = 1$ .

In other words, the spacetime is asymptotically flat. Substituting Equations (7) and (8) into the Einstein Equations (3), (4) and (5), we find Equations (3), (4) and (5) become

$$0 = \frac{\alpha_2}{r^2} + \frac{\alpha_3}{r^3} + \frac{\alpha_4}{r^4} + \frac{\alpha_5}{r^5} + \frac{\alpha_6}{r^6} + O\left(\frac{1}{r^6}\right), \quad (9)$$

$$0 = \frac{\beta_1}{r} + \frac{\beta_2}{r^2} + \frac{\beta_3}{r^3} + \frac{\beta_4}{r^4} + \frac{\beta_5}{r^5} + \frac{\beta_6}{r^6} + O\left(\frac{1}{r^6}\right), \quad (10)$$

$$0 = \frac{\gamma_1}{r} + \frac{\gamma_2}{r^2} + \frac{\gamma_3}{r^3} + \frac{\gamma_4}{r^4} + \frac{\gamma_5}{r^5} + \frac{\gamma_6}{r^6} + O\left(\frac{1}{r^6}\right), \quad (11)$$

where  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$  are constants whose concrete expressions are determined by  $a_i$ ,  $b_i$  and  $M$ . We need to find the expressions for 12 coefficients,  $a_i$  and  $b_i$  in Equations (7) and (8). So, we want 12 algebraic equations.

By setting the leading terms in Equations (9), (10) and (11) to zero, we obtain 12 algebraic equations

$$\alpha_2 = 0, \quad \alpha_3 = 0, \quad \alpha_4 = 0, \quad \alpha_5 = 0, \quad (12)$$

$$\beta_1 = 0, \quad \beta_2 = 0, \quad \beta_3 = 0, \quad \beta_4 = 0, \quad (13)$$

$$\beta_5 = 0, \quad \gamma_3 = 0, \quad \gamma_4 = 0, \quad \gamma_5 = 0. \quad (14)$$

We find  $\gamma_1 = \beta_1$ ,  $\gamma_2 = \beta_2$  and  $\gamma_1 = 0$ ,  $\gamma_2 = 0$ . Solving the 12 equations, we arrive at

$$a_1 = -b_1, \quad b_1 = b_1, \quad (15)$$

$$a_2 = \frac{b_1^2}{2}, \quad b_2 = \frac{b_1^2}{4}, \quad (16)$$

$$a_3 = -\frac{b_1^3}{4}, \quad b_3 = 0, \quad (17)$$

$$a_4 = \frac{b_1^4}{8}, \quad b_4 = 0, \quad (18)$$

$$a_5 = -\frac{b_1^5}{16}, \quad b_5 = 0, \quad (19)$$

$$a_6 = \frac{b_1^6}{32}, \quad b_6 = 0. \quad (20)$$

We remember  $b_1 = 1$  in the Newtonian limit of Einstein equations. Therefore, we finally have

$$a_1 = -1, \quad b_1 = 1, \quad (21)$$

$$a_2 = \frac{1}{2}, \quad b_2 = \frac{1}{4}, \quad (22)$$

$$a_3 = -\frac{1}{4}, \quad b_3 = 0, \quad (23)$$

$$a_4 = \frac{1}{8}, \quad b_4 = 0, \quad (24)$$

$$a_5 = -\frac{1}{16}, \quad b_5 = 0, \quad (25)$$

$$a_6 = \frac{1}{32}, \quad b_6 = 0. \quad (26)$$

On the other hand, if we expand the Schwarzschild solution, Equation (6), using the Taylor series, we have

$$A = 1 - \frac{M}{r} + \frac{M^2}{2r^2} - \frac{M^3}{4r^3} + \frac{M^4}{8r^4} - \frac{M^5}{16r^5} + \frac{M^6}{32r^6} + O\left(\frac{1}{r^6}\right), \quad (27)$$

$$B = 1 + \frac{M}{r} + \frac{M^2}{4r^2}. \quad (28)$$

Equations (21)–(26) are exactly consistent with Equations (27)–(28). Thus we conclude that there are no problems in seeking the *exact* solution to any order by using the Taylor series method.

### 3 SOLUTION IN THE STAROBINSKY MODEL

In this section, we solve the field equations in the Starobinsky model by using the Taylor series method. The action of the Starobinsky model in the absence of matters is given by

$$S = \int d^4x \sqrt{-g} \frac{1}{16\pi} (R + sR^2). \quad (29)$$

Here  $s$  is a coupling constant which has the dimension of the square of length. The corresponding equations of motion take the form of

$$G_{\mu\nu} + 2sR \left( G_{\mu\nu} + \frac{1}{4} R g_{\mu\nu} \right) + 2s (g_{\mu\nu} \nabla^2 R - \nabla_\mu \nabla_\nu R) = 0. \quad (30)$$

Given the metric, Equation (2), the equations of motion are then obtained. Then we face a daunting task of resolving the equations. Therefore, we resort to the Taylor series expansion method. Substituting Equations (7) and (8) into the equations of motion, we obtain

$$0 = \frac{\alpha_2}{r^2} + \frac{\alpha_3}{r^3} + \frac{\alpha_4}{r^4} + \frac{\alpha_5}{r^5} + \frac{\alpha_6}{r^6} + O\left(\frac{1}{r^6}\right), \quad (31)$$

$$0 = \frac{\beta_1}{r} + \frac{\beta_2}{r^2} + \frac{\beta_3}{r^3} + \frac{\beta_4}{r^4} + \frac{\beta_5}{r^5} + \frac{\beta_6}{r^6} + O\left(\frac{1}{r^6}\right), \quad (32)$$

$$0 = \frac{\gamma_1}{r} + \frac{\gamma_2}{r^2} + \frac{\gamma_3}{r^3} + \frac{\gamma_4}{r^4} + \frac{\gamma_5}{r^5} + \frac{\gamma_6}{r^6} + O\left(\frac{1}{r^6}\right), \quad (33)$$

where  $\alpha_i$ ,  $\beta_i$  and  $\gamma_i$  are determined by  $a_i$ ,  $b_i$  and  $M$ . In this case, we also have  $\gamma_1 = \beta_1$  and  $\gamma_2 = \beta_2$ . Thus we have the following algebraic equations

$$\alpha_2 = 0, \quad \alpha_3 = 0, \quad \alpha_4 = 0, \quad \alpha_5 = 0, \quad (34)$$

$$\beta_1 = 0, \quad \beta_2 = 0, \quad \beta_3 = 0, \quad \beta_4 = 0, \quad (35)$$

$$\beta_5 = 0, \quad \gamma_3 = 0, \quad \gamma_4 = 0, \quad \gamma_5 = 0. \quad (36)$$

Solving these equations, we obtain

$$a_1 = -b_1, \quad b_1 = b_1, \quad (37)$$

$$a_2 = \frac{b_1^2}{2}, \quad b_2 = \frac{b_1^2}{4}, \quad (38)$$

$$a_3 = -\frac{b_1^3}{4}, \quad b_3 = 0, \quad (39)$$

$$a_4 = \frac{b_1^4}{8}, \quad b_4 = 0, \quad (40)$$

$$a_5 = -\frac{b_1^5}{16}, \quad b_5 = 0, \quad (41)$$

$$a_6 = \frac{b_1^6}{32}, \quad b_6 = 0. \quad (42)$$

They are exact for the Schwarzschild solution when  $b_1 = 1$ . We note that the coupling constant  $s$  does not emerge in the expressions of the coefficients. This means the Ricci-squared term  $sR^2$  makes no contribution in the equations of motion. This observation is consistent with the proof of Nelson. In fact, Nelson showed that any static black-hole solution of the Starobinsky model must have a vanishing Ricci scalar. Taking this into consideration, the equations of motion defined by Equation (30) are simply

$$G_{\mu\nu} = 0. \quad (43)$$

Thus the Ricci-squared term  $sR^2$  makes no contribution to the equations of motion. Up to this point, we confirm that Nelson's conclusion that *the unique, static, spherically symmetric and asymptotically flat black hole solution in the Starobinsky model which is nothing but the Schwarzschild solution* is correct, despite the flaw in his proof. Compared to Nelson's verification, we can go further. Using the Taylor series expansion method, we have checked that for more general modified gravity

$$S = \int d^4x \sqrt{-g} \frac{1}{16\pi} \times \left[ R + s_2 R^2 + s_3 R^3 + s_4 R^4 + O(R^4) \right], \quad (44)$$

the unique, static, spherically symmetric and asymptotically flat black hole solution remains the Schwarzschild solution. Then the applicability of the conclusion is extended. It should be noted that Cañate recently presented a no-hair theorem which discards the existence of four dimensional asymptotically flat, static and spherically symmetric or stationary axially symmetric, non-trivial black holes in the frame of  $f(R)$  gravity (Cañate 2018). This represents huge support for our examination.

#### 4 CONCLUSION AND DISCUSSION

In conclusion, we have investigated the solution to the problem in the Starobinsky gravity model. By using the Taylor series expansion method, we show the static, spherically symmetric and asymptotically flat spacetime is uniquely the Schwarzschild solution. This is in agreement with the proof of Nelson who uses the method of extending the Lichnerowicz and Israel theorems from Einstein gravity to higher order theories of gravity, despite the flaw in his proof. But compared to Nelson's verification, we can go further. We find the conclusion can be extended to the more generalized modified gravity. Thus it is an important conclusion.

We can also understand the conclusion in another way. We remember that the  $f(R)$  theory of modified gravity is equivalent to the scalar tensor theory

$$\tilde{S} = \int d^4x \sqrt{-\tilde{g}} \times \left[ \frac{\tilde{R}}{16\pi} + \frac{1}{2} \nabla_\mu \tilde{\phi} \nabla^\mu \tilde{\phi} + V(\tilde{\phi}) \right], \quad (45)$$

and by making variable redefinitions and conformal transformations (Sotiriou & Faraoni 2010)

$$g_{\mu\nu} = e^{-\sqrt{\frac{16\pi}{3}} \tilde{\phi}} \tilde{g}_{\mu\nu}. \quad (46)$$

Heusler (1992); Sudarsky (1995) have shown that black holes in the above scalar tensor theory cannot carry scalar charge for arbitrary potentials. This is known as the black-hole no-hair theorem. The theorem guarantees the scalar field  $\tilde{\phi}$  is constant in this spacetime. Then the scalar potential plays the role of the cosmological constant. The no-hair theorem requires that the Schwarzschild-de Sitter solution is the unique static spherically symmetric spacetime. If one assumes the spacetime is asymptotically flat, we are left with only the Schwarzschild solution  $\tilde{g}_{\mu\nu}$ . From Equation (46) we see the solution  $g_{\mu\nu}$  of  $f(R)$  gravity is exactly the

Schwarzschild solution since it is proportional to  $\tilde{g}_{\mu\nu}$  by a constant. In all, we have good reason to believe that *the Schwarzschild solution is the unique static, spherically symmetric and asymptotically flat spacetime solution in the Starobinsky gravity.*

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