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Revisiting gamma-ray burst afterglows with time-dependent parameters

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Abstract The relativistic external shock model of gamma-ray burst (GRB) afterglows has been established with five free parameters, i.e., the total kinetic energy E, the equipartition parameters for electrons $\epsilon_{\rm e}$ and for the magnetic field $\epsilon_{\rm B}$, the number density of the environment n and the index of the power-law distribution of shocked electrons p. A lot of modified models have been constructed to consider the variety of GRB afterglows, such as: the wind medium environment by letting n change with radius, the energy injection model by letting kinetic energy change with time and so on. In this paper, by assuming all four parameters (except p) change with time, we obtain a set of formulas for the dynamics and radiation, which can be used as a reference for modeling GRB afterglows. Some interesting results are obtained. For example, in some spectral segments, the radiated flux density does not depend on the number density or the profile of the environment. As an application, through modeling the afterglow of GRB 060607A, we find that it can be interpreted in the framework of the time dependent parameter model within a reasonable range.

Key words: (stars:) gamma-ray burst: general — methods: analytical

1 INTRODUCTION

The external forward shock afterglow model, which is often taken as the standard model of gamma-ray burst (GRB) afterglows, was established in the late 1990s (Sari et al. 1996; Mészáros & Rees 1997; Katz & Piran 1997; Goodman 1997; Sari et al. 1998), based on the fireball model of GRBs (e.g., Rees & Meszaros 1992, 1994). It has been summarized into the standard form (e.g., Sari et al. 1998; Wijers & Galama 1999; Gruzinov & Waxman 1999). The picture is that the afterglows are the results of a collimated relativistic jet interacting with an external medium and producing a collisionless shock, in which electrons are accelerated and emit photons via synchrotron radiation.

The standard model has achieved great success: it explains well the observations of late time afterglows in broad-band (e.g., Panaitescu & Kumar 2000, 2001; Yost et al. 2003). But after the launch of the *Swift* satellite in 2004, some observed data were surprising: a great part of them cannot be explained in the framework of the standard model, especially early X-ray afterglow data. Some issues are still open, such as: what are the details of the geometry and dynamics of the outflows, how are electrons accelerated by the shocks, how is the magnetic field amplified by the shock and so on (Kumar & Zhang 2015). Thus, various kinds of models based on the standard model have been constructed. In general, the standard model has been extended in the following aspects: (a) reverse shock; (b) geometry; (c) ambient circumburst medium; (d) outflow compositions; (e) radiation processes. We list some correlative and selected references in the following (see Kumar & Zhang (2015) for a comprehensive review).

(a) Considering the reverse shock, authors have predicted the very early afterglow and the optical flash of GRB afterglows (Sari & Piran 1999b). They found that observations of the GRB 990123 afterglow agreed with predictions from the theory (Sari & Piran 1999a). Afterwards, details about the dynamics and radiation were established for an interstellar environment (Kobayashi 2000) and wind environment (Zhang et al. 2003; Kobayashi & Zhang 2003; Wu et al. 2003; Zou et al. 2005), with more applications to individual GRBs (e.g., Fan et al. 2002; Shao & Dai 2005). A more complex model based on it has been constructed, such as reverse shock from magnetized ejecta (Zhang & Kobayashi 2005), and reverse-forward shock from a differently shaped jet (Yan et al. 2007). As an application, for the unusual case of GRB 130427A, authors used a reverse shock to fit its afterglow (Laskar et al. 2013).

(b) Considering the effects of geometry, authors have improved the jet models (Sari et al. 1999; Wu et al. 2004b), including the off-axis model (e.g., Heise 2003; van Eerten & MacFadyen 2013) and the structured jet model (Mészáros et al. 1998; Zhang & Mészáros 2002a). In particular, the structured jet model is divided into many types, such as: the bulk Lorentz factor depending on the direction angle θ as power laws (Mészáros et al. 1998), angular energy distributions following a Gaussian distribution (Zhang & Mészáros 2002a), and a two-component jet (Berger et al. 2003; Huang et al. 2004; Wu et al. 2005b; Gao & Wei 2005; Xie et al. 2012). For more such topics, refer to Lipunov et al. (2001); Dai & Gou (2001); Rossi et al. (2002); Wei & Jin (2003); Jin & Wei (2004); Jin et al. (2007) and Zhang et al. 2004. Some more erratic shapes of the jet have been suggested, such as a ring-shaped jet (Zou & Dai 2006; Xu et al. 2008; Xu & Huang 2010) and a cylindrical jet (Cheng et al. 2001).

(c) For the ambient circumburst medium, authors considered effects of GRB surroundings, such as: (1) Wind medium rather than interstellar medium (ISM) (e.g., Dai & Lu 1998b; Dai & Wu 2003; Chevalier & Li 1999, 2000); (2) More complicated surroundings, such as a density-jump medium (Dai et al. 2001; Dai & Lu 2002; Feng & Dai 2011; Geng et al. 2014), or a general decline in the density profile (Yi et al. 2013). Later on, the combination of different environments was considered, such as a jet in a wind environment with lateral expansion (Wu et al. 2004a).

(d) Considering outflow compositions, mainly including neutron-rich, magnetization or magnetic field structure, authors did a lot of work, such as: (1) The neutron component in a fireball was introduced by Derishev et al. (1999), and some authors developed it (e.g., Beloborodov 2003; Fan & Wei 2004; Fan et al. 2005b,a); (2) Magnetized jet (e.g., Fan et al. 2004b,c, 2005b; Zhang & Yan 2011); (3) The radial decay magnetic field was discussed in Uhm & Zhang (2014) and Zhang et al. (2016); (4) Electron-positron pairs in the afterglow were investigated, such as pair loading in the prompt and earlier afterglow stage (Li et al. 2003; Fan et al. 2004a) and the wind bubble in the afterglow (Donaghy 2006; Yu & Dai 2007). Other considerations were also discussed, such as a flat spectrum of the electron's power law with index $p < 2$ (Dai & Cheng 2001; Wang et al. 2012).

(e) Authors extended the model by incorporating more physical processes, which mainly affect the dynamic evolution of GRBs and their afterglows. These include: (1) The inverse Compton (IC) scattering in GRB afterglows was introduced by Wei & Lu (1998), and detailedly discussed in Sari & Esin (2001); for the latest progresses, refer to Uhm & Zhang (2014) and Zhao et al. (2014). Authors have extended the model in different aspects, such as in terms of the light curves (Wei & Lu 2000). (2) The relativistic shock in the model is usually assumed to be radiative or quasi-adiabatic. As an improvement, a realistic model involving radiation loss for GRB afterglows has been developed (Huang et al. 2000a). Nevertheless, other effects based on radiation processes were discussed extensively, such as the light curve breaks caused by the spectrum crossing effect (Wei & Lu 2002), afterglows in the radiative regime (Li et al. 2002), a comprehensive consideration on the environment, IC scattering and radiative efficiency (Wu et al. 2005a), and Klein-Nishina effects on high-energy emission (Wang et al. 2010).

Nevertheless, more considerations based on the standard afterglow model have been proposed. Huang et al. (1998, 1999b,a) considered the overall evolution of an afterglow, especially including the non-relativistic stage of the afterglow, and later on, different aspects of non-relativistic effects were discussed, such as the beamed jet effect (Wei & Lu 2001). Sari (1998); Li et al. (2000); Huang et al. (2007); Geng et al. (2013) considered effects of the equal arrival time surface. To explain the variety of afterglows, Dai & Lu (1998a, 2000) proposed the energy injection model, and the energy injected into different environments was considered (Wang & Dai 2001). Many applications to individual GRBs have been applied, such as Knust et al. (2017) to GRB 150424A, Fan & Xu (2006) to GRB 051221A, Xu et al. (2009) to GRB 060729, and Liu et al. (2009) to the shallow decay of X-ray afterglows. Especially after the launch of *Swift*, more and more late activities have been observed, and consequently related models have been proposed, such as the late internal shock model for X-ray flares (Fan & Wei 2005; Wu et al. 2006; Zou et al. 2006; Yu & Dai 2009). The dust scattering effect on the X-ray afterglow has been extensively considered (Shao & Dai 2007).

Instead of extending the standard model, an opposite problem appeared: what is the available range of the standard model? In other words, how bad/good is the external forward shock model? Wang et al. (2015) performed a systematic study on this question. Very recently, De Pasquale et al. (2016) argued that the late X-ray afterglow of GRB 130427A challenges the standard forward shock model.

The relativistic external shock model of GRB afterglows has five free parameters, i.e., the total kinetic energy E, the equipartition parameters $\epsilon_{\rm e}$ for electrons and $\epsilon_{\rm B}$ for the magnetic field, the number density of the environment n and the index of the power-law distribution of the shocked electrons p. All the above models involve constant microphysics parameters of $\epsilon_{\rm e}$ and $\epsilon_{\rm B}$. Yet, in principle, these parameters should depend on time. Some authors have considered such more complicated models (e.g., Ioka et al. 2006; Fan & Piran 2006), but their schemes either involve less than four parameters that change or just give a part of the formulas (e.g., Maselli et al. 2014; van Eerten et al. 2012). No one has ever provided a fully parameterized solution by releasing all the parameters evolving with time. We modify the four parameters as $\{E(t), \epsilon_{\rm B}(t), \epsilon_{\rm e}(t), n(R)\}$ depending on time to develop a set of formulas, which can be used as a reference for modeling the GRB afterglows. One can allow any of these parameters to change with time, while leaving others constant, and the related formulas can be read simply from the set of formulas.

In Section 2, we briefly review the fireball scaling relations of the dynamics and synchrotron radiation from the external shock model of GRB afterglows. In Section 3, we derive the flux densities according to different cases of typical frequency combinations. In Section 4, we plot the typical light curves and other quantities varying with time, and apply them to GRB 060607A. Conclusions and discussion are presented in Section 5.

2 SCALES AND CHARACTERISTIC PARAMETERS

2.1 Scaling-laws of the Fireball Model

GRB afterglows are produced by collimated relativistic jets, the dynamics of which can be described by evolution of the fireball. In the afterglow standard model, a spherical relativistic adiabatic blast wave propagates into the external medium. The basic formulas of fireball evolution are the scaling relations. Considering the synchrotron radiation formulas as well as the Lorentz transformation, one can obtain the critical parameters and predict the observable quantities, and compare them with observations (Sari et al. 1998, hereafter SPN98).

In the ultra-relativistic phase, the shocked shell can be roughly described by the Blandford-McKee solution (Kobayashi & Sari 2000; Wu et al. 2003; Kobayashi & Zhang 2003; Kobayashi 2000). Following Blandford & McKee (1976) (hereafter BM76) and Panaitescu & Kumar (2000), as the first extension of the standard model, to describe more kinds of external media, one can assume the external medium (proton) number density can be modeled by a power law

$$n(R) = AR^{-k}, \tag{1}$$

where A is defined as $A = n_0$ if $k = 0$, or generally defined as $n(R) = A_* R_{18}^{-k}$, in which A_* indicates the number density at $R = 10^{18}$ cm, R is the radial distance from the center of the source and k is in the range of $[0, 4)$ (for $k > 4$, refer to Best & Sari (2000), in which the Blandford-McKee solution is not suitable in this paper). In particular, $k = 0$ means a homogeneous ISM and $k = 2$ means free wind. Thus the mass of medium swept by the shock wave is

$$m(R) = \frac{4\pi}{3-k} m_{\rm p} n(R) R^3, \tag{2}$$

where $m_{\rm p}$ is the mass of a proton. There are two radiation limits in the fireball model: adiabatic and radiative. Although radiative models have been discussed by authors, one can ignore the radiation case because radiation time scale is much larger than the fireball dynamic evolutionary time scale. What is more, authors found that the blast wave became an adiabatic blast wave at later stages (e.g., Huang et al. 2000a,b; Pe'er & Zhang 2006). Thus, here we only consider the adiabatic case (e.g., SPN98).

For simplicity, one can ignore energy injection by considering total energy as constant, namely, assuming that the energy of the blast wave contained in the fireball (BM76) is $E = \frac{8\pi}{17-4k} n(R) R^3 \gamma^2 m_{\rm p} c^2 = {\rm const}$, where γ is the bulk Lorentz factor of the shocked material relative to the medium. For detailed calculation, refer to BM76. The relation between radius R, Lorentz factor γ and observer's time t is

$${\rm d}R = \frac{4}{1+z} \gamma^2 c {\rm d}t, \tag{3}$$

where z is the redshift. One can derive the dynamic relations of R and γ, e.g. Granot & Sari (2002); Gao et al. (2013).

Considering energy injection, when the injected energy in the form of a Poynting flux and a reverse shock does not exist or is very weak, one can approximately treat the blast wave as a system with continuous energy increase. Phenomenologically, one can assume that the central engine has a power-law luminosity $L(t) = L_0 \left(t/\tau_{\rm E}\right)^{-q_{\rm E}}$, where L_0 is the luminosity before injection, $\tau_{\rm E}$ is initial time of energy injection and $q_{\rm E}$ indicates how fast the energy is being injected. Thus, the injected energy is $E_{\rm inj} = L_0 \tau_{\rm E}^{q_{\rm E}} t^{1-q_{\rm E}} \big/ (1-q_{\rm E})$(Gao et al. 2013; van Eerten 2014; Kumar & Zhang 2015). Therefore, the total energy in an engine-fireball system is

$$E_{\rm tot}(t) = E_{\rm bef} + \int_0^t L(t){\rm d}t = E_{\rm bef} + E_{\rm inj} = E_{\rm bef} + \frac{L_0 \tau_{\rm E}^{q_{\rm E}}}{1-q_{\rm E}} t^{1-q_{\rm E}}\,, \tag{4}$$

where $E_{\rm bef}$ is the energy of the fireball before injection. The energy injection effect becomes significant when $E_{\rm inj} \gg E_{\rm bef}$. Alteratively, one can assume that $E_{tot}(t) = E_0(\frac{t}{\tau_{\rm E}})^{1-q_{\rm E}}$ for $t > \tau_{\rm E}$ and $E(t) = E_0$ for $t \le \tau_{\rm E}$, where $E_0 \equiv \frac{L_0\tau_{\rm E}^{q_{\rm E}}}{1-q_{\rm E}}$. The two methods are equivalent (see Kumar & Zhang 2015, for example). Note that although the injection is slow, it can significantly affect the dynamics of the fireball. Adopting the method of BM76, one can modify the scaling-laws of the dynamics as

$$R = \left[\frac{(4-k)(17-4k)}{4\pi m_{\rm p} c(2-q_{\rm E})}\right]^{\frac{1}{4-k}} \times (1+z)^{\frac{2-q_{\rm E}}{k-4}} \times A^{-\frac{1}{4-k}} E_0^{\frac{1}{4-k}} \tau_{\rm E}^{\frac{q_{\rm E}-1}{4-k}} t^{\frac{2-q_{\rm E}}{4-k}}, \tag{5}$$

and

$$\gamma = \left[\frac{(2-q_{\rm E})^{3-k}(17-4k)}{4^{5-k}\,(4-k)^{3-k}\,\pi m_{\rm p} c^{5-k}}\right]^{\frac{1}{8-2k}} \times (1+z)^{\frac{2+q_{\rm E}-k}{8-2k}} \times A^{-\frac{1}{8-2k}} E_0^{\frac{1}{8-2k}} \tau_{\rm E}^{\frac{1-q_{\rm E}}{2k-8}} t^{\frac{2+q_{\rm E}-k}{2k-8}}. \tag{6}$$

In particular, one has

$$R = \left[\frac{17}{\pi m_{\rm p} c(2-q_{\rm E})}\right]^{\frac{1}{4}} \times (1+z)^{\frac{q_{\rm E}-2}{4}} \times A^{-\frac{1}{4}} E_0^{\frac{1}{4}} \tau_{\rm E}^{\frac{q_{\rm E}-1}{4}} t^{\frac{2-q_{\rm E}}{4}}, \tag{7}$$

and

$$\gamma = \left[\frac{17(2-q_{\rm E})^3}{4^8 \pi m_{\rm p} c^5}\right]^{\frac{1}{8}} \times (1+z)^{\frac{2+q_{\rm E}}{8}} \times A^{-\frac{1}{8}} E_0^{\frac{1}{8}} \tau_{\rm E}^{\frac{q_{\rm E}-1}{8}} t^{-\frac{2+q_{\rm E}}{8}}, \tag{8}$$

for k = 0 (ISM),

$$R = \left[\frac{9}{2\pi m_{\rm p} c(2-q_{\rm E})}\right]^{\frac{1}{2}} \times (1+z)^{\frac{q_{\rm E}-2}{2}} \times A^{-\frac{1}{2}} E_0^{\frac{1}{2}} \tau_{\rm E}^{\frac{q_{\rm E}-1}{2}} t^{\frac{2-q_{\rm E}}{2}}, \tag{9}$$

and

$$\gamma = \left[\frac{9(2-q_{\rm E})}{4^3 2\pi m_{\rm p} c^3}\right]^{\frac{1}{4}} \times (1+z)^{\frac{q_{\rm E}}{4}} \times A^{-\frac{1}{4}} E_0^{\frac{1}{4}} \tau_{\rm E}^{\frac{q_{\rm E}-1}{4}} t^{-\frac{q_{\rm E}}{4}}, \tag{10}$$

for k = 2 (free wind).

The parameter $q_{\rm E}$, being much larger than 1, means the energy injection is not important, and Equation (5) and Equation (6) return to the cases without energy injection, for which $q_{\rm E} = 1$ in the scaling laws.

2.2 Typical Frequency and Cooling Frequency

The strength of a post-shock magnetic field in its comoving frame is given by SPN98: $B = (32\pi m_{\rm p}\epsilon_{\rm B} n)^{\frac{1}{2}}\gamma c$. A relativistic electron with random Lorentz factor $\gamma_{\rm e} \gg 1$ in a magnetic field emits synchrotron radiation. Respectively, the radiation power and characteristic frequency of an electron are given by (e.g., Rees 1967; Rybicki & Lightman 1979; Ghisellini 2013) $P(\gamma_{\rm e}) = \frac{4}{3}\sigma_{\rm T} c\gamma^2\gamma_{\rm e}^2\frac{B^2}{8\pi}$ and $\nu(\gamma_{\rm e}) = \gamma\gamma_{\rm e}^2\frac{qB}{2\pi m_{\rm e} c}$.

The spectral power varies as $\nu^{\frac{1}{3}}$ for $\nu < \nu(\gamma_{\rm e})$, and cuts off exponentially for $\nu > \nu(\gamma_{\rm e})$. Approximately, the peak spectral power of a single electron $P_{\nu,\max}$ at $\nu(\gamma_{\rm e})$ is

$$P_{\nu,\max} \simeq \frac{P(\gamma_{\rm e})}{\nu(\gamma_{\rm e})} = \frac{m_{\rm e} c^2 \sigma_{\rm T}}{3q}\gamma B = \frac{\sqrt{3}\phi_{\rm p} q^3}{m_{\rm e} c}\gamma B.$$

Here $\phi_{\rm p}$ is a parameter introduced by Wijers & Galama (1999): $\phi_{\rm p} \equiv F_\nu(x_{\rm p})$ is the dimensionless flux at this point and $x_{\rm p}$ is the dimensionless maximum point of the spectrum. Thus the peak flux density of the afterglow is $F_{\nu,\max} = \frac{N_{\rm e} P_{\nu,\max}}{4\pi D_{\rm L}^2}(1+z)$, where $D_{\rm L}$ is the luminosity distance to the observer and $N_{\rm e} = \frac{4\pi}{3-k} A R^{3-k}$ is the total number of radiating electrons.

Assuming that the distribution of accelerated electrons is a power law, e.g., $n(\gamma_{\rm e}){\rm d}\gamma_{\rm e} \propto \gamma_{\rm e}^{-{\rm p}}{\rm d}\gamma_{\rm e}$ for $\gamma_{\rm e} \ge \gamma_{\rm m}$, where $\gamma_{\rm m}$ is minimum Lorentz factor of the accelerated electrons, one has $\gamma_{\rm m} = \epsilon_{\rm e}\frac{p-2}{p-1}\frac{m_{\rm p}}{m_{\rm e}}\gamma$, where $\epsilon_{\rm e}$ is energy fraction of the total internal energy carried by electrons, and $p > 2$ is required. To represent analytically, SPN98 introduced the cooling Lorentz factor $\gamma_{\rm c}$ defined by $\gamma\gamma_{\rm c} m_{\rm e} c^2 = P(\gamma_{\rm c})\,t$, namely $\gamma_{\rm c} = \frac{6\pi m_{\rm e} c}{\sigma_{\rm T}\gamma B^2 t} = \frac{3m_{\rm e}}{16\epsilon_{\rm B}\sigma_{\rm T} m_{\rm p} c}\frac{1}{t\gamma^3 n_{\rm e}}$, where $\sigma_{\rm T}$ is the Thomson cross section. The electron Lorentz factors $\gamma_{\rm c}$ and $\gamma_{\rm m}$ define two characteristic emission frequencies $\nu_{\rm c}$ and $\nu_{\rm m}$ in the synchrotron spectrum, respectively. Substituting the expression of $\gamma_{\rm c}$ into the typical synchrotron frequency, one can obtain the cooling frequency $\nu_{\rm c}$, which has the physical meaning that if synchrotron frequency is higher than it, electron cooling should be considered. Substituting the expression of $\gamma_{\rm m}$ into the synchrotron frequency above, one has the typical synchrotron frequency $\nu_{\rm m}$. Following Wijers & Galama (1999), one has

$$\nu_{\rm m} = \frac{3x_{\rm p}}{4\pi}\frac{qB}{m_{\rm e} c}\gamma_{\rm m}^2\gamma = \frac{3x_{\rm p}}{4\pi}\frac{qB}{m_{\rm e} c}\epsilon_{\rm e}^2\left(\frac{p-2}{p-1}\right)^2\left(\frac{m_{\rm p}}{m_{\rm e}}\right)^2\gamma^3.$$

The $\nu_{\rm m}$ and $\nu_{\rm c}$, along with latter $\nu_{\rm a}$, correspond to breaks in the spectrum, which consists of several power law segments. They are also known as the 'break frequencies.'

The continuity equation for electrons in energy space is (e.g., Rybicki & Lightman 1979): $\frac{\partial}{\partial t}n(\gamma_{\rm e},t)+\frac{\partial}{\partial\gamma}\left(\dot{\gamma_{\rm e}}n(\gamma_{\rm e},t)\right)=Q\left(\gamma_{\rm e}\right)$, where $n(\gamma_{\rm e},t)$ is the distribution of electrons, $Q\left(\gamma_{\rm e}\right)$ is the rate of electron injection and $\dot{\gamma_{\rm e}}$ is the rate of change of $\gamma_{\rm e}$ due to radiation. In the case of adiabatic steady injection of electrons, one has $Q(\gamma_{\rm e})\propto\gamma_{\rm e}^{-p}$ (electrons accelerated by shock). By solving the equations in the condition $\frac{\partial}{\partial t}n(\gamma_{\rm e},t)=0$, one can obtain a stable distribution of the electrons due to continuous injection.

We generalize parameters $\{\epsilon_{\rm B},\epsilon_{\rm e}\}$ as $\{\epsilon_{\rm B}(t),\epsilon_{\rm e}(t)\}$ phenomenologically, namely $\epsilon_{\rm B}(t)=\epsilon_{\rm B,0}$ when $t<\tau_{\rm B}$, $\epsilon_{\rm B}(t)=\epsilon_{\rm B,0}\left(t/\tau_{\rm B}\right)^{-q_{\rm B}}$ when $t\geq\tau_{\rm B}$, and $\epsilon_{\rm e}(t)=\epsilon_{\rm e,0}$ when $t<\tau_{\rm e}$, $\epsilon_{\rm e}(t)=\epsilon_{\rm e,0}\left(t/\tau_{\rm e}\right)^{-q_{\rm e}}$ when $t\geq\tau_{\rm e}$, for which $\epsilon_{\rm B,0}$ and $\epsilon_{\rm e,0}$ are fixed values, $\tau_{\rm B}$ and $\tau_{\rm e}$ are characteristic decay times, and $q_{\rm B}$ and $q_{\rm e}$ are the corresponding slopes. Note that $q_{\rm B}$ and $q_{\rm e}$ are assumed to be so small that they do not affect the dynamics. It is worth mentioning that Warren et al. (2017) concluded that the evolution of $\epsilon_{\rm e}$ is close to a power law decay.

In GRB afterglows, normally, the effect of IC could be important (Sari et al. 1996). Formally, one can include the effect by inserting an appropriate power of $(1+{\rm Y})$ into the formulas for break frequencies, where $Y=(\epsilon_{\rm e}/\epsilon_{\rm B})^{\frac{1}{2}}$ in fast cooling phases, while $Y=(\epsilon_{\rm e}/\epsilon_{\rm B})^{\frac{1}{2}}(\nu_{\rm m}/\nu_{\rm c})^{\frac{p-2}{4}}$ in slow cooling phases (Sari & Esin 2001). Using the formulas, one can calculate the analytical expressions of generalized characteristic frequency as

$$\begin{aligned}\nu_{\rm m}=\;&3^{1}2^{-\frac{9}{2}}\pi^{-1}c^{-\frac{5}{2}}x_{\rm p}m_{\rm e}^{-3}m_{\rm p}^{2}q\left(\frac{p-2}{p-1}\right)^{2}\\&\epsilon_{\rm B,0}^{\frac{1}{2}}\epsilon_{\rm e,0}^{2}\left(17-4k\right)^{\frac{1}{2}}\left(2-q_{\rm E}\right)^{\frac{3}{2}}\left(4-k\right)^{-\frac{3}{2}}\\&\left(1+z\right)^{\frac{1}{2}q_{\rm E}+\frac{1}{2}q_{\rm B}+2q_{\rm e}}\\&E_{0}^{\frac{1}{2}}\tau_{\rm E}^{\frac{1}{2}(q_{\rm E}-1)}\tau_{\rm B}^{\frac{1}{2}q_{\rm B}}\tau_{\rm e}^{2q_{\rm e}}t^{-\frac{1}{2}q_{\rm E}-1-\frac{1}{2}q_{\rm B}-2q_{\rm e}},\end{aligned}\tag{11}$$

and

$$\begin{aligned}\nu_{\rm c}=\;&3^{3}2^{\frac{20-k}{2k-8}}\pi^{\frac{k}{k-4}}c^{\frac{3k-4}{2k-8}}x_{\rm p}m_{\rm e}m_{\rm p}^{\frac{4}{k-4}}q\sigma_{\rm T}^{-2}\epsilon_{\rm B,0}^{-\frac{3}{2}}\\&(4-k)^{\frac{k-12}{2k-8}}\left(17-4k\right)^{\frac{4-3k}{2k-8}}\left(2-q_{\rm E}\right)^{\frac{12-k}{2k-8}}\\&\left(1+Y\right)^{-2}\left(1+z\right)^{\frac{(4-3k)q_{\rm E}+4k}{2k-8}-\frac{3}{2}q_{\rm B}}\\&A^{\frac{4}{k-4}}E_{0}^{\frac{4-3k}{2k-8}}\tau_{\rm E}^{\frac{3k-4}{2k-8}(1-q_{\rm E})}\tau_{\rm B}^{-\frac{3}{2}q_{\rm B}}\\&t^{\frac{(2-q_{\rm E})(4-3k)}{2k-8}+\frac{3}{2}q_{\rm B}}.\end{aligned}\tag{12}$$

The analytical expressions are complicated. Usually, the numerical equations are obtained by substituting multiple parameters into the general formulas. We take $p=2.5$ if p has to be set as a number to get the analytical expression, and the time $t_{\rm d}$ is in the unit of day.

For $k=0$ (ISM), one has

$$\begin{aligned}\nu_{\rm m}=\;&1.67\times10^{15}\eta^{-2q_{\rm e}-\frac{1}{2}q_{\rm E}-\frac{1}{2}q_{\rm B}}\\&\epsilon_{\rm e,-0.5}^{2}\epsilon_{\rm B,-1}^{\frac{1}{2}}\left(2-q_{\rm E}\right)^{\frac{3}{2}}\\&(1+z)^{\frac{1}{2}q_{\rm E}+\frac{1}{2}q_{\rm B}+2q_{\rm e}}\\&E_{53}^{\frac{1}{2}}\tau_{\rm E,4}^{\frac{1}{2}(q_{\rm E}-1)}\tau_{\rm B,4}^{\frac{1}{2}q_{\rm B}}\tau_{\rm e,4}^{2q_{\rm e}}\\&t_{\rm d}^{-2q_{\rm e}-\frac{1}{2}q_{\rm E}-\frac{1}{2}q_{\rm B}-1}\ {\rm Hz},\end{aligned}\tag{13}$$

where $\eta=8.64$, which occurs with the time converted to the unit of day, and

$$\begin{aligned}\nu_{\rm c}=\;&5.70\times10^{11}\eta^{\frac{1}{2}q_{\rm E}+\frac{3}{2}q_{\rm B}}\epsilon_{\rm B,-1}^{-\frac{3}{2}}\left(2-q_{\rm E}\right)^{-\frac{3}{2}}\\&(1+z)^{-\frac{1}{2}q_{\rm E}-\frac{3}{2}q_{\rm B}}\left(1+Y\right)^{-2}\\&n_{0}^{-1}E_{53}^{-\frac{1}{2}}\tau_{\rm E,4}^{\frac{1}{2}(1-q_{\rm E})}\tau_{\rm B}^{-\frac{3}{2}q_{\rm B}}t_{\rm d}^{\frac{1}{2}q_{\rm E}+\frac{3}{2}q_{\rm B}-1}\ {\rm Hz}.\end{aligned}\tag{14}$$

For $k=2$ (wind medium), one has

$$\begin{aligned}\nu_{\rm m}=\;&3.42\times10^{15}\eta^{-2q_{\rm e}-\frac{1}{2}q_{\rm E}-\frac{1}{2}q_{\rm B}}\epsilon_{\rm e,-0.5}^{2}\epsilon_{\rm B,-1}^{\frac{1}{2}}\\&(2-q_{\rm E})^{\frac{3}{2}}\left(1+z\right)^{\frac{1}{2}q_{\rm E}+\frac{1}{2}q_{\rm B}+2q_{\rm e}}E_{53}^{\frac{1}{2}}\\&\tau_{\rm E,4}^{\frac{1}{2}(q_{\rm E}-1)}\tau_{\rm B,4}^{\frac{1}{2}q_{\rm B}}\tau_{\rm e,4}^{2q_{\rm e}}t_{\rm d}^{-2q_{\rm e}-\frac{1}{2}q_{\rm E}-\frac{1}{2}q_{\rm B}-1}\ {\rm Hz},\end{aligned}\tag{15}$$

and

$$\begin{aligned}\nu_{\rm c}=\;&5.91\times10^{15}\eta^{\frac{3}{2}q_{\rm B}-\frac{1}{2}q_{\rm E}}\epsilon_{\rm B,-1}^{-\frac{3}{2}}\left(2-q_{\rm E}\right)^{-\frac{5}{2}}\\&(1+z)^{\frac{1}{2}q_{\rm E}-\frac{3}{2}q_{\rm B}-2}\left(1+Y\right)^{-2}\\&E_{53}^{\frac{1}{2}}\tau_{\rm E,4}^{\frac{1}{2}(q_{\rm E}-1)}\tau_{\rm B}^{-\frac{3}{2}q_{\rm B}}t_{\rm d}^{1-\frac{1}{2}q_{\rm E}+\frac{3}{2}q_{\rm B}}\ {\rm Hz}.\end{aligned}\tag{16}$$

2.3 Synchrotron Self-Absorption Frequencies

The third characteristic frequency $\nu_{\rm a}$ is defined by synchrotron self-absorption (SSA), below which the synchrotron photons are absorbed in the condition of the optical depth $\tau\geq1$ (see Rybicki & Lightman 1979, for example). While some authors define the condition as $\tau\equiv e^{-1}\simeq0.35$ (e.g. Wijers & Galama 1999), we adopt $\tau=1$ in this paper.

In the late afterglow, $\nu_{\rm a}$ is usually the lowest among the three characteristic frequencies (e.g. SPN98), while the broad-band synchrotron spectrum can fall into two broad categories depending on the order of $\gamma_{\rm m}$ and $\gamma_{\rm c}$, namely, in the fast cooling regime if $\gamma_{\rm c}<\gamma_{\rm m}$, and in the slow cooling regime if $\gamma_{\rm c}>\gamma_{\rm m}$ (see Section 2.2). For completeness, according to Wu et al. (2003), we list all possible expressions of $\nu_{\rm a}$ in these two cases.

(1) Self-absorption in the fast cooling region ($\gamma_{\rm c}<\gamma_{\rm m}$).

In comoving frames, the self-absorption coefficient is

$$\alpha_{\nu,\mathrm{f}} \approx \begin{cases} c_1 \frac{q}{B}\left(n_\mathrm{e}\gamma_\mathrm{c}\right)\gamma_c^{-6}\left(\frac{\nu_\mathrm{obs}}{\nu_\mathrm{c}}\right)^{-\frac{5}{3}}, \ \nu_\mathrm{obs}<\nu_\mathrm{c}<\nu_\mathrm{m}, \\ c_2 \frac{q}{B}\left(n_\mathrm{e}\gamma_\mathrm{c}\right)\gamma_c^{-6}\left(\frac{\nu_\mathrm{obs}}{\nu_\mathrm{c}}\right)^{-3}, \ \nu_\mathrm{c}<\nu_\mathrm{obs}<\nu_\mathrm{m}, \\ c_2 \frac{q}{B}\left(n_\mathrm{e}\gamma_\mathrm{c}\right)\gamma_\mathrm{m}^{-6}\left(\frac{\nu_\mathrm{obs}}{\nu_\mathrm{m}}\right)^{-\frac{p+5}{2}}, \ \nu_\mathrm{c}<\nu_\mathrm{m}<\nu_\mathrm{obs}, \end{cases} \tag{17}$$

where ν_obs is observed frequency, $c_1 = \frac{2^{\frac{14}{3}}\pi^2}{3^2\Gamma(1/3)}\frac{p+2}{p+\frac{2}{3}}$, $c_2 = 2^{\frac{p+10}{2}}3^{-2}\pi g(p)$ and $g(p) = 2^4 3^{\frac{1}{2}}\Gamma(\frac{3p+2}{12})\Gamma(\frac{3p+10}{12})\left(p+\frac{10}{3}\right)$ (e.g. Rybicki & Lightman 1979). Approximatively, when $p = 2.5$, $c_1 \simeq 14.78$ and $c_2 \simeq 17.80$. Here $\Gamma(x)$ is the gamma function. For convenience, we denote the first formula as $\alpha_{\nu,\mathrm{f1}}$, the second formula as $\alpha_{\nu,\mathrm{f2}}$ and the third formula as $\alpha_{\nu,\mathrm{f3}}$. For the first case of fast cooling, i.e., $\nu_\mathrm{obs} < \nu_\mathrm{c} < \nu_\mathrm{m}$, the absorption frequency corresponding to $\alpha_{\nu,\mathrm{f1}}$ is

$$\nu_\mathrm{a,f1} = \left(\frac{c_1}{3-k}\frac{qnR}{B\gamma_\mathrm{c}^5}\right)^{\frac{3}{5}}\nu_\mathrm{c}.$$

Thus, for the general case, one has

$$\begin{aligned}\nu_\mathrm{a,f1} = {} & 2^{\frac{56-25k}{5k-20}}\left(17-4k\right)^{\frac{9k-14}{5k-20}}\left(3-k\right)^{-\frac{3}{5}} \\ & \left(4-k\right)^{\frac{30-2k}{5k-20}}\pi^{\frac{30-13k}{5k-20}}c^{\frac{50-18k}{5k-20}}x_\mathrm{p}m_\mathrm{p}^{\frac{10+3k}{20-5k}} \\ & m_\mathrm{e}^{-2}q^{\frac{8}{5}}c_1^{\frac{3}{5}}\sigma_\mathrm{T}\epsilon_\mathrm{B,0}^{\frac{6}{5}}\left(1+Y\right)\left(2-q_\mathrm{E}\right)^{\frac{2k-30}{5k-20}} \\ & \left(1+z\right)^{\frac{(9k-14)q_\mathrm{E}-17k+24}{5k-20}+\frac{6}{5}q_\mathrm{B}}A^{\frac{22}{20-5k}} \\ & E_0^{\frac{9k-14}{5k-20}}\tau_\mathrm{E}^{\frac{9k-14}{5k-20}(q_\mathrm{E}-1)}\tau_\mathrm{B}^{\frac{6}{5}q_\mathrm{B}} \\ & t^{-\frac{(9k-14)q_\mathrm{E}-12k+4}{5k-20}-\frac{6}{5}q_\mathrm{B}}.\end{aligned} \tag{18}$$

Separately, one has

$$\begin{aligned}\nu_\mathrm{a,f1} = {} & 2.01\times10^{12}\eta^{-\frac{7}{10}q_E-\frac{6}{5}q_\mathrm{B}}\epsilon_{B,-1}^{\frac{6}{5}}\left(1+Y\right) \\ & \left(2-q_\mathrm{E}\right)^{\frac{3}{2}}\left(1+z\right)^{\frac{7}{10}q_\mathrm{E}+\frac{6}{5}q_\mathrm{B}-\frac{6}{5}}n_0^{\frac{11}{10}} \\ & E_{53}^{\frac{7}{10}}\tau_\mathrm{E}^{\frac{7}{10}(q_\mathrm{E}-1)}\tau_\mathrm{B}^{\frac{6}{5}q_\mathrm{B}}t_\mathrm{d}^{\frac{1}{5}-\frac{7}{10}q_\mathrm{E}-\frac{6}{5}q_\mathrm{B}}\ \mathrm{Hz}\end{aligned} \tag{19}$$

for ISM ($k=0$), and

$$\begin{aligned}\nu_\mathrm{a,f1} = {} & 1.22\times10^{8}\eta^{\frac{2}{5}q_\mathrm{E}-\frac{6}{5}q_\mathrm{B}}\epsilon_{B,-1}^{\frac{6}{5}}\left(1+Y\right) \\ & \left(2-q_\mathrm{E}\right)^{\frac{13}{5}}\left(1+z\right)^{\frac{6}{5}q_\mathrm{B}-\frac{2}{5}q_\mathrm{E}+1} \\ & E_{53}^{-\frac{2}{5}}\tau_\mathrm{E}^{\frac{2}{5}(1-q_\mathrm{E})}\tau_\mathrm{B}^{\frac{6}{5}q_\mathrm{B}}t_\mathrm{d}^{\frac{2}{5}q_\mathrm{E}-2-\frac{6}{5}q_\mathrm{B}}\ \mathrm{Hz}\end{aligned} \tag{20}$$

for wind medium ($k=2$).

Analogously, for the second case of fast cooling, i.e., $\nu_\mathrm{c} < \nu_\mathrm{obs} < \nu_\mathrm{m}$, the absorption frequency corresponding to $\alpha_{\nu,\mathrm{f2}}$ is $\nu_\mathrm{a,f2} = \left(\frac{c_2}{3-k}\frac{qnR}{B\gamma_\mathrm{c}^5}\right)^{\frac{1}{3}}\nu_\mathrm{c}$, and one can obtain the general formula

$$\begin{aligned}\nu_\mathrm{a,f2} = {} & 3^{\frac{4}{3}}2^{\frac{32-9k}{3k-12}}\pi^{\frac{10-3k}{3k-12}}c^{\frac{14-4k}{3k-12}}c_2^{\frac{1}{3}}x_\mathrm{p}m_\mathrm{e}^{-\frac{2}{3}} \\ & m_\mathrm{p}^{\frac{2-k}{3k-12}}q^{\frac{4}{3}}\sigma_\mathrm{T}^{-\frac{1}{3}}\left(4-k\right)^{\frac{2}{3k-12}} \\ & \left(17-4k\right)^{\frac{k-2}{3k-12}}\left(3-k\right)^{-\frac{1}{3}} \\ & \left(2-q_\mathrm{E}\right)^{-\frac{2}{3k-12}}\left(1+Y\right)^{-\frac{1}{3}} \\ & \left(1+z\right)^{\frac{(k-2)q_\mathrm{E}-3k+8}{3(k-4)}}A^{\frac{2}{12-3k}} \\ & E_0^{\frac{k-2}{3k-12}}\tau_\mathrm{E}^{\frac{k-2}{3k-12}(q_\mathrm{E}-1)}t^{\frac{(2-k)q_\mathrm{E}+4}{3(k-4)}}.\end{aligned} \tag{21}$$

For ISM ($k=0$) and wind medium ($k=2$), one has

$$\begin{aligned}\nu_\mathrm{a,f2} = {} & 1.22\times10^{12}\eta^{-\frac{1}{6}q_\mathrm{E}}\left(2-q_\mathrm{E}\right)^{\frac{1}{6}}\left(1+Y\right)^{-\frac{1}{3}} \\ & \left(1+z\right)^{\frac{1}{6}q_\mathrm{E}-\frac{2}{3}}n_0^{\frac{1}{6}}E_{53}^{\frac{1}{6}} \\ & \tau_\mathrm{E}^{\frac{1}{6}(q_\mathrm{E}-1)}t_\mathrm{d}^{-\frac{1}{6}q_\mathrm{E}-\frac{1}{3}}\ \mathrm{Hz},\end{aligned} \tag{22}$$

$$\begin{aligned}\nu_\mathrm{a,f2} = {} & 3.38\times10^{11}\left(2-q_\mathrm{E}\right)^{\frac{1}{3}}\left(1+Y\right)^{-\frac{1}{3}} \\ & \left(1+z\right)^{-\frac{1}{3}}t_\mathrm{d}^{-\frac{2}{3}}\ \mathrm{Hz},\end{aligned} \tag{23}$$

respectively.

For the third case of fast cooling, i.e., $\nu_\mathrm{c} < \nu_\mathrm{m} < \nu_\mathrm{obs}$, the absorption frequency corresponding to $\alpha_{\nu,\mathrm{f3}}$ is $\nu_\mathrm{a,f3} = \left(\frac{c_2}{3-k}\frac{qnR}{B\gamma_\mathrm{c}^5}\right)^{\frac{2}{p+5}}\left(\frac{\nu_m}{\nu_\mathrm{c}}\right)^{\frac{p-1}{p+5}}\nu_\mathrm{c}$. The general formula is

$$\begin{aligned}\nu_\mathrm{a,f3} = {} & 3^{\frac{p+7}{p+5}}2^{-\frac{(9k-36)p+7k+8}{2(p+5)(k-4)}}\pi^{\frac{16+4p-5k-kp}{(p+5)(k-4)}}c^{\frac{36+20p-5kp-11k}{2(p+5)(k-4)}} \\ & x_\mathrm{p}m_\mathrm{e}^{\frac{-1-3p}{p+5}}m_\mathrm{p}^{\frac{2(kp-2k-4p+6)}{(p+5)(k-4)}}q^{\frac{p+7}{p+5}}\sigma_\mathrm{T}^{-\frac{2}{p+5}}\epsilon_\mathrm{B,0}^{\frac{p-1}{2(p+5)}}\epsilon_\mathrm{e,0}^{\frac{2p-2}{p+5}} \\ & \left(4-k\right)^{\frac{4}{(p+5)(k-4)}-\frac{3(p-1)}{2(p+5)}}\left(17-4k\right)^{\frac{2k-4}{(p+5)(k-4)}+\frac{p-1}{2(p+5)}} \\ & \left(2-q_\mathrm{E}\right)^{\frac{p-1}{2(p+5)}+\frac{kp-k-4p}{(p+5)(k-4)}}c_2^{\frac{2}{p+5}}\left(1+Y\right)^{-\frac{2}{p+5}} \\ & \left(1+z\right)^{\frac{(kp+3k-4p-4)q_\mathrm{E}-12k+32}{2(p+5)(k-4)}+\frac{p-1}{2p+10}q_\mathrm{B}+\frac{2p-2}{p+5}q_\mathrm{e}} \\ & \left(p-1\right)^{\frac{2-2p}{p+5}}\left(p-2\right)^{\frac{2p-2}{p+5}}A^{-\frac{4}{(p+5)(k-4)}}\left(3-k\right)^{-\frac{2}{p+5}} \\ & E_0^{\frac{(k-4)p+3k-4}{2(p+5)(k-4)}}\tau_\mathrm{E}^{\frac{(kp+3k-4p-4)}{2(p+5)(k-4)}(q_\mathrm{E}-1)}\tau_\mathrm{B}^{\frac{p-1}{2p+10}q_\mathrm{B}}\tau_\mathrm{e}^{\frac{2p-2}{p+5}q_\mathrm{e}} \\ & t^{\frac{(q_\mathrm{E}+2)(4-k)p+(2-3q_\mathrm{E})k+4q_\mathrm{E}+8}{2(p+5)(k-4)}-\frac{q_\mathrm{B}(p-1)}{2p+10}-\frac{q_\mathrm{e}(2p-2)}{p+5}}.\end{aligned} \tag{24}$$

Separately, one has

$$\begin{aligned}\nu_\mathrm{a,f3} = {} & 5.17\times10^{12}\eta^{-\frac{7}{30}q_\mathrm{E}-\frac{2}{5}q_\mathrm{e}-\frac{1}{10}q_\mathrm{B}}\left(2-q_\mathrm{E}\right)^{\frac{13}{30}} \\ & \epsilon_\mathrm{B,-1}^{\frac{1}{10}}\epsilon_\mathrm{e,-0.5}^{\frac{2}{5}}\left(1+Y\right)^{-\frac{4}{15}} \\ & \left(1+z\right)^{\frac{7}{30}q_\mathrm{E}-\frac{8}{15}+\frac{1}{10}q_\mathrm{B}+\frac{2}{5}q_\mathrm{e}} \\ & n_0^{\frac{2}{15}}E_{53}^{\frac{7}{30}}\tau_\mathrm{E}^{\frac{7}{30}(q_\mathrm{E}-1)}\tau_\mathrm{B}^{\frac{1}{10}q_\mathrm{B}}\tau_\mathrm{e}^{\frac{2}{5}q_\mathrm{e}} \\ & t_\mathrm{d}^{-\frac{7}{30}q_\mathrm{E}-\frac{7}{15}-\frac{1}{10}q_\mathrm{B}-\frac{2}{5}q_\mathrm{e}}\ \mathrm{Hz}\end{aligned} \tag{25}$$

for ISM ($k = 0$), and

$$\begin{aligned}\nu_{\rm a,f3} = {} & 2.14\times10^{12}\eta^{-\frac{1}{10}q_{\rm E}-\frac{1}{10}q_{\rm B}-\frac{2}{5}q_{\rm e}}(2-q_{\rm E})^{\frac{17}{30}} \\ & \epsilon_{\rm B,-1}^{\frac{1}{10}}\epsilon_{\rm e,-0.5}^{\frac{2}{5}}(1+Y)^{-\frac{4}{15}} \\ & (1+z)^{\frac{3}{10}q_{\rm E}-\frac{4}{15}+\frac{1}{10}q_{\rm E}+\frac{2}{5}q_{\rm e}} \\ & E_{53}^{\frac{1}{10}}\tau_{\rm E}^{\frac{1}{10}(q_{\rm E}-1)}\tau_{\rm B}^{\frac{1}{10}q_{\rm B}}\tau_{\rm e}^{\frac{2}{5}q_{\rm e}} \\ & t_{\rm d}^{-\frac{1}{10}q_{\rm E}-\frac{11}{15}-\frac{1}{10}q_{\rm B}-\frac{2}{5}q_{\rm e}}\ {\rm Hz}\end{aligned} \tag{26}$$

for wind medium ($k = 2$).

(2) Self-absorption in the slow cooling region ($\gamma_{\rm c} > \gamma_{\rm m}$)

$$\alpha_{\nu,{\rm s}} \approx \begin{cases} c_1(p-1)\frac{q}{B}\left(n_{\rm e}\gamma_{\rm m}^{p-1}\right)\gamma_m^{-(p+4)}\left(\frac{\nu_{\rm obs}}{\nu_{\rm m}}\right)^{-\frac{5}{3}}, \\ \qquad \nu_{\rm obs}<\nu_{\rm m}<\nu_{\rm c}, \\ c_2(p-1)\frac{q}{B}\left(n_{\rm e}\gamma_{\rm m}^{p-1}\right)\gamma_m^{-(p+4)}\left(\frac{\nu_{\rm obs}}{\nu_{\rm m}}\right)^{-\frac{p+4}{2}}, \\ \qquad \nu_{\rm m}<\nu_{\rm obs}<\nu_{\rm c}, \\ c_3(p-1)\frac{q}{B}\left(n_{\rm e}\gamma_{\rm m}^{p-1}\right)\gamma_{\rm c}^{-(p+4)}\left(\frac{\nu_{\rm obs}}{\nu_{\rm c}}\right)^{-\frac{p+5}{2}}, \\ \qquad \nu_{\rm m}<\nu_{\rm c}<\nu_{\rm obs}, \end{cases} \tag{27}$$

where $c_3 = (\frac{2\pi}{3})^{\frac{3}{2}}(p+2)$. Approximatively, when $p = 2.5$, $c_3 \simeq 13.64$. For convenience, we denote the first formula as $\alpha_{\nu,{\rm s1}}$, the second formula as $\alpha_{\nu,{\rm s2}}$ and the third formula as $\alpha_{\nu,{\rm s3}}$. For the first case of slow cooling, i.e., $\nu_{\rm obs} < \nu_{\rm m} < \nu_{\rm c}$, the absorption frequency corresponding to $\alpha_{\nu,{\rm s1}}$ is $\nu_{\rm a,s1} = \left[\frac{c_1(p-1)}{3-k}\frac{qnR}{B\gamma_{\rm m}^5}\right]^{\frac{3}{5}}\nu_{\rm m}$.

For the general formula, one has

$$\begin{aligned}\nu_{\rm a,s1} = {} & 3^1 2^{\frac{36-15k}{5k-20}}\pi^{\frac{20-8k}{5k-20}}c^{\frac{20-8k}{5k-20}}x_{\rm p}m_{\rm p}^{\frac{20-8k}{5k-20}}q^{\frac{8}{5}}c_1^{\frac{3}{5}} \\ & (p-1)^{\frac{8}{5}}(p-2)^{-1}(3-k)^{-\frac{3}{5}}\epsilon_{\rm e,0}^{-1}\epsilon_{\rm B,0}^{\frac{1}{5}} \\ & (4-k)^{\frac{5k-6}{5k-20}}(17-4k)^{\frac{4k-4}{5k-20}}(2-q_{\rm E})^{\frac{3k}{20-5k}} \\ & (1+z)^{\frac{4(kq_{\rm E}-3k-q_{\rm E}+6)}{5k-20}-q_{\rm e}+\frac{1}{5}q_{\rm B}} \\ & A^{-\frac{12}{5k-20}}E_0^{\frac{4k-4}{5k-20}}\tau_{\rm E}^{\frac{4(k-1)}{5(k-4)}(q_{\rm E}-1)}\tau_{\rm B}^{\frac{1}{5}q_{\rm B}}\tau_{\rm e}^{-q_{\rm e}} \\ & t^{-\frac{4(k-1)q_{\rm E}-7k+4}{5(k-4)}+q_{\rm e}-\frac{1}{5}q_{\rm B}}.\end{aligned} \tag{28}$$

Separately, one has

$$\begin{aligned}\nu_{\rm a,s1} = {} & 4.74\times10^{10}\eta^{-\frac{1}{5}q_{\rm B}+q_{\rm e}}\epsilon_{\rm e,-0.5}^{-1}\epsilon_{\rm B,-1}^{\frac{1}{5}} \\ & (1+z)^{\frac{1}{5}q_{\rm E}-\frac{6}{5}-q_{\rm e}+\frac{1}{5}q_{\rm B}}n_0^{\frac{3}{5}}E_{53}^{\frac{1}{5}} \\ & \tau_{\rm E}^{\frac{1}{5}(q_{\rm E}-1)}\tau_{\rm B}^{\frac{1}{5}q_{\rm B}}\tau_{\rm e}^{-q_{\rm e}}t_{\rm d}^{\frac{1}{5}-\frac{1}{5}q_{\rm E}-\frac{1}{5}q_{\rm B}+q_{\rm e}}\ {\rm Hz}\end{aligned} \tag{29}$$

for ISM ($k = 0$), and

$$\begin{aligned}\nu_{\rm a,s1} = {} & 2.04\times10^{8}\eta^{\frac{2}{5}q_{\rm E}-1-\frac{1}{5}q_{\rm B}+q_{\rm e}}\epsilon_{\rm e,-0.5}^{-1}\epsilon_{\rm B,-1}^{\frac{1}{5}} \\ & (2-q_{\rm E})^{\frac{3}{5}}(1+z)^{-\frac{2}{5}q_{\rm E}-q_{\rm e}+\frac{1}{5}q_{\rm B}}E_{53}^{-\frac{2}{5}} \\ & \tau_{\rm E}^{\frac{2}{5}(1-q_{\rm E})}\tau_{\rm B}^{\frac{1}{5}q_{\rm B}}\tau_{\rm e}^{-q_{\rm e}}t_{\rm d}^{\frac{2}{5}q_{\rm E}-1-\frac{1}{5}q_{\rm B}+q_{\rm e}}\ {\rm Hz}\end{aligned} \tag{30}$$

for wind medium ($k = 2$).

For the second case of slow cooling, i.e., $\nu_{\rm m} < \nu_{\rm obs} < \nu_{\rm c}$, the absorption frequency corresponding to $\alpha_{\nu,{\rm s2}}$ is $\nu_{\rm a,s2} = \left[\frac{c_2(p-1)}{3-k}\frac{qnR}{B\gamma_{\rm m}^5}\right]^{\frac{2}{p+4}}\nu_{\rm m}$.

For the general formula, one has

$$\begin{aligned}& \nu_{\rm a,s2} = \\ & 3^1 2^{\frac{72+36p-26k-9kp}{2(p+4)(k-4)}}\pi^{\frac{(4-k)p-6k+16}{(p+4)(k-4)}}c^{\frac{(20-5k)p-14k+40}{2(p+4)(k-4)}} \\ & x_{\rm p}m_{\rm e}^{-\frac{3p+2}{p+4}}m_{\rm p}^{\frac{(2k-8)p-4k+8}{(p+4)(k-4)}}q^{\frac{p+6}{p+4}}c_2^{\frac{2}{p+4}} \\ & (4-k)^{-\frac{(3k-12)p-2k-8}{2(p+4)(k-4)}}(17-4k)^{\frac{(k-4)p+6k-8}{2(p+4)(k-4)}} \\ & (3-k)^{-\frac{2}{p+4}}(p-1)^{\frac{4-2p}{p+4}}(p-2)^{\frac{2p-2}{p+4}} \\ & (2-q_{\rm E})^{\frac{(12-5k)p-10k+16}{2(p+4)(k-4)}} \\ & (1+z)^{\frac{(kp+6k-4p-8)q_{\rm E}-16k+32}{2(p+4)(k-4)}+\frac{p+2}{2p+8}q_{\rm B}+\frac{2p-2}{p+4}q_{\rm e}} \\ & \epsilon_{\rm B,0}^{\frac{p+2}{2p+8}}\epsilon_{\rm e,0}^{\frac{2p-2}{p+4}}A^{\frac{8}{(p+4)(4-k)}}E^{\frac{(k-4)p+6k-8}{2(p+4)(k-4)}} \\ & \tau_{\rm E}^{\frac{(kp+6k-4p-8)(q_{\rm E}-1)}{2(p+4)(k-4)}}\tau_{\rm B}^{\frac{p+2}{2p+8}q_{\rm B}}\tau_{\rm e}^{\frac{2p-2}{p+4}q_{\rm e}} \\ & t^{-\frac{(kp+6k-4p-8)q_{\rm E}+2kp-8k-8p}{2(p+4)(k-4)}-\frac{p+2}{2p+8}q_{\rm B}-\frac{2p-2}{p+4}q_{\rm e}}.\end{aligned} \tag{31}$$

Separately, one has

$$\begin{aligned}\nu_{\rm a,s2} = {} & 8.22\times10^{12}\eta^{-\frac{3}{16}q_{\rm e}-\frac{9}{26}q_{\rm E}-\frac{9}{26}q_{\rm B}} \\ & (2-q_{\rm E})^{\frac{19}{26}}\epsilon_{\rm B,-1}^{\frac{9}{26}}\epsilon_{\rm e,-0.5}^{\frac{6}{13}} \\ & (1+z)^{\frac{9}{26}q_{\rm E}-\frac{8}{13}+\frac{9}{26}q_{\rm B}+\frac{6}{13}q_{\rm e}} \\ & n_0^{\frac{4}{13}}E_{53}^{\frac{9}{26}}\tau_{\rm E}^{\frac{9}{26}(q_{\rm E}-1)}\tau_{\rm B}^{\frac{9}{26}q_{\rm B}}\tau_{\rm e}^{\frac{3}{16}q_{\rm e}} \\ & t_{\rm d}^{-\frac{9}{26}q_{\rm B}-\frac{9}{26}q_{\rm E}-\frac{6}{13}q_{\rm e}-\frac{5}{13}}\ {\rm Hz},\end{aligned} \tag{32}$$

for ISM ($k = 0$), and

$$\begin{aligned}\nu_{\rm a,s2} = {} & 7.15\times10^{11}\eta^{-\frac{9}{26}q_{\rm B}-\frac{1}{26}q_{\rm E}-\frac{6}{13}q_{\rm e}} \\ & (2-q_{\rm E})^{\frac{27}{26}}\epsilon_{\rm B,-1}^{\frac{9}{26}}\epsilon_{\rm e,-0.5}^{\frac{6}{13}} \\ & (1+z)^{\frac{1}{26}q_{\rm E}+\frac{9}{26}q_{\rm B}+\frac{6}{13}q_{\rm e}}E_{53}^{\frac{1}{26}} \\ & \tau_{\rm E}^{\frac{1}{26}(q_{\rm E}-1)}\tau_{\rm B}^{\frac{9}{26}q_{\rm B}}\tau_{\rm e}^{\frac{3}{16}q_{\rm e}} \\ & t_{\rm d}^{-\frac{9}{26}q_{\rm B}-\frac{1}{26}q_{\rm E}-\frac{6}{13}q_{\rm e}-1}\ {\rm Hz},\end{aligned} \tag{33}$$

for wind medium ($k = 2$).

For the third case of slow cooling, i.e., $\nu_{\rm m} < \nu_{\rm c} < \nu_{\rm obs}$, the absorption frequency corresponding to $\alpha_{\nu,{\rm s3}}$ is $\nu_{\rm a,s3} = \left[\frac{c_2(p-1)}{3-k}\frac{qnR}{B\gamma_{\rm m}^5}\right]^{\frac{2}{p+5}}\left(\frac{\nu_{\rm c}}{\nu_{\rm m}}\right)^{\frac{1}{p+5}}\nu_{\rm m}$.

For the general formula, one has

$$\begin{aligned}&\nu_{\rm a,s3} = \\ &3^{\frac{p+7}{p+5}} 2^{\frac{36-3kp-7k+15p}{(p+5)(k-4)}} \pi^{\frac{(4-k)p-5k+16}{(p+5)(k-4)}} c^{\frac{(20-5k)p-11k+36}{2(p+5)(k-4)}} \\ &c_2^{\frac{2}{p+5}} x_{\rm p} m_{\rm e}^{-\frac{3p+1}{p+5}} m_{\rm p}^{\frac{(2k-8)p-4k+12}{(p+5)(k-4)}} q^{\frac{p+7}{p+5}} \sigma_{\rm T}^{-\frac{2}{p+5}} \\ &\epsilon_{\rm B,0}^{\frac{p-1}{2p+10}} \epsilon_{\rm e,0}^{\frac{2p-2}{p+5}} (p-1)^{\frac{4-2p}{p+5}} (p-2)^{\frac{2p-2}{p+5}} \\ &(4-k)^{-\frac{3(p+4)}{2(p+5)}} (17-4k)^{\frac{12-k}{2(p+5)(k-4)}-\frac{p+4}{2p+10}} \\ &(3-k)^{-\frac{2}{p+5}} (2-q_{\rm E})^{\frac{3k(p-1)-12p+4}{2(p+5)(k-4)}} (1+Y)^{-\frac{2}{p+5}} \\ &(1+z)^{\frac{(kp+3k-4p-4)q_{\rm E}-12k+32}{2(p+5)(k-4)}+\frac{p-1}{2p+10}q_{\rm B}+\frac{2p-2}{p+5}q_{\rm e}} \\ &A^{-\frac{4}{(p+5)(k-4)}} E_0^{\frac{(k-4)p-4+3k}{2(p+5)(k-4)}} \tau_{\rm E}^{\frac{(kp+3k-4p-4)(q_{\rm E}-1)}{2(p+5)(k-4)}} \\ &\tau_{\rm B}^{\frac{p-1}{2p+10}q_{\rm B}} \tau_{\rm e}^{\frac{2p-2}{p+5}q_{\rm e}} \\ &t^{\frac{(q_{\rm E}+2)(4-k)p+(2-3q_{\rm E})k+4q_{\rm E}+8}{2(p+5)(k-4)}-\frac{p-1}{2p+10}q_{\rm B}-\frac{2(p-1)}{p+5}q_{\rm e}}.\end{aligned} \tag{34}$$

Separately, one has

$$\begin{aligned}\nu_{\rm a,s3} =\ & 5.76\times 10^{12} \eta^{-\frac{7}{30}q_{\rm E}-\frac{1}{10}q_{\rm B}-\frac{2}{5}q_{\rm e}} (2-q_{\rm E})^{\frac{13}{30}} \\ & (1+z)^{\frac{7}{30}q_{\rm E}-\frac{8}{15}+\frac{1}{10}q_{\rm B}+\frac{2}{5}q_{\rm e}} \\ & (1+Y)^{-\frac{4}{15}} \epsilon_{\rm B,-1}^{\frac{1}{10}} \epsilon_{\rm e,-0.5}^{\frac{2}{5}} \\ & n_0^{\frac{2}{15}} E_{53}^{\frac{7}{30}} \tau_{\rm E}^{\frac{7}{30}(q_{\rm E}-1)} \tau_{\rm B}^{\frac{1}{10}q_{\rm B}} \tau_{\rm e}^{\frac{2}{5}q_{\rm e}} \\ & t_{\rm d}^{-\frac{7}{15}-\frac{7}{30}q_{\rm E}-\frac{1}{10}q_{\rm B}-\frac{2}{5}q_{\rm e}}\ {\rm Hz},\end{aligned} \tag{35}$$

for ISM ($k = 0$), and

$$\begin{aligned}\nu_{\rm a,s3} =\ & 9.92\times 10^{15} \eta^{\frac{3}{16}q_{\rm e}-\frac{9}{26}q_{\rm E}-\frac{9}{26}q_{\rm B}} (2-q_{\rm E})^{\frac{17}{30}} \\ & (1+z)^{\frac{1}{10}q_{\rm E}-\frac{4}{15}+\frac{1}{10}q_{\rm B}+\frac{2}{5}q_{\rm e}} \\ & \epsilon_{\rm B,-1}^{\frac{1}{10}} \epsilon_{\rm e,-0.5}^{\frac{2}{5}} E_{53}^{\frac{1}{10}} \tau_{\rm E}^{\frac{1}{10}(q_{\rm E}-1)} \tau_{\rm B}^{\frac{1}{10}q_{\rm B}} \tau_{\rm e}^{\frac{2}{5}q_{\rm e}} \\ & t_{\rm d}^{-\frac{7}{15}-\frac{7}{30}q_{\rm E}-\frac{1}{10}q_{\rm B}-\frac{2}{5}q_{\rm e}}\ {\rm Hz},\end{aligned} \tag{36}$$

for wind medium ($k = 2$).

Above are all the six cases of SSA, in which we give general analytical expressions with variable parameters. To get the numerical solution, such as $k = 1.5$ and $q_{\rm E} = 0.1$, one only needs to substitute values into the corresponding general formulas.

3 A VARIETY OF CASES OF FLUX DENSITIES

Following the same process as in Mészáros & Rees (1997); Sari et al. (1998); Gao et al. (2013) and the related references, one can take the characteristic parameters as the basis of classification. There are two kinds of situations: fast cooling and slow cooling. To rearrange the other two parameters, one has six kinds of cases. For the energy distribution of electrons expressed in Section 2.2, the observed synchrotron radiation flux densities are shown in the following.

3.1 The Fast-Cooling Phase

There are three cases, namely I(1) $\nu_{\rm a} < \nu_{\rm c} < \nu_{\rm m}$, I(2) $\nu_{\rm c} < \nu_{\rm a} < \nu_{\rm m}$ and I(3) $\nu_{\rm c} < \nu_{\rm m} < \nu_{\rm a}$. In the following, we list the formulas for each case one by one.

In the case of I(1) $\nu_{\rm a} < \nu_{\rm c} < \nu_{\rm m}$, for the formulation of the flux density $F_{\nu,\rm f11}$, namely in the condition $\nu_{\rm obs} < \nu_{\rm a} < \nu_{\rm c} < \nu_{\rm m}$, $F_{\nu,\rm f11} = \left(\frac{\nu_{\rm obs}}{\nu_{\rm a}}\right)^2 \left(\frac{\nu_{\rm a}}{\nu_{\rm c}}\right)^{\frac{1}{3}} F_{\nu,\rm max}$, where the subscript f11 means the first case of fast cooling.

For the analytical expression, one has

$$\begin{aligned}F_{\nu,\rm f11} =\ & 3^{-\frac{1}{2}} 2^{\frac{16-6k}{4-k}} \epsilon_{\rm B,0}^{-1} \sigma_{\rm T}^{-1} D_{\rm L}^{-2} (1+Y)^{-1} \\ & (1+z)^{\frac{(4-q_{\rm E})k-8}{k-4}-q_{\rm B}} c_1^{-1} \phi_{\rm p} x_{\rm p}^{-2} \\ & \pi^{\frac{8-3k}{4-k}} c^{\frac{2k-4}{k-4}} m_{\rm p}^{\frac{4}{k-4}} m_{\rm e}^2 \nu_{\rm obs}^2 \\ & (4-k)^{\frac{k-8}{k-4}} (2-q_{\rm E})^{\frac{k-8}{4-k}} \\ & (17-4k)^{-\frac{k}{k-4}} A^{\frac{4}{k-4}} E_0^{-\frac{k}{k-4}} \\ & \tau_{\rm E}^{\frac{k}{k-4}(1-q_{\rm E})} \tau_{\rm B}^{-q_{\rm B}} t^{\frac{(q_{\rm E}-1)k-4}{k-4}+q_{\rm B}}.\end{aligned} \tag{37}$$

Separately, one has

$$\begin{aligned}F_{\nu,\rm f11} =\ & 4.44\times 10^8 \eta^{q_{\rm B}} (2-q_{\rm E})^{-2} D_{28}^{-2} \\ & (1+Y)^{-1} (1+z)^{2-q_{\rm B}} \nu_{14.5}^2 \\ & n_0^{-1} \epsilon_{\rm B,-1}^{-1} \tau_{\rm B,4}^{-q_{\rm B}} t_{\rm d}^{1+q_{\rm B}}\ {\rm Jy},\end{aligned} \tag{38}$$

for ISM, and

$$\begin{aligned}F_{\nu,\rm f11} =\ & 2.36\times 10^{12} \eta^{q_{\rm B}-q_{\rm E}} (2-q_{\rm E})^{-3} D_{28}^{-2} \\ & (1+Y)^{-1} (1+z)^{q_{\rm E}-q_{\rm B}} \nu_{14.5}^2 \epsilon_{\rm B,-1}^{-1} \\ & E_{53} \tau_{\rm E,4}^{q_{\rm E}-1} \tau_{\rm B,4}^{-q_{\rm B}} t_{\rm d}^{3-q_{\rm E}+q_{\rm B}}\ {\rm Jy},\end{aligned} \tag{39}$$

for wind medium.

Analogously, for the expression of $F_{\nu,\rm f12}$, which means the second case of fast cooling I(1), namely in the condition $\nu_{\rm a} < \nu_{\rm obs} < \nu_{\rm c}$, one has $F_{\nu,\rm f12} = \left(\frac{\nu_{\rm obs}}{\nu_{\rm c}}\right)^{\frac{1}{3}} F_{\nu,\rm max}$.

For the analytical expression, one has

$$\begin{aligned}F_{\nu,\rm f12} =\ & 3^{-\frac{1}{2}} 2^{\frac{8-7k}{3(4-k)}} (4-k)^{\frac{k+6}{3k-12}} (k-3)^{-1} \\ & (2-q_{\rm E})^{\frac{-6-k}{3k-12}} \\ & (17-4k)^{\frac{6k-14}{3k-12}} c^{\frac{38-12k}{3k-12}} \pi^{\frac{6-4k}{3k-12}} \\ & \phi_{\rm p} x_{\rm p}^{-\frac{1}{3}} m_{\rm e}^{-\frac{4}{3}} m_{\rm p}^{\frac{2-3k}{3k-12}} D_{\rm L}^{-2} \\ & (1+z)^{\frac{(6k-14)q_{\rm E}-5k}{3k-12}+q_{\rm B}} (1+Y)^{\frac{2}{3}} \\ & \sigma_{\rm T}^{\frac{2}{3}} q^{\frac{8}{3}} \nu_{\rm obs}^{\frac{1}{3}} \epsilon_{\rm B,0} A^{\frac{10}{12-3k}} E_0^{\frac{6k-14}{3k-12}} \\ & \tau_{\rm E}^{\frac{2(q_{\rm E}-1)(3k-7)}{3k-12}} \tau_{\rm B}^{q_{\rm B}} t^{\frac{(14-6k)q_{\rm E}+9k-16}{3k-12}-q_{\rm B}}.\end{aligned} \tag{40}$$

Separately, one has

$$F_{\nu,\mathrm{f12}}=9.66\times10^{4}\eta^{-q_{\rm B}-\frac{7}{6}q_{\rm E}}\left(2-q_{\rm E}\right)^{\frac{1}{2}}D_{28}^{-2}\left(1+z\right)^{\frac{7}{6}q_{\rm E}+q_{\rm B}}\left(1+Y\right)^{\frac{2}{3}}\nu_{14.5}^{\frac{1}{3}}\epsilon_{\rm B,-1}n_0^{\frac{5}{6}}E_{53}^{\frac{7}{6}}\tau_{\rm E}^{\frac{7}{6}(q_{\rm E}-1)}\tau_{\rm B}^{q_{\rm B}}t_{\rm d}^{-\frac{7}{6}q_{\rm E}+\frac{4}{3}-q_{\rm B}}\ \mathrm{Jy}, \tag{41}$$

for ISM, and

$$F_{\nu,\mathrm{f12}}=4.82\times10\eta^{-\frac{1}{3}q_{\rm E}-q_{\rm B}}(2-q_{\rm E})^{\frac{4}{3}}D_{28}^{-2}\left(1+z\right)^{\frac{5}{3}+\frac{1}{3}q_{\rm E}+q_{\rm B}}\left(1+Y\right)^{\frac{2}{3}}\nu_{14.5}^{\frac{1}{3}}\epsilon_{\rm B,-1}E_{53}^{\frac{1}{3}}\tau_{\rm E}^{\frac{1}{3}(q_{\rm E}-1)}\tau_{\rm B}^{q_{\rm B}}t_{\rm d}^{-\frac{1}{3}q_{\rm E}-\frac{1}{3}-q_{\rm B}}\ \mathrm{Jy}, \tag{42}$$

for wind medium.

For the expression of $F_{\nu,\mathrm{f13}}$, namely in the condition $\nu_{\rm a}<\nu_{\rm c}<\nu_{\rm obs}<\nu_{\rm m}$, one has $F_{\nu,\mathrm{f13}}=\left(\frac{\nu_{\rm obs}}{\nu_{\rm c}}\right)^{-\frac{1}{2}}F_{\nu,\max}$.

For the analytical expression, one has

$$F_{\nu,\mathrm{f13}}=3^2 2^{-\frac{11}{4}}\pi^{-\frac{1}{2}}\phi_{\rm p}x_{\rm p}^{\frac{1}{2}}\left(4-k\right)^{\frac{3}{4}}\left(17-4k\right)^{\frac{3}{4}}\left(1+z\right)^{\frac{3}{4}q_{\rm E}-\frac{1}{4}q_{\rm B}}q^{\frac{7}{2}}\nu_{\rm obs}^{-\frac{1}{2}}\sigma_{\rm T}^{-1}\left(1+Y\right)^{-1}\left(2-q_{\rm E}\right)^{-\frac{3}{4}}c^{-\frac{11}{4}}m_{\rm p}^{-1}m_{\rm e}^{-\frac{1}{2}}\epsilon_{\rm B,0}^{-\frac{1}{4}}\left(3-k\right)^{-1}D_{\rm L}^{-2}E_0^{\frac{3}{4}}\tau_{\rm E}^{\frac{3}{4}(q_{\rm E}-1)}\tau_{\rm B}^{-\frac{1}{4}q_{\rm B}}t^{\frac{2-3q_{\rm E}+q_{\rm B}}{4}}. \tag{43}$$

Separately, one has

$$F_{\nu,\mathrm{f13}}=2.04\times10^{2}\eta^{\frac{1}{4}q_{\rm B}-\frac{3}{4}q_{\rm E}}\left(1+z\right)^{\frac{3}{4}q_{\rm E}-\frac{1}{4}q_{\rm B}}\left(1+Y\right)^{-1}D_{28}^{-2}\nu_{14.5}^{-\frac{1}{2}}\epsilon_{\rm B,-1}^{-\frac{1}{4}}E_{53}^{\frac{3}{4}}\tau_{\rm E}^{\frac{3}{4}(q_{\rm E}-1)}\tau_{\rm B}^{-\frac{1}{4}q_{\rm B}}t_{\rm d}^{\frac{1}{4}q_{\rm B}-\frac{3}{4}q_{\rm E}+\frac{1}{2}}\ \mathrm{Jy}, \tag{44}$$

for ISM, and

$$F_{\nu,\mathrm{f13}}=2.26\times10^{2}\eta^{\frac{1}{4}q_{\rm B}-\frac{3}{4}q_{\rm E}}\left(1+z\right)^{\frac{3}{4}q_{\rm E}-\frac{1}{4}q_{\rm B}}\left(1+Y\right)^{-1}D_{28}^{-2}\nu_{14.5}^{-\frac{1}{2}}\epsilon_{\rm B,-1}^{-\frac{1}{4}}E_{53}^{\frac{3}{4}}\tau_{\rm E}^{\frac{3}{4}(q_{\rm E}-1)}\tau_{\rm B}^{-\frac{1}{4}q_{\rm B}}t_{\rm d}^{\frac{1}{4}q_{\rm B}-\frac{3}{4}q_{\rm E}+\frac{1}{2}}\ \mathrm{Jy}, \tag{45}$$

for wind medium.

For the expression of $F_{\nu,\mathrm{f14}}$, namely in the condition $\nu_{\rm a}<\nu_{\rm c}<\nu_{\rm m}<\nu_{\rm obs}$, one has $F_{\nu,\mathrm{f14}}=\left(\frac{\nu_{\rm obs}}{\nu_{\rm m}}\right)^{-\frac{p}{2}}\left(\frac{\nu_{\rm m}}{\nu_{\rm c}}\right)^{-\frac{1}{2}}F_{\nu,\max}$.

For the analytical expression, one has

$$F_{\nu,\mathrm{f14}}=3^{\frac{3+p}{2}}2^{-\frac{9}{4}p-\frac{1}{2}}\left(2-q_{\rm E}\right)^{\frac{3}{4}p-\frac{3}{2}}\pi^{-\frac{p}{2}}\left(p-2\right)^{p-1}\left(p-1\right)^{1-p}\phi_{\rm p}x_{\rm p}^{\frac{p}{2}}\sigma_{\rm T}^{-1}\left(k-3\right)^{-1}\left(1+z\right)^{\frac{1}{4}(p+2)q_{\rm E}+\frac{p-2}{4}q_{\rm B}+(p-1)q_{\rm e}}\left(1+Y\right)^{-1}D_{\rm L}^{-2}c^{-\frac{5}{4}p-\frac{3}{2}}\left(17-4k\right)^{\frac{p+3}{4}}m_{\rm e}^{-\frac{3}{2}p+1}q^{\frac{1}{2}p+3}m_{\rm p}^{p-2}\epsilon_{\rm B,0}^{\frac{p}{4}-\frac{1}{2}}\epsilon_{\rm e,0}^{p-1}\left(4-k\right)^{\frac{6-3p}{4}}\nu_{\rm obs}^{-\frac{p}{2}}E_0^{\frac{p+2}{4}}\tau_{\rm E}^{\frac{(p+2)(q_{\rm E}-1)}{4}}\tau_{\rm B}^{\frac{p-2}{4}q_{\rm B}}\tau_{\rm e}^{(p-1)q_{\rm e}}t^{\frac{(p+2)(1-q_{\rm E})-q_{\rm B}(p-2)+(2-3p)}{4}-(p-1)q_{\rm e}}. \tag{46}$$

Separately, one has

$$F_{\nu,\mathrm{f14}}=1.74\times10^{3}\eta^{-\frac{9}{8}q_{\rm E}-\frac{1}{8}q_{\rm B}-\frac{3}{2}q_{\rm e}}(2-q_{\rm E})^{\frac{3}{8}}\left(1+z\right)^{\frac{9}{8}q_{\rm E}+\frac{1}{8}q_{\rm B}+\frac{3}{2}q_{\rm e}}\left(1+Y\right)^{-1}D_{28}^{-2}\nu_{14.5}^{-\frac{5}{4}}\epsilon_{\rm B,-1}^{\frac{1}{8}}\epsilon_{\rm e,-0.5}^{\frac{3}{2}}E_{53}^{\frac{9}{8}}\tau_{\rm E,4}^{\frac{9}{8}(q_{\rm E}-1)}\tau_{\rm B,4}^{\frac{1}{8}q_{\rm B}}\tau_{\rm e,4}^{\frac{3}{2}q_{\rm e}}t_{\rm d}^{-\frac{9}{8}q_{\rm E}-\frac{1}{4}-\frac{1}{8}q_{\rm B}-\frac{3}{2}q_{\rm e}}\ \mathrm{Jy}, \tag{47}$$

for the case of ISM, and

$$F_{\nu,\mathrm{f14}}=8.61\times10^{2}\eta^{-\frac{9}{8}q_{\rm E}-\frac{1}{8}q_{\rm B}-\frac{3}{2}q_{\rm e}}(2-q_{\rm E})^{\frac{3}{8}}\left(1+z\right)^{\frac{9}{8}q_{\rm E}+\frac{1}{8}q_{\rm B}+\frac{3}{2}q_{\rm e}}\left(1+Y\right)^{-1}D_{28}^{-2}\nu_{14.5}^{-\frac{5}{4}}\epsilon_{\rm B,-1}^{\frac{1}{8}}\epsilon_{\rm e,-0.5}^{\frac{3}{2}}E_{53}^{\frac{9}{8}}\tau_{\rm E,4}^{\frac{9}{8}(q_{\rm E}-1)}\tau_{\rm B,4}^{\frac{q_{\rm B}}{8}}\tau_{\rm e,4}^{\frac{3}{2}q_{\rm e}}t_{\rm d}^{-\frac{9}{8}q_{\rm E}-\frac{1}{4}-\frac{1}{8}q_{\rm B}-\frac{3}{2}q_{\rm e}}\ \mathrm{Jy}, \tag{48}$$

for wind medium.

In the case of I(2) $\nu_{\rm c}<\nu_{\rm a}<\nu_{\rm m}$, for the expression of $F_{\nu,\mathrm{f21}}$, namely in the condition $\nu_{\rm obs}<\nu_{\rm c}<\nu_{\rm a}<\nu_{\rm m}$, one has $F_{\nu,\mathrm{f21}}=\left(\frac{\nu_{\rm obs}}{\nu_{\rm c}}\right)^{2}\left(\frac{\nu_{\rm c}}{\nu_{\rm a}}\right)^{3}F_{\nu,\max}$.

For the analytical expression, one has

$$F_{\nu,\mathrm{f21}}=3^{-\frac{1}{2}}2^{\frac{6k-16}{k-4}}c_2^{-1}\left(1+Y\right)^{-1}\pi^{\frac{8-3k}{4-k}}\sigma_{\rm T}^{-1}\phi_{\rm p}x_{\rm p}^{-2}m_{\rm p}^{\frac{4}{k-4}}D_{\rm L}^{-2}\left(2-q_{\rm E}\right)^{\frac{8-k}{k-4}}\left(4-k\right)^{\frac{k-8}{k-4}}\nu_{\rm obs}^{2}m_{\rm e}^{2}c^{\frac{2k-4}{k-4}}\left(1+z\right)^{\frac{k(4-q_{\rm E})-8}{k-4}-q_{\rm B}}\left(17-4k\right)^{\frac{k}{4-k}}\epsilon_{\rm B,0}^{-1}A^{\frac{4}{k-4}}E_0^{\frac{k}{4-k}}\tau_{\rm E}^{\frac{k}{k-4}(1-q_{\rm E})}\tau_{\rm B}^{-q_{\rm B}}t^{\frac{(q_{\rm E}-1)k-4}{k-4}+q_{\rm B}}. \tag{49}$$

Separately, one has

$$F_{\nu,\mathrm{f21}}=3.68\times10^{8}\eta^{q_{\rm B}}(2-q_{\rm E})^{-2}(1+z)^{2-q_{\rm B}}\left(1+Y\right)^{-1}D_{28}^{-2}\nu_{14.5}^{2}\epsilon_{\rm B,-1}^{-1}n_0^{-1}\tau_{\rm B,4}^{-q_{\rm B}}t_{\rm d}^{1+q_{\rm B}}\ \mathrm{Jy}, \tag{50}$$

for ISM, and

$$F_{\nu,\mathrm{f21}} = 1.96\times10^{12}\eta^{q_\mathrm{B}-q_\mathrm{E}}(2-q_\mathrm{E})^{-3} (1+z)^{q_\mathrm{E}-q_\mathrm{B}}(1+Y)^{-1}D_{28}^{-2}\nu_{14.5}^{2} \epsilon_{\mathrm{B},-1}^{-1}E_{53}\tau_{\mathrm{E},4}^{q_\mathrm{E}-1}\tau_{\mathrm{B},4}^{-q_\mathrm{B}}t_\mathrm{d}^{3-q_\mathrm{E}+q_\mathrm{B}}\ \mathrm{Jy}, \tag{51}$$

for wind medium.

For the expression of $F_{\nu,\mathrm{f22}}$, namely in the condition $\nu_\mathrm{c}<\nu_\mathrm{obs}<\nu_\mathrm{a}<\nu_\mathrm{m}$, one has $F_{\nu,\mathrm{f22}}=\left(\frac{\nu_\mathrm{obs}}{\nu_\mathrm{a}}\right)^{\frac{5}{2}}\left(\frac{\nu_\mathrm{a}}{\nu_\mathrm{c}}\right)^{-\frac{1}{2}}F_{\nu,\max}$.

For the analytical expression, one has

$$\begin{aligned}F_{\nu,\mathrm{f22}} =\ & 3^{-2}2^{\frac{25k-84}{4k-16}}\pi^{\frac{5k-16}{2k-8}}D_\mathrm{L}^{-2}c_2^{-1}q^{-\frac{1}{2}}x_\mathrm{p}^{-\frac{5}{2}}\phi_\mathrm{p}\\ &(4-k)^{\frac{4-5k}{4k-16}}(2-q_\mathrm{E})^{\frac{20-3k}{4k-16}}(17-4k)^{\frac{-4-k}{4k-16}}\\ &(1+z)^{\frac{12k-(k+4)q_\mathrm{E}-32}{4k-16}-\frac{1}{4}q_\mathrm{B}}\\ &c^{\frac{5k-12}{4k-16}}\epsilon_{\mathrm{B},0}^{-\frac{1}{4}}m_\mathrm{e}^{\frac{3}{2}}m_\mathrm{p}^{\frac{2}{k-4}}\nu_\mathrm{obs}^{\frac{5}{2}}A^{\frac{2}{k-4}}E_0^{\frac{4+k}{16-4k}}\\ &\tau_\mathrm{E}^{\frac{k+4}{4k-16}(1-q_\mathrm{E})}\tau_\mathrm{B}^{-\frac{1}{4}q_\mathrm{B}}t^{\frac{(k+4)q_\mathrm{E}+2k-24}{4k-16}+\frac{1}{4}q_\mathrm{B}}.\end{aligned} \tag{52}$$

Separately, one has

$$F_{\nu,\mathrm{f22}} = 8.67\times10^{9}\eta^{\frac{1}{4}q_\mathrm{B}-\frac{1}{4}q_\mathrm{E}}(2-q_\mathrm{E})^{-\frac{5}{4}} (1+z)^{\frac{1}{4}q_\mathrm{E}+2-\frac{1}{4}q_\mathrm{B}}D_{28}^{-2}\nu_{14.5}^{\frac{5}{2}} \epsilon_{\mathrm{B},-1}^{-\frac{1}{4}}n_0^{-\frac{1}{2}}E_{53}^{\frac{1}{4}}\tau_{\mathrm{B},4}^{-\frac{1}{4}q_\mathrm{B}}\tau_{\mathrm{E},4}^{\frac{1}{4}(q_\mathrm{E}-1)} t_\mathrm{d}^{-\frac{1}{4}q_\mathrm{E}+\frac{3}{2}+\frac{1}{4}q_\mathrm{B}}\ \mathrm{Jy}, \tag{53}$$

for ISM, and

$$F_{\nu,\mathrm{f22}} = 1.85\times10^{11}\eta^{\frac{1}{4}q_\mathrm{B}-\frac{3}{4}q_\mathrm{E}}(2-q_\mathrm{E})^{-\frac{7}{4}} (1+z)^{\frac{3}{4}q_\mathrm{E}+1-\frac{1}{4}q_\mathrm{B}}D_{28}^{-2}\nu_{14.5}^{\frac{5}{2}} \epsilon_{\mathrm{B},-1}^{-\frac{1}{4}}E_{53}^{\frac{3}{4}}\tau_{\mathrm{E},4}^{\frac{3}{4}(q_\mathrm{E}-1)}\tau_{\mathrm{B},4}^{-\frac{1}{4}q_\mathrm{B}} t_\mathrm{d}^{\frac{5}{2}+\frac{1}{4}q_\mathrm{B}-\frac{3}{4}q_\mathrm{E}}\ \mathrm{Jy}, \tag{54}$$

for wind medium.

For the expression of $F_{\nu,\mathrm{f23}}$, namely in the condition $\nu_\mathrm{c}<\nu_\mathrm{a}<\nu_\mathrm{obs}<\nu_\mathrm{m}$, one has $F_{\nu,\mathrm{f23}}=\left(\frac{\nu_\mathrm{obs}}{\nu_\mathrm{c}}\right)^{-\frac{1}{2}}F_{\nu,\max},\nu_\mathrm{a}<\nu_\mathrm{obs}<\nu_\mathrm{m}$.

For the analytical expression, one has

$$\begin{aligned}F_{\nu,\mathrm{f23}} =\ & 3^2 2^{-\frac{11}{4}}\pi^{-\frac{1}{2}}(4-k)^{\frac{3}{4}}(2-q_\mathrm{E})^{-\frac{3}{4}}D_\mathrm{L}^{-2}\\ &c^{-\frac{11}{4}}(17-4k)^{\frac{3}{4}}(k-3)^{-1}q^{\frac{7}{2}}x_\mathrm{p}^{\frac{1}{2}}\phi_\mathrm{p}\\ &\sigma_\mathrm{T}^{-1}(1+Y)^{-1}609[-(1+z)^{\frac{3q_\mathrm{E}}{4}-\frac{1}{4}q_\mathrm{B}}\\ &m_\mathrm{p}^{-1}m_\mathrm{e}^{-\frac{1}{2}}\epsilon_{\mathrm{B},0}^{-\frac{1}{4}}\nu_\mathrm{obs}^{-\frac{1}{2}}E_0^{\frac{3}{4}}\tau_\mathrm{B}^{-\frac{1}{4}q_\mathrm{B}}\\ &\tau_\mathrm{E}^{\frac{3}{4}(q_\mathrm{E}-1)}t^{\frac{q_\mathrm{B}-3q_\mathrm{E}+2}{4}}.\end{aligned} \tag{55}$$

Separately, one has

$$F_{\nu,\mathrm{f23}} = 2.04\times10^{2}\eta^{\frac{1}{4}q_\mathrm{B}-\frac{3}{4}q_\mathrm{E}}(2-q_\mathrm{E})^{-\frac{3}{4}} (1+z)^{\frac{3}{4}q_\mathrm{E}-\frac{1}{4}q_\mathrm{B}}(1+Y)^{-1} D_{28}^{-2}\nu_{14.5}^{-\frac{1}{2}}\epsilon_{\mathrm{B},-1}^{-\frac{1}{4}}E_{53}^{\frac{3}{4}}\tau_{\mathrm{B},4}^{-\frac{1}{4}q_\mathrm{B}}\tau_{\mathrm{E},4}^{\frac{3}{4}(q_\mathrm{E}-1)} t_\mathrm{d}^{\frac{1}{4}q_\mathrm{B}-\frac{3}{4}q_\mathrm{E}+\frac{1}{2}}\ \mathrm{Jy}, \tag{56}$$

for the case of ISM, and

$$F_{\nu,\mathrm{f23}} = 2.26\times10^{2}\eta^{\frac{1}{4}q_\mathrm{B}-\frac{3}{4}q_\mathrm{E}}(2-q_\mathrm{E})^{-\frac{3}{4}} (1+z)^{\frac{3}{4}q_\mathrm{E}-\frac{1}{4}q_\mathrm{B}}(1+Y)^{-1} D_{28}^{-2}\nu_{14.5}^{-\frac{1}{2}}\epsilon_{\mathrm{B},-1}^{-\frac{1}{4}}E_{53}^{\frac{3}{4}}\tau_{\mathrm{E},4}^{\frac{3}{4}(q_\mathrm{E}-1)}\tau_{\mathrm{B},4}^{-\frac{1}{4}q_\mathrm{B}} t_\mathrm{d}^{\frac{1}{4}q_\mathrm{B}-\frac{3}{4}q_\mathrm{E}+\frac{1}{2}}\ \mathrm{Jy}, \tag{57}$$

for wind medium.

For the expression of $F_{\nu,\mathrm{f24}}$, namely in the condition $\nu_\mathrm{c}<\nu_\mathrm{a}<\nu_\mathrm{m}<\nu_\mathrm{obs}$, one has $F_{\nu,\mathrm{f24}}=\left(\frac{\nu_\mathrm{obs}}{\nu_\mathrm{m}}\right)^{-\frac{p}{2}}\left(\frac{\nu_\mathrm{m}}{\nu_\mathrm{c}}\right)^{-\frac{1}{2}}F_{\nu,\max}$.

For the analytical expression, one has

$$\begin{aligned}F_{\nu,\mathrm{f24}} =\ & 3^{\frac{3+p}{2}}2^{-\frac{9}{4}p-\frac{1}{2}}\pi^{-\frac{p}{2}}\sigma_\mathrm{T}^{-1}(3-k)^{-1}(1+Y)^{-1}\\ &D_\mathrm{L}^{-2}(4-k)^{\frac{6-3p}{4}}(p-2)^{p-1}(p-1)^{1-p}\\ &(2-q_\mathrm{E})^{\frac{3}{4}p-\frac{3}{2}}c^{-\frac{5p+6}{4}}(17-4k)^{\frac{1}{4}p+\frac{1}{2}}\\ &(p-2)^p x_\mathrm{p}^{\frac{p}{2}}\phi_\mathrm{p}q^{\frac{p}{2}+3}m_\mathrm{e}^{1-\frac{3}{2}p}m_\mathrm{p}^{p-2}\epsilon_\mathrm{e}^{p-1}\\ &\epsilon_{\mathrm{B},0}^{\frac{p-2}{4}}(1+z)^{\frac{p+2}{4}q_\mathrm{E}+\frac{p-2}{4}q_\mathrm{B}+(p-1)q_\mathrm{e}}\nu_\mathrm{obs}^{-\frac{p}{2}}\\ &E_0^{\frac{p+2}{4}}\tau_\mathrm{E}^{\frac{p+2}{4}(q_\mathrm{E}-1)}\tau_\mathrm{B}^{\frac{p-2}{4}q_\mathrm{B}}\tau_\mathrm{e}^{(p-1)q_\mathrm{e}}\\ &t^{1-\frac{p+2}{4}q_\mathrm{E}-\frac{p}{2}-(p-1)q_\mathrm{e}-\frac{p-2}{4}q_\mathrm{B}}.\end{aligned} \tag{58}$$

Separately, one has

$$F_{\nu,\mathrm{f24}} = 4.53\times10^{2}\eta^{-\frac{9}{8}q_\mathrm{E}-\frac{3}{2}q_\mathrm{E}-\frac{1}{8}q_\mathrm{B}}(2-q_\mathrm{E})^{\frac{3}{8}} (1+z)^{\frac{9}{8}q_\mathrm{E}+\frac{1}{8}q_\mathrm{B}+\frac{3}{2}q_\mathrm{e}}(1+Y)^{-1} D_{28}^{-2}\nu_{14.52}^{-\frac{5}{4}}\epsilon_{\mathrm{B},-1}^{\frac{1}{8}}\epsilon_{\mathrm{e},-0.5}^{\frac{3}{2}}E_{53}^{\frac{9}{8}}\tau_{\mathrm{E},4}^{\frac{9}{8}(q_\mathrm{E}-1)} \tau_{\mathrm{B},4}^{\frac{q_\mathrm{B}}{8}}\tau_{\mathrm{e},4}^{\frac{3}{2}q_\mathrm{e}}t_\mathrm{d}^{-\frac{9}{8}q_\mathrm{E}-\frac{3}{2}q_\mathrm{e}-\frac{1}{8}q_\mathrm{B}-\frac{1}{4}}\ \mathrm{Jy}, \tag{59}$$

for ISM, and

$$F_{\nu,\mathrm{f24}} = 8.61\times10^{2}\eta^{-\frac{9}{8}q_\mathrm{E}-\frac{3}{2}q_\mathrm{E}-\frac{1}{8}q_\mathrm{B}}(2-q_\mathrm{E})^{\frac{3}{8}} (1+z)^{\frac{9}{8}q_\mathrm{E}+\frac{1}{8}q_\mathrm{B}+\frac{3}{2}q_\mathrm{e}}(1+Y)^{-1} D_{28}^{-2}\nu_{14.5}^{-\frac{5}{4}}\epsilon_{\mathrm{B},-1}^{\frac{1}{8}}\epsilon_{\mathrm{e},-0.5}^{\frac{3}{2}}E_{53}^{\frac{9}{8}}\tau_{\mathrm{E},4}^{\frac{9}{8}(q_\mathrm{E}-1)} \tau_{\mathrm{B},4}^{\frac{q_\mathrm{B}}{8}}\tau_{\mathrm{e},4}^{\frac{3}{2}q_\mathrm{e}}t_\mathrm{d}^{-\frac{9}{8}q_\mathrm{E}-\frac{3}{2}q_\mathrm{e}-\frac{1}{8}q_\mathrm{B}-\frac{1}{4}}\ \mathrm{Jy}, \tag{60}$$

for wind medium.

In case of I(3) $\nu_{\rm c} < \nu_{\rm m} < \nu_{\rm a}$, for the expression of $F_{\nu,{\rm f31}}$, namely in the condition $\nu_{\rm obs} < \nu_{\rm c} < \nu_{\rm m} < \nu_{\rm a}$, one has $F_{\nu,{\rm f31}} = \left(\frac{\nu_{\rm obs}}{\nu_{\rm c}}\right)^2 \left(\frac{\nu_{\rm c}}{\nu_{\rm a}}\right)^3 \left(\frac{\nu_{\rm a}}{\nu_{\rm m}}\right)^{-\frac{p-1}{2}} F_{\nu,{\rm max}}$.

For the analytical expression, one has

$$\begin{aligned} &F_{\nu,{\rm f31}} = \\ &3^{-\frac{1}{2}} 2^{\frac{11-k}{k-4}} \pi^{\frac{3k-8}{k-4}} \phi_{\rm p} x_{\rm p}^{-2} \sigma_{\rm T}^{-1} (1+Y)^{-1} D_{28}^{-2} \\ &\epsilon_{\rm B,0}^{-1} c_2^{-1} c^{\frac{2k-4}{k-4}} (4-k)^{\frac{k-8}{k-4}} (17-4k)^{-\frac{k}{k-4}} \\ &(2-q_{\rm E})^{\frac{8-k}{k-4}} (1+z)^{\frac{8-(4-q_{\rm E})k}{k-4} - q_{\rm B}} \\ &m_{\rm p}^{\frac{4}{k-4}} m_{\rm e}^2 \nu_{\rm obs}^2 A^{\frac{4}{k-4}} E_0^{\frac{k}{4-k}} \tau_{\rm E}^{\frac{k}{k-4}(1-q_{\rm E})} \\ &\tau_{\rm B}^{-q_{\rm B}} t^{\frac{(q_{\rm E}-1)k-4}{k-4} + q_{\rm B}}. \end{aligned} \tag{61}$$

Separately, one has

$$\begin{aligned} F_{\nu,{\rm f31}} = {} & 3.68 \times 10^8 \eta^{q_{\rm B}} (2-q_{\rm E})^{-2} (1+z)^{2-q_{\rm B}} \\ & (1+Y)^{-1} D_{28}^{-2} \nu_{14.5}^2 \\ & \epsilon_{\rm B,-1}^{-1} n_0^{-1} \tau_{\rm B,4}^{-q_{\rm B}} t_{\rm d}^{1+q_{\rm B}} \,{\rm Jy}, \end{aligned} \tag{62}$$

for ISM, and

$$\begin{aligned} F_{\nu,{\rm f31}} = {} & 1.96 \times 10^{12} \eta^{q_{\rm B}-q_{\rm E}} (2-q_{\rm E})^{-3} \\ & (1+z)^{q_{\rm E}-q_{\rm B}} (1+Y)^{-1} D_{28}^{-2} \nu_{14.5}^2 \\ & \epsilon_{\rm B,-1}^{-1} E_{53} \tau_{\rm E,4}^{q_{\rm E}-1} \tau_{\rm B,4}^{-q_{\rm B}} t_{\rm d}^{3-q_{\rm E}+q_{\rm B}} \,{\rm Jy}, \end{aligned} \tag{63}$$

for wind medium.

For the expression of $F_{\nu,{\rm f32}}$, namely in the condition $\nu_{\rm c} < \nu_{\rm obs} < \nu_{\rm a}$, one has $F_{\nu,{\rm f32}} = \left(\frac{\nu_{\rm obs}}{\nu_{\rm a}}\right)^{\frac{5}{2}} \left(\frac{\nu_{\rm a}}{\nu_{\rm m}}\right)^{-\frac{p}{2}} \left(\frac{\nu_{\rm m}}{\nu_{\rm c}}\right)^{-\frac{1}{2}} F_{\nu,{\rm max}}$, $\nu_{\rm c} < \nu_{\rm obs} < \nu_{\rm a}$.

For the analytical expression, one has

$$\begin{aligned} &F_{\nu,{\rm f32}} = \\ &3^{-2} 2^{\frac{3k-7}{k-4}} D_{\rm L}^{-2} c^{\frac{5k-12}{4k-16}} c_2^{-1} (17-4k)^{\frac{-4-k}{4k-16}} \\ &(4-k)^{\frac{3k-20}{4k-16}} \pi^{\frac{5k-16}{2k-8}} x_{\rm p}^{-\frac{5}{2}} \phi_{\rm p} q^{-\frac{1}{2}} m_{\rm e}^{\frac{3}{2}} m_{\rm p}^{\frac{2}{k-4}} \\ &\epsilon_{\rm B,0}^{-\frac{1}{4}} (2-q_{\rm E})^{\frac{20-3k}{4k-16}} \\ &(1+z)^{\frac{12k-32-(4+k)q_{\rm E}}{4k-16} - \frac{1}{4}q_{\rm B}} \nu_{\rm obs}^{\frac{5}{2}} A^{\frac{2}{k-4}} E_0^{\frac{4+k}{16-4k}} \\ &\tau_{\rm E}^{\frac{k+4}{4k-16}(1-q_{\rm E})} \tau_{\rm B}^{-\frac{1}{4}q_{\rm B}} t^{\frac{(k+4)q_{\rm E}+2k-24}{4k-16} + \frac{q_{\rm B}}{4}}. \end{aligned} \tag{64}$$

Separately, one has

$$\begin{aligned} F_{\nu,{\rm f32}} = {} & 8.67 \times 10^9 \eta^{\frac{1}{4}q_{\rm B}-\frac{1}{4}q_{\rm E}} (2-q_{\rm E})^{-\frac{5}{4}} \\ & (1+z)^{\frac{1}{4}q_{\rm E}-\frac{1}{4}q_{\rm B}+2} D_{28}^{-2} \nu_{14.5}^{\frac{5}{2}} \\ & \epsilon_{\rm B,-1}^{-\frac{1}{4}} n_0^{-\frac{1}{2}} E_{53}^{\frac{1}{4}} \tau_{\rm E,4}^{\frac{1}{4}(q_{\rm E}-1)} \tau_{\rm B,4}^{-\frac{1}{4}q_{\rm B}} \\ & t_{\rm d}^{\frac{1}{4}q_{\rm B}-\frac{1}{4}q_{\rm E}+\frac{3}{2}} \,{\rm Jy}, \end{aligned} \tag{65}$$

for ISM, and

$$\begin{aligned} F_{\nu,{\rm f32}} = {} & 1.85 \times 10^{11} \eta^{\frac{1}{4}q_{\rm B}-\frac{3}{4}q_{\rm E}} (2-q_{\rm E})^{-\frac{7}{4}} \\ & (1+z)^{\frac{3}{4}q_{\rm E}-\frac{1}{4}q_{\rm B}+1} D_{28}^{-2} \nu_{14.5}^{\frac{5}{2}} \epsilon_{\rm B,-1}^{-\frac{1}{4}} \\ & E_{53}^{\frac{3}{4}} \tau_{\rm E,4}^{\frac{3}{4}(q_{\rm E}-1)} \tau_{\rm B,4}^{-\frac{1}{4}q_{\rm B}} t_{\rm d}^{\frac{5}{2}-\frac{3}{4}q_{\rm E}+\frac{1}{4}q_{\rm B}} \,{\rm Jy}, \end{aligned} \tag{66}$$

for wind medium.

For the expression of $F_{\nu,{\rm f33}}$, namely in the condition $\nu_{\rm c} < \nu_{\rm m} < \nu_{\rm a} < \nu_{\rm obs}$, one has $F_{\nu,{\rm f33}} = \left(\frac{\nu_{\rm obs}}{\nu_{\rm m}}\right)^{-\frac{p}{2}} \left(\frac{\nu_{\rm m}}{\nu_{\rm c}}\right)^{-\frac{1}{2}} F_{\nu,{\rm max}}$.

For the analytical expression, one has

$$\begin{aligned} &F_{\nu,{\rm f33}} = \\ &3^{\frac{3+p}{2}} 2^{-\frac{9}{4}p-\frac{1}{2}} \pi^{-\frac{p}{2}} (3-k)^{-1} (1+Y)^{-1} D_{\rm L}^{-2} \\ &(17-4k)^{\frac{p+2}{4}} (p-1)^{1-p} (p-2)^{p-1} \\ &(4-k)^{\frac{3(1-p)}{4}} (2-q_{\rm E})^{\frac{3}{4}p-\frac{3}{2}} \sigma_{\rm T}^{-1} m_{\rm e}^{\frac{2-3p}{2}} m_{\rm p}^{p-2} \\ &c^{-\frac{5p+6}{4}} \epsilon_{\rm e}^{p-1} \epsilon_{\rm B,0}^{\frac{p-2}{4}} \phi_{\rm p} x_{\rm p}^{\frac{p}{2}} q^{\frac{p}{2}+3} \\ &(1+z)^{\frac{p+2}{4}q_{\rm E}+\frac{p-2}{4}q_{\rm B}+(p-1)q_{\rm e}} \nu_{\rm obs}^{-\frac{p}{2}} E_0^{\frac{p+2}{4}} \\ &\tau_{\rm E}^{\frac{p+2}{4}(q_{\rm E}-1)} \tau_{\rm B}^{\frac{p-2}{4}q_{\rm B}} \tau_{\rm e}^{(p-1)q_{\rm e}} \\ &t^{(1-p)q_{\rm e}-\frac{p+2}{4}q_{\rm E}-\frac{p-2}{4}q_{\rm B}+1-\frac{p}{2}}. \end{aligned} \tag{67}$$

Separately, one has

$$\begin{aligned} F_{\nu,{\rm f33}} = {} & 6.97 \times 10^3 \eta^{-\frac{9}{8}q_{\rm E}-\frac{1}{8}q_{\rm B}-\frac{3}{2}q_{\rm e}} (2-q_{\rm E})^{\frac{3}{8}} \\ & (1+z)^{\frac{9}{8}q_{\rm E}+\frac{1}{8}q_{\rm B}+\frac{3}{2}q_{\rm e}} (1+Y)^{-1} D_{28}^{-2} \\ & \nu_{14.5}^{-\frac{5}{4}} \epsilon_{\rm B,-1}^{\frac{1}{8}} \epsilon_{\rm e,-0.5}^{\frac{3}{2}} E_{53}^{\frac{9}{8}} \tau_{\rm E,4}^{\frac{9}{8}(q_{\rm E}-1)} \tau_{\rm B,4}^{\frac{1}{8}q_{\rm B}} \\ & \tau_{\rm e,4}^{\frac{3}{2}q_{\rm E}} t_{\rm d}^{-\frac{9}{8}q_{\rm E}-\frac{1}{4}-\frac{1}{8}q_{\rm B}-\frac{3}{2}q_{\rm e}} \,{\rm Jy}, \end{aligned} \tag{68}$$

for ISM, and

$$\begin{aligned} F_{\nu,{\rm f33}} = {} & 8.61 \times 10^2 \eta^{-\frac{9}{8}q_{\rm E}-\frac{1}{8}q_{\rm B}-\frac{3}{2}q_{\rm e}} (2-q_{\rm E})^{\frac{3}{8}} \\ & (1+z)^{\frac{9}{8}q_{\rm E}+\frac{1}{8}q_{\rm B}+\frac{3}{2}q_{\rm e}} (1+Y)^{-1} D_{28}^{-2} \\ & \nu_{14.5}^{-\frac{5}{4}} \epsilon_{\rm B,-1}^{\frac{1}{8}} \epsilon_{\rm e,-0.5}^{\frac{3}{2}} E_{53}^{\frac{9}{8}} \tau_{\rm E,4}^{\frac{9}{8}(q_{\rm E}-1)} \tau_{\rm B,4}^{\frac{q_{\rm B}}{8}} \\ & \tau_{\rm e,4}^{\frac{3}{2}q_{\rm e}} t_{\rm d}^{-\frac{9}{8}q_{\rm E}-\frac{1}{4}-\frac{1}{8}q_{\rm B}-\frac{3}{2}q_{\rm e}} \,{\rm Jy}, \end{aligned} \tag{69}$$

for wind medium.

3.2 The Slow Cooling Phase

There are three cases, namely II(1) $\nu_{\rm a} < \nu_{\rm m} < \nu_{\rm c}$, II(2) $\nu_{\rm m} < \nu_{\rm a} < \nu_{\rm c}$ and II(3) $\nu_{\rm m} < \nu_{\rm c} < \nu_{\rm a}$:
In the case of II(1) $\nu_{\rm a} < \nu_{\rm m} < \nu_{\rm c}$, for the expression of $F_{\nu,{\rm s11}}$, namely in the condition $\nu_{\rm obs} < \nu_{\rm a} < \nu_{\rm m} < \nu_{\rm c}$, one has $F_{\nu,{\rm s11}} = \left(\frac{\nu_{\rm obs}}{\nu_{\rm a}}\right)^2 \left(\frac{\nu_{\rm a}}{\nu_{\rm m}}\right)^{\frac{1}{3}} F_{\nu,{\rm max}}$, where the index s11 means the first case of slow cooling.

For the analytical expression, one has

$$F_{\nu,\mathrm{s11}} = 3^{-\frac{3}{2}} 2^{\frac{2k-6}{k-4}} \pi^{\frac{2k-6}{k-4}} c^{\frac{2}{k-4}} (p-1)^{-2} (p-2) \phi_{\mathrm{p}} \epsilon_{\mathrm{e}} x_{\mathrm{p}}^{-2} D_{\mathrm{L}}^{-2} c_1^{-1} m_{\mathrm{p}}^{\frac{k-2}{k-4}} (17-4k)^{\frac{2}{4-k}} (4-k)^{\frac{2}{4-k}} (2-q_{\mathrm{E}})^{\frac{2}{k-4}} (1+z)^{\frac{3k-8}{k-4}-\frac{2}{k-4}q_{\mathrm{E}}+q_{\mathrm{e}}} \nu_{\mathrm{obs}}^2 A^{\frac{2}{k-4}} E_0^{\frac{2}{4-k}} \tau_{\mathrm{E}}^{\frac{2}{k-4}(1-q_{\mathrm{E}})} \tau_{\mathrm{e}}^{q_{\mathrm{e}}} t^{\frac{2q_{\mathrm{E}}-4}{k-4}-q_{\mathrm{e}}}. \tag{70}$$

Separately, one has

$$F_{\nu,\mathrm{s11}} = 1.85\times10^5 \eta^{-q_{\mathrm{e}}-\frac{1}{2}q_{\mathrm{E}}} (2-q_{\mathrm{E}})^{-\frac{1}{2}} (1+z)^{\frac{1}{2}q_{\mathrm{E}}+q_{\mathrm{e}}+2} D_{28}^{-2} \nu_{14.5}^2 \epsilon_{\mathrm{e},-0.5} n_0^{-\frac{1}{2}} E_{53}^{\frac{1}{2}} \tau_{\mathrm{E},4}^{\frac{1}{2}(q_{\mathrm{E}}-1)} \tau_{\mathrm{e},4}^{q_{\mathrm{e}}} t_{\mathrm{d}}^{1-q_{\mathrm{e}}-\frac{1}{2}q_{\mathrm{E}}}\ \mathrm{Jy}, \tag{71}$$

for ISM, and

$$F_{\nu,\mathrm{s11}} = 1.20\times10^{12} \eta^{-q_{\mathrm{E}}-q_{\mathrm{e}}} (2-q_{\mathrm{E}})^{-1} (1+z)^{q_{\mathrm{E}}+q_{\mathrm{e}}+1} D_{28}^{-2} \nu_{14.5}^2 \epsilon_{\mathrm{e},-0.5} E_{53} \tau_{\mathrm{E},4}^{q_{\mathrm{E}}-1} \tau_{\mathrm{e},4}^{q_{\mathrm{E}}} t_{\mathrm{d}}^{2-q_{\mathrm{E}}-q_{\mathrm{e}}}\ \mathrm{Jy}, \tag{72}$$

for wind medium.

For the expression of $F_{\nu,\mathrm{s12}}$, namely in the condition $\nu_{\mathrm{a}} < \nu_{\mathrm{obs}} < \nu_{\mathrm{m}} < \nu_{\mathrm{c}}$, one has $F_{\nu,\mathrm{s12}} = \left(\frac{\nu_{\mathrm{obs}}}{\nu_{\mathrm{m}}}\right)^{\frac{1}{3}} F_{\nu,\mathrm{max}}$.

For the analytical expression, one has

$$F_{\nu,\mathrm{s12}} = 3^{\frac{1}{6}} 2^{\frac{k}{4-k}} \pi^{\frac{2-2k}{3k-12}} D_{\mathrm{L}}^{-2} \phi_{\mathrm{p}} x_{\mathrm{p}}^{-\frac{1}{3}} (k-3)^{-1} c^{\frac{26-8k}{3k-12}} q^{\frac{8}{3}} m_{\mathrm{p}}^{\frac{14-5k}{3k-12}} (p-2)^{-\frac{2}{3}} (p-1)^{\frac{2}{3}} (4-k)^{\frac{k-2}{k-4}} (17-4k)^{\frac{4k-10}{3k-12}} (2-q_{\mathrm{E}})^{\frac{2-k}{k-4}} (1+z)^{\frac{(4q_{\mathrm{E}}-4)k-9q_{\mathrm{E}}+1}{3k-12}-\frac{2}{3}q_{\mathrm{e}}+\frac{1}{3}q_{\mathrm{B}}} \epsilon_{\mathrm{B},0}^{\frac{1}{3}} \epsilon_{\mathrm{e},0}^{-\frac{2}{3}} \nu_{\mathrm{obs}}^{\frac{1}{3}} A^{\frac{2}{4-k}} E_0^{\frac{4k-10}{3k-12}} \tau_{\mathrm{E}}^{\frac{2(q_{\mathrm{E}}-1)(2k-5)}{3(k-4)}} \tau_{\mathrm{B}}^{\frac{1}{3}q_{\mathrm{B}}} \tau_{\mathrm{e}}^{-\frac{2}{3}q_{\mathrm{e}}} t^{\frac{(10-4k)q_{\mathrm{E}}+7k-16}{3k-12}+\frac{2}{3}q_{\mathrm{e}}-\frac{1}{3}q_{\mathrm{B}}}. \tag{73}$$

Separately, one has

$$F_{\nu,\mathrm{s12}} = 6.77\times10^3 \eta^{-\frac{1}{3}q_{\mathrm{B}}-\frac{5}{6}q_{\mathrm{E}}+\frac{2}{3}q_{\mathrm{E}}} (2-q_{\mathrm{E}})^{-\frac{1}{2}} (1+z)^{\frac{5}{6}q_{\mathrm{E}}-\frac{2}{3}q_{\mathrm{e}}+\frac{1}{3}q_{\mathrm{B}}} D_{28}^{-2} \nu_{14.5}^{\frac{1}{3}} \epsilon_{\mathrm{e},-0.5}^{-\frac{2}{3}} \epsilon_{\mathrm{B},-1}^{\frac{1}{3}} n_0^{\frac{1}{2}} E_{53}^{\frac{5}{6}} \tau_{\mathrm{E},4}^{\frac{5}{6}(q_{\mathrm{E}}-1)} \tau_{\mathrm{B},4}^{\frac{1}{3}q_{\mathrm{B}}} \tau_{\mathrm{e},4}^{-\frac{2}{3}q_{\mathrm{e}}} t_{\mathrm{d}}^{\frac{4}{3}-\frac{1}{3}q_{\mathrm{B}}-\frac{5}{6}q_{\mathrm{E}}+\frac{2}{3}q_{\mathrm{e}}}\ \mathrm{Jy}, \tag{74}$$

for ISM, and

$$F_{\nu,\mathrm{s12}} = 5.78\times10 \eta^{\frac{2}{3}q_{\mathrm{e}}-\frac{1}{3}q_{\mathrm{E}}-\frac{1}{3}q_{\mathrm{B}}} (1+z)^{\frac{1}{3}q_{\mathrm{E}}+1-\frac{2}{3}q_{\mathrm{e}}+\frac{1}{3}q_{\mathrm{B}}} D_{28}^{-2} \nu_{14.5}^{\frac{1}{3}} \epsilon_{\mathrm{e},-0.5}^{-\frac{2}{3}} \epsilon_{\mathrm{B},-1}^{\frac{1}{3}} E_{53}^{\frac{1}{3}} \tau_{\mathrm{E},4}^{\frac{1}{3}(q_{\mathrm{E}}-1)} \tau_{\mathrm{B},4}^{\frac{1}{3}q_{\mathrm{B}}} \tau_{\mathrm{e},4}^{-\frac{2}{3}q_{\mathrm{e}}} t_{\mathrm{d}}^{\frac{1}{3}+\frac{2}{3}q_{\mathrm{e}}-\frac{1}{3}q_{\mathrm{E}}-\frac{1}{3}q_{\mathrm{B}}}\ \mathrm{Jy}, \tag{75}$$

for wind medium.

For the expression of $F_{\nu,\mathrm{s13}}$, namely in the condition $\nu_{\mathrm{a}} < \nu_{\mathrm{m}} < \nu_{\mathrm{obs}} < \nu_{\mathrm{c}}$, one has $F_{\nu,\mathrm{s13}} = \left(\frac{\nu_{\mathrm{obs}}}{\nu_{\mathrm{m}}}\right)^{\frac{1-p}{2}} F_{\nu,\mathrm{max}}$.

For the analytical expression, one has

$$F_{\nu,\mathrm{s13}} = 3^{\frac{p}{2}} 2^{\frac{36p-9kp-k-12}{4(4-k)}} \pi^{\frac{4p-(p+1)k}{2k-8}} (17-4k)^{\frac{3k-8}{2k-8}+\frac{p-1}{4}} (2-q_{\mathrm{E}})^{\frac{3kp-5k-12p+12}{4(k-4)}} (3-k)^{-1} D_{\mathrm{L}}^{-2} \phi_{\mathrm{p}} x_{\mathrm{p}}^{\frac{p-1}{2}} (p-2)^{p-1} (p-1)^{1-p} (4-k)^{\frac{3-p}{4}+\frac{k}{2k-8}} m_{\mathrm{e}}^{1-\frac{3}{2}p} m_{\mathrm{p}}^{\frac{(p-2)k-4p+6}{k-4}} q^{\frac{p+5}{2}} c^{\frac{(-5p-9)k+20p+28}{4k-16}} \epsilon_{\mathrm{B},0}^{\frac{p+1}{4}} \epsilon_{\mathrm{e}}^{p-1} \nu_{\mathrm{obs}}^{\frac{1-p}{2}} (1+z)^{\frac{(kp+5k-4p-12)q_{\mathrm{E}}-4k}{4k-16}+\frac{p+1}{4}q_{\mathrm{B}}+(p-1)q_{\mathrm{e}}} A^{-\frac{2}{k-4}} E_0^{\frac{(p+5)k-4p-12}{4k-16}} \tau_{\mathrm{E}}^{\frac{(kp+5k-4p-12)}{4k-16}(q_{\mathrm{E}}-1)} \tau_{\mathrm{B}}^{\frac{p+1}{4}q_{\mathrm{B}}} \tau_{\mathrm{e}}^{(1-p)q_{\mathrm{e}}} t^{\frac{(kp+5k-4p-12)q_{\mathrm{E}}-4k}{16-4k}-\frac{p+1}{4}q_{\mathrm{B}}-(p-1)q_{\mathrm{e}}}. \tag{76}$$

Separately, one has

$$F_{\nu,\mathrm{s13}} = 2.43\times10^4 \eta^{-\frac{7}{8}q_{\mathrm{B}}-\frac{11}{8}q_{\mathrm{E}}-\frac{3}{2}q_{\mathrm{e}}} (2-q_{\mathrm{E}})^{\frac{9}{8}} (1+z)^{\frac{11}{8}q_{\mathrm{E}}+\frac{7}{8}q_{\mathrm{B}}+\frac{3}{2}q_{\mathrm{e}}} D_{28}^{-2} \nu_{14.5}^{-\frac{3}{4}} \epsilon_{\mathrm{e},-0.5}^{\frac{3}{2}} \epsilon_{\mathrm{B},-1}^{\frac{7}{8}} n_0^{\frac{1}{2}} E_{53}^{\frac{11}{8}} \tau_{\mathrm{B},4}^{\frac{7}{8}q_{\mathrm{B}}} \tau_{\mathrm{E},4}^{\frac{11}{8}(q_{\mathrm{E}}-1)} \tau_{\mathrm{e},4}^{\frac{3}{2}q_{\mathrm{e}}} t_{\mathrm{d}}^{\frac{1}{4}-\frac{7}{8}q_{\mathrm{B}}-\frac{11}{8}q_{\mathrm{E}}-\frac{3}{2}q_{\mathrm{e}}}\ \mathrm{Jy}, \tag{77}$$

for ISM, and

$$F_{\nu,\mathrm{s13}} = 5.10\times10^2 \eta^{-\frac{7}{8}q_{\mathrm{B}}-\frac{7}{8}q_{\mathrm{E}}-\frac{3}{2}q_{\mathrm{e}}} (2-q_{\mathrm{E}})^{\frac{13}{8}} (1+z)^{\frac{7}{8}q_{\mathrm{E}}+\frac{7}{8}q_{\mathrm{B}}+\frac{3}{2}q_{\mathrm{e}}+1} D_{28}^{-2} \nu_{14.5}^{-\frac{3}{4}} \epsilon_{\mathrm{e},-0.5}^{\frac{3}{2}} \epsilon_{\mathrm{B},-1}^{\frac{7}{8}} E_{53}^{\frac{7}{8}} \tau_{\mathrm{B},4}^{\frac{7}{8}q_{\mathrm{B}}} \tau_{\mathrm{E},4}^{\frac{7}{8}(q_{\mathrm{E}}-1)} \tau_{\mathrm{e},4}^{\frac{3}{2}q_{\mathrm{e}}} t_{\mathrm{d}}^{-\frac{3}{4}-\frac{7}{8}q_{\mathrm{B}}-\frac{7}{8}q_{\mathrm{E}}-\frac{3}{2}q_{\mathrm{e}}}\ \mathrm{Jy}, \tag{78}$$

for wind medium.

For the expression of $F_{\nu,\mathrm{s14}}$, namely in the condition $\nu_{\mathrm{a}} < \nu_{\mathrm{m}} < \nu_{\mathrm{c}} < \nu_{\mathrm{obs}}$, one has $F_{\nu,\mathrm{s14}} = \left(\frac{\nu_{\mathrm{obs}}}{\nu_{\mathrm{c}}}\right)^{-\frac{p}{2}} \left(\frac{\nu_{\mathrm{c}}}{\nu_{\mathrm{m}}}\right)^{-\frac{p-1}{2}} F_{\nu,\mathrm{max}}$.

For the analytical expression, one has

$$\begin{aligned} &F_{\nu,\mathrm{s14}} = \\ &3^{\frac{3+p}{2}} 2^{-\frac{9}{4}p-\frac{1}{2}} \pi^{-\frac{p}{2}} \sigma_{\mathrm{T}}^{-1} \phi_{\mathrm{p}} x_{\mathrm{p}}^{\frac{p}{2}} c^{-\frac{5p+6}{4}} (1+Y)^{-1} \\ &(1+z)^{\frac{p+2}{4}q_{\mathrm{E}}+\frac{p-2}{4}q_{\mathrm{B}}+(p-1)q_{\mathrm{e}}} (3-k)^{-1} \\ &(4-k)^{\frac{3p-6}{4}} (p-1)^{1-p} (p-2)^{p-1} \\ &(2-q_{\mathrm{E}})^{\frac{3p-6}{4}} (17-4k)^{\frac{p+2}{4}} q^{\frac{p+6}{2}} \\ &m_{\mathrm{e}}^{\frac{2-3p}{2}} m_{\mathrm{p}}^{p-2} \epsilon_{\mathrm{B},0}^{\frac{p-2}{4}} \epsilon_{\mathrm{e}}^{p-1} D_{\mathrm{L}}^{-2} \nu_{\mathrm{obs}}^{-\frac{p}{2}} \\ &E_0^{\frac{p+2}{4}} \tau_{\mathrm{E}}^{\frac{p+2}{4}(q_{\mathrm{E}}-1)} \tau_{\mathrm{B}}^{\frac{p-2}{4}q_{\mathrm{B}}} \tau_{\mathrm{e}}^{(p-1)q_{\mathrm{e}}} \\ &t^{1-\frac{p+2}{4}q_{\mathrm{E}}-\frac{p}{2}+\frac{2-p}{4}q_{\mathrm{B}}+(1-p)q_{\mathrm{e}}}. \end{aligned} \quad (79)$$

Separately, one has

$$\begin{aligned} F_{\nu,\mathrm{s14}} = {}& 1.74\times 10^{3} \eta^{-\frac{1}{8}q_{\mathrm{B}}-\frac{9}{8}q_{\mathrm{E}}-\frac{3}{2}q_{\mathrm{e}}} (2-q_{\mathrm{E}})^{\frac{3}{8}} \\ & (1+z)^{\frac{9}{8}q_{\mathrm{E}}+\frac{7}{8}q_{\mathrm{B}}+\frac{3}{2}q_{\mathrm{e}}} (1+Y)^{-1} \\ & D_{28}^{-2} \nu_{14.5}^{-\frac{5}{4}} \epsilon_{\mathrm{e},-0.5}^{\frac{3}{2}} \epsilon_{\mathrm{B},-1}^{\frac{1}{8}} E_{53}^{\frac{9}{8}} \tau_{\mathrm{E},4}^{\frac{9}{8}(q_{\mathrm{E}}-1)} \\ & \tau_{\mathrm{B},4}^{\frac{q_{\mathrm{B}}}{8}} \tau_{\mathrm{e},4}^{\frac{3}{2}q_{\mathrm{e}}} t_{\mathrm{d}}^{-\frac{1}{4}-\frac{1}{8}q_{\mathrm{B}}-\frac{9}{8}q_{\mathrm{E}}-\frac{3}{2}q_{\mathrm{e}}} \ \mathrm{Jy}, \end{aligned} \quad (80)$$

for ISM, and

$$\begin{aligned} F_{\nu,\mathrm{s14}} = {}& 8.61\times 10^{2} \eta^{-\frac{1}{8}q_{\mathrm{B}}-\frac{9}{8}q_{\mathrm{E}}-\frac{3}{2}q_{\mathrm{e}}} (2-q_{\mathrm{E}})^{\frac{3}{8}} \\ & (1+z)^{\frac{9}{8}q_{\mathrm{E}}+\frac{7}{8}q_{\mathrm{B}}+\frac{3}{2}q_{\mathrm{e}}} (1+Y)^{-1} \\ & D_{28}^{-2} \nu_{14.5}^{-\frac{5}{4}} \epsilon_{\mathrm{e},-0.5}^{\frac{3}{2}} \epsilon_{\mathrm{B},-1}^{\frac{1}{8}} E_{53}^{\frac{9}{8}} \tau_{\mathrm{E},4}^{\frac{9}{8}(q_{\mathrm{E}}-1)} \\ & \tau_{\mathrm{B},4}^{\frac{1}{8}} \tau_{\mathrm{e},4}^{\frac{3}{2}q_{\mathrm{e}}} t_{\mathrm{d}}^{-\frac{1}{4}-\frac{1}{8}q_{\mathrm{B}}-\frac{9}{8}q_{\mathrm{E}}-\frac{3}{2}q_{\mathrm{e}}} \ \mathrm{Jy}, \end{aligned} \quad (81)$$

for wind medium.

In case of II(2) $\nu_{\mathrm{m}} < \nu_{\mathrm{a}} < \nu_{\mathrm{c}}$, for $F_{\nu,\mathrm{s21}}$, namely $\nu_{\mathrm{obs}} < \nu_{\mathrm{m}} < \nu_{\mathrm{a}} < \nu_{\mathrm{c}}$, one has $F_{\nu,\mathrm{s21}} = \left(\frac{\nu_{\mathrm{obs}}}{\nu_{\mathrm{m}}}\right)^{2} \left(\frac{\nu_{\mathrm{m}}}{\nu_{\mathrm{a}}}\right)^{\frac{p+4}{2}} F_{\nu,\max}$.

For the analytical expression, one has

$$\begin{aligned} &F_{\nu,\mathrm{s21}} = 3^{-\frac{3}{2}} 2^{\frac{4(3-k)}{4-k}} c^{\frac{2}{k-4}} \phi_{\mathrm{p}} x_{\mathrm{p}}^{-2} D_{\mathrm{L}}^{-2} c_2^{-1} \pi^{\frac{2k-6}{k-4}} \\ &(17-4k)^{\frac{2}{4-k}} (4-k)^{\frac{2}{4-k}} (2-q_{\mathrm{E}})^{\frac{2}{k-4}} \\ &(1+z)^{\frac{3k-2q_{\mathrm{E}}-8}{k-4}+q_{\mathrm{e}}} (p-2)(p-1)^{-2} \epsilon_{\mathrm{e},0} \\ &m_{\mathrm{p}}^{\frac{k-2}{k-4}} \nu_{\mathrm{obs}}^{2} A^{\frac{2}{k-4}} E_0^{-\frac{2}{k-4}} \tau_{\mathrm{E}}^{\frac{2(1-q_{\mathrm{E}})}{k-4}} \tau_{\mathrm{e}}^{q_{\mathrm{e}}} t^{\frac{2q_{\mathrm{E}}-4}{k-4}-q_{\mathrm{e}}}. \end{aligned} \quad (82)$$

Separately, one has

$$\begin{aligned} F_{\nu,\mathrm{s21}} = {}& 1.53\times 10^{5} \eta^{-\frac{1}{2}q_{\mathrm{E}}-q_{\mathrm{e}}} (2-q_{\mathrm{E}})^{-\frac{1}{2}} \\ & (1+z)^{2+\frac{1}{2}q_{\mathrm{E}}+q_{\mathrm{e}}} D_{28}^{-2} \nu_{14.5}^{2} \epsilon_{\mathrm{e},-0.5} \\ & n_0^{-\frac{1}{2}} E_{53}^{\frac{1}{2}} \tau_{\mathrm{E},4}^{\frac{1}{2}(q_{\mathrm{E}}-1)} \tau_{\mathrm{e},4}^{q_{\mathrm{e}}} t_{\mathrm{d}}^{1-\frac{1}{2}q_{\mathrm{E}}-q_{\mathrm{e}}} \ \mathrm{Jy}, \end{aligned} \quad (83)$$

for ISM, and

$$\begin{aligned} F_{\nu,\mathrm{s21}} = {}& 9.96\times 10^{11} \eta^{-q_{\mathrm{E}}-q_{\mathrm{e}}} (2-q_{\mathrm{E}})^{-1} \\ & (1+z)^{q_{\mathrm{E}}+q_{\mathrm{e}}+1} D_{28}^{-2} \nu_{14.5}^{2} \epsilon_{\mathrm{e},-0.5} \\ & E_{53} \tau_{\mathrm{E},4}^{q_{\mathrm{E}}-1} \tau_{\mathrm{e},4}^{q_{\mathrm{e}}} t_{\mathrm{d}}^{2-q_{\mathrm{E}}-q_{\mathrm{e}}} \ \mathrm{Jy}, \end{aligned} \quad (84)$$

for wind medium.

For the expression of $F_{\nu,\mathrm{s22}}$, namely in the condition $\nu_{\mathrm{m}} < \nu_{\mathrm{obs}} < \nu_{\mathrm{a}} < \nu_{\mathrm{c}}$, one has $F_{\nu,\mathrm{s22}} = \left(\frac{\nu_{\mathrm{obs}}}{\nu_{\mathrm{a}}}\right)^{\frac{5}{2}} \left(\frac{\nu_{\mathrm{a}}}{\nu_{\mathrm{m}}}\right)^{-\frac{p-1}{2}} F_{\nu,\max}$.

For the analytical expression, one has

$$\begin{aligned} &F_{\nu,\mathrm{s22}} = \\ &3^{-2} 2^{\frac{41k-132}{4(k-4)}} c_2^{-1} D_{\mathrm{L}}^{-2} m_{\mathrm{e}}^{-\frac{3}{2}} m_{\mathrm{p}}^{\frac{2}{k-4}} q^{-\frac{1}{2}} \phi_{\mathrm{p}} x_{\mathrm{p}}^{-\frac{5}{2}} \\ &\pi^{\frac{5k-16}{2k-8}} c^{\frac{5k-12}{4k-16}} (17-4k)^{\frac{20-3k}{4(k-4)}} \\ &(1+z)^{\frac{(12-q_{\mathrm{E}})k-4q_{\mathrm{E}}-32}{4k-16}-\frac{1}{4}q_{\mathrm{B}}} \\ &(4-k)^{\frac{3k-20}{4(k-4)}} (2-q_{\mathrm{E}})^{\frac{20-3k}{4(k-4)}} (p-1)^{-1} \\ &\epsilon_{\mathrm{B},0}^{-\frac{1}{4}} \nu_{\mathrm{obs}}^{\frac{5}{2}} A^{\frac{2}{k-4}} E_0^{\frac{4+k}{16-4k}} \tau_{\mathrm{E}}^{\frac{k+4}{4k-16}(1-q_{\mathrm{E}})} \\ &\tau_{\mathrm{B}}^{-\frac{1}{4}q_{\mathrm{B}}} t^{\frac{(q_{\mathrm{E}}+2)k+4q_{\mathrm{E}}-24}{4k-16}+\frac{1}{4}q_{\mathrm{B}}}. \end{aligned} \quad (85)$$

Separately, one has

$$\begin{aligned} F_{\nu,\mathrm{s22}} = {}& 5.78\times 10^{9} \eta^{\frac{1}{4}q_{\mathrm{B}}-\frac{1}{4}q_{\mathrm{E}}} (2-q_{\mathrm{E}})^{-\frac{5}{4}} \\ & (1+z)^{2+\frac{1}{4}q_{\mathrm{E}}-\frac{1}{4}q_{\mathrm{B}}} D_{28}^{-2} \nu_{14.5}^{\frac{5}{2}} \epsilon_{\mathrm{B},-1}^{-\frac{1}{4}} n_0^{-\frac{1}{2}} \\ & E_{53}^{\frac{1}{4}} \tau_{\mathrm{E},4}^{\frac{1}{4}(q_{\mathrm{E}}-1)} \tau_{\mathrm{B},4}^{-\frac{1}{4}q_{\mathrm{B}}} t_{\mathrm{d}}^{\frac{3}{2}+\frac{1}{4}q_{\mathrm{B}}-\frac{1}{4}q_{\mathrm{E}}} \ \mathrm{Jy}, \end{aligned} \quad (86)$$

for ISM, and

$$\begin{aligned} F_{\nu,\mathrm{s22}} = {}& 3.02\times 10^{11} \eta^{\frac{1}{4}q_{\mathrm{B}}-\frac{3}{4}q_{\mathrm{E}}} (2-q_{\mathrm{E}})^{-\frac{7}{4}} \\ & (1+z)^{\frac{3}{4}q_{\mathrm{E}}+1-\frac{1}{4}q_{\mathrm{B}}} D_{28}^{-2} \nu_{14.5}^{\frac{5}{2}} \epsilon_{\mathrm{B},-1}^{-\frac{1}{4}} \\ & E_{53}^{\frac{3}{4}} \tau_{\mathrm{E},4}^{\frac{3}{4}(q_{\mathrm{E}}-1)} \tau_{\mathrm{B},4}^{-\frac{1}{4}q_{\mathrm{B}}} t_{\mathrm{d}}^{\frac{5}{2}+\frac{1}{4}q_{\mathrm{B}}-\frac{3}{4}q_{\mathrm{E}}} \ \mathrm{Jy}, \end{aligned} \quad (87)$$

for wind medium.

For the expression of $F_{\nu,\mathrm{s23}}$, namely in the condition $\nu_{\mathrm{m}} < \nu_{\mathrm{a}} < \nu_{\mathrm{obs}} < \nu_{\mathrm{c}}$, one has $F_{\nu,\mathrm{s23}} = \left(\frac{\nu_{\mathrm{obs}}}{\nu_{\mathrm{m}}}\right)^{-\frac{p-1}{2}} F_{\nu,\max}$.

For the analytical expression, one has

$$\begin{aligned} &F_{\nu,\mathrm{s23}} = \\ &3^{\frac{p}{2}} 2^{\frac{9kp+k-36p+12}{4(4-k)}} \pi^{\frac{4p-(p+1)k}{2k-8}} c^{\frac{20p-(5p+9)k+28}{4(k-4)}} \\ &(3-k)^{-1} (p-2)^{p-1} (p-1)^{1-p} \phi_{\mathrm{p}} x_{\mathrm{p}}^{\frac{p-1}{2}} \\ &(4-k)^{\frac{3(1-p)}{4}+\frac{k}{2(k-4)}} D_{\mathrm{L}}^{-2} \\ &(17-4k)^{\frac{3k-8}{2(k-4)}+\frac{p-1}{4}} (2-q_{\mathrm{E}})^{\frac{3kp-12p-5k+12}{4(k-4)}} \\ &m_{\mathrm{e}}^{\frac{1-p}{2}} m_{\mathrm{p}}^{\frac{(p-2)k-4p+6}{k-4}} q^{\frac{1}{2}p+\frac{5}{2}} \epsilon_{\mathrm{e},0}^{p-1} \epsilon_{\mathrm{B},0}^{\frac{p+1}{4}} \nu_{\mathrm{obs}}^{\frac{1-p}{2}} \\ &(1+z)^{\frac{(kp+5k-4p-12)q_{\mathrm{E}}-4k}{4(k-4)}+\frac{p+1}{4}q_{\mathrm{B}}+(p-1)q_{\mathrm{e}}} \\ &A^{\frac{2}{4-k}} E_0^{\frac{(p+5)k-4p-12}{4k-16}} \tau_{\mathrm{E}}^{\frac{kp+5k-4p-12}{4k-16}(q_{\mathrm{E}}-1)} \\ &\tau_{\mathrm{B}}^{\frac{p+1}{4}q_{\mathrm{B}}} \tau_{\mathrm{e}}^{(p-1)q_{\mathrm{e}}} \\ &t^{\frac{(kp+5k-4p-12)q_{\mathrm{E}}+2kp-10k-8p+24}{16-4k}-\frac{p+1}{4}q_{\mathrm{B}}-(p-1)q_{\mathrm{e}}}. \end{aligned} \quad (88)$$

Separately, one has

$$F_{\nu,{\rm s23}} = 2.43\times10^{4}\eta^{-\frac{7}{8}q_{\rm B}-\frac{11}{8}q_{\rm E}-\frac{3}{2}q_{\rm e}}(2-q_{\rm E})^{\frac{9}{8}} (1+z)^{\frac{11}{8}q_{\rm E}+\frac{7}{8}q_{\rm B}+\frac{3}{2}q_{\rm e}} D_{28}^{-2}\nu_{14.5}^{-\frac{3}{4}} \epsilon_{\rm e,-0.5}^{\frac{3}{2}}\epsilon_{\rm B,-1}^{\frac{7}{8}} n_0^{\frac{1}{2}} E_{53}^{\frac{11}{8}} \tau_{\rm E,4}^{\frac{11}{8}(q_{\rm E}-1)}\tau_{\rm B,4}^{\frac{1}{8}q_{\rm B}}\tau_{\rm e,4}^{\frac{3}{2}q_{\rm e}} t_{\rm d}^{\frac{1}{4}-\frac{7}{8}q_{\rm B}-\frac{11}{8}q_{\rm E}-\frac{3}{2}q_{\rm e}}\ {\rm Jy}, \quad (89)$$

for ISM, and

$$F_{\nu,{\rm s23}} = 5.11\times10^{2}\eta^{-\frac{7}{8}q_{\rm B}-\frac{7}{8}q_{\rm E}-\frac{3}{2}q_{\rm e}}(2-q_{\rm E})^{\frac{13}{8}} (1+z)^{\frac{7}{8}q_{\rm E}+1+\frac{7}{8}q_{\rm B}+\frac{3}{2}q_{\rm e}} D_{28}^{-2}\nu_{14.5}^{-\frac{3}{4}} \epsilon_{\rm e,-0.5}^{\frac{3}{2}}\epsilon_{\rm B,-1}^{\frac{7}{8}} E_{53}^{\frac{7}{8}}\tau_{\rm E,4}^{\frac{7}{8}(q_{\rm E}-1)}\tau_{\rm B,4}^{\frac{7}{8}q_{\rm B}}\tau_{\rm e,4}^{\frac{3}{2}q_{\rm e}} t_{\rm d}^{-\frac{3}{4}-\frac{7}{8}q_{\rm B}-\frac{7}{8}q_{\rm E}-\frac{3}{2}q_{\rm e}}\ {\rm Jy}, \quad (90)$$

for wind medium.

For the expression of $F_{\nu,{\rm s24}}$, namely in the condition $\nu_{\rm m}<\nu_{\rm a}<\nu_{\rm c}<\nu_{\rm obs}$, one has $F_{\nu,{\rm s24}} = \left(\frac{\nu_{\rm obs}}{\nu_{\rm c}}\right)^{-\frac{p}{2}}\left(\frac{\nu_{\rm c}}{\nu_{\rm m}}\right)^{-\frac{p-1}{2}} F_{\nu,{\rm max}}$.

For the analytical expression, one has

$$F_{\nu,{\rm s24}} = 3^{\frac{3+p}{2}}2^{-\frac{9p}{4}-\frac{1}{2}}\pi^{-\frac{p}{2}}(3-k)^{-1}(1+Y)^{-1} D_{\rm L}^{-2}\sigma_{\rm T}^{-1}m_{\rm e}^{1-\frac{3p}{2}}m_{\rm p}^{p-2}q^{3+\frac{1}{2}p}c^{-\frac{5p}{4}-\frac{3}{2}}\phi_{\rm p}x_{\rm p}^{\frac{p}{2}} (p-1)^{1-p}(p-2)^{p-1}(17-4k)^{\frac{p+3}{4}} (4-k)^{\frac{6-3p}{4}}(2-q_{\rm E})^{\frac{3p-6}{4}}\epsilon_{\rm B,0}^{\frac{p-2}{4}}\epsilon_{\rm e}^{p-1} (1+z)^{\frac{p+2}{4}q_{\rm E}+(p-1)q_{\rm e}+\frac{p-2}{4}q_{\rm B}}\nu_{\rm obs}^{-\frac{p}{2}}E_0^{\frac{p+2}{4}} \tau_{\rm E}^{\frac{p+2}{4}(q_{\rm E}-1)}\tau_{\rm B}^{\frac{p-2}{4}q_{\rm B}}\tau_{\rm e}^{(p-1)q_{\rm e}} t^{1-\frac{p+2}{4}q_{\rm E}-\frac{p}{2}+\frac{2-p}{4}q_{\rm B}-(p-1)q_{\rm e}}. \quad (91)$$

Separately, one has

$$F_{\nu,{\rm s24}} = 1.74\times10^{3}\eta^{-\frac{1}{8}q_{\rm B}-\frac{9}{8}q_{\rm E}-\frac{3}{2}q_{\rm e}}(2-q_{\rm E})^{\frac{3}{8}} (1+z)^{\frac{9}{8}q_{\rm E}+\frac{1}{8}q_{\rm B}+\frac{3}{2}q_{\rm e}}(1+Y)^{-1}D_{28}^{-2} \nu_{14.5}^{-\frac{5}{4}}\epsilon_{\rm e,-0.5}^{\frac{3}{2}}\epsilon_{\rm B,-1}^{\frac{1}{8}}E_{53}^{\frac{9}{8}}\tau_{\rm E,4}^{\frac{9}{8}(q_{\rm E}-1)}\tau_{\rm B,4}^{\frac{1}{8}q_{\rm B}}\tau_{\rm e,4}^{\frac{3}{2}q_{\rm E}} t_{\rm d}^{-\frac{1}{4}-\frac{1}{8}q_{\rm B}-\frac{9}{8}q_{\rm E}-\frac{3}{2}q_{\rm e}}\ {\rm Jy}, \quad (92)$$

for ISM, and

$$F_{\nu,{\rm s24}} = 8.61\times10^{2}\eta^{-\frac{1}{8}q_{\rm B}-\frac{9}{8}q_{\rm E}-\frac{3}{2}q_{\rm e}}(2-q_{\rm E})^{\frac{3}{8}} (1+z)^{\frac{9}{8}q_{\rm E}+\frac{1}{8}q_{\rm B}+\frac{3}{2}q_{\rm e}}(1+Y)^{-1}D_{28}^{-2} \nu_{14.5}^{-\frac{5}{4}}\epsilon_{\rm e,-0.5}^{\frac{3}{2}}\epsilon_{\rm B,-1}^{\frac{1}{8}}E_{53}^{\frac{9}{8}}\tau_{\rm E,4}^{\frac{9}{8}(q_{\rm E}-1)}\tau_{\rm B,4}^{\frac{1}{8}q_{\rm B}}\tau_{\rm e,4}^{\frac{3}{2}q_{\rm e}} t_{\rm d}^{-\frac{1}{4}-\frac{1}{8}q_{\rm B}-\frac{9}{8}q_{\rm E}-\frac{3}{2}q_{\rm e}}\ {\rm Jy}, \quad (93)$$

for wind medium.

In case of II(3) $\nu_{\rm m}<\nu_{\rm c}<\nu_{\rm a}$, for $F_{\nu,{\rm s31}}$, namely in the condition $\nu_{\rm obs}<\nu_{\rm m}<\nu_{\rm c}<\nu_{\rm a}$, one has $F_{\nu,{\rm s31}} = \left(\frac{\nu_{\rm obs}}{\nu_{\rm m}}\right)^{2}\left(\frac{\nu_{\rm m}}{\nu_{\rm a}}\right)^{\frac{p+4}{2}}\left(\frac{\nu_{\rm a}}{\nu_{\rm c}}\right)^{-\frac{1}{2}} F_{\nu,{\rm max}}$.

For the analytical expression, one has

$$F_{\nu,{\rm s31}} = 3^{-\frac{3}{2}}2^{\frac{4k-12}{k-4}}\pi^{\frac{2k-6}{k-4}}c^{\frac{2}{k-4}}(p-1)^{-2} (p-2)\phi_{\rm p}x_{\rm p}^{-2}m_{\rm p}^{\frac{k-2}{k-4}}D_{\rm L}^{-2}c_2^{-1} (17-4k)^{\frac{2}{4-k}}(2-q_{\rm E})^{\frac{2}{k-4}}(4-k)^{\frac{2}{4-k}} (1+z)^{\frac{3k-8-2q_{\rm E}}{k-4}+q_{\rm e}}\epsilon_{\rm e} \nu_{\rm obs}^{2}A^{\frac{2}{k-4}}E_0^{-\frac{2}{k-4}}\tau_{\rm E}^{\frac{2}{k-4}(1-q_{\rm E})}\tau_{\rm e}^{q_{\rm e}} t^{\frac{2q_{\rm E}-4}{k-4}-q_{\rm e}}. \quad (94)$$

Separately, one has

$$F_{\nu,{\rm s31}} = 1.33\times10^{10}\eta^{-\frac{1}{2}q_{\rm E}-q_{\rm e}}(2-q_{\rm E})^{-\frac{1}{2}} (1+z)^{\frac{1}{2}q_{\rm E}+2+q_{\rm e}}D_{28}^{-2}\nu_{14.5}^{2}\epsilon_{\rm e,-0.5} n_0^{-\frac{1}{2}}E_{53}^{\frac{1}{2}}\tau_{\rm E,4}^{\frac{1}{2}(q_{\rm E}-1)}\tau_{\rm e,4}^{q_{\rm e}}t_{\rm d}^{1-\frac{1}{2}q_{\rm E}-q_{\rm e}}\ {\rm Jy}, \quad (95)$$

for ISM, and

$$F_{\nu,{\rm s31}} = 9.96\times10^{11}\eta^{-q_{\rm E}-q_{\rm e}}(2-q_{\rm E})^{-1} (1+z)^{q_{\rm E}+q_{\rm e}+1}D_{28}^{-2}\nu_{14.5}^{2} \epsilon_{\rm e,-0.5}E_{53}\tau_{\rm E,4}^{q_{\rm E}-1}\tau_{\rm e,4}^{q_{\rm e}}t_{\rm d}^{2-q_{\rm E}-q_{\rm e}}\ {\rm Jy}, \quad (96)$$

for wind medium.

For the expression of $F_{\nu,{\rm s32}}$, namely in the condition $\nu_{\rm m}<\nu_{\rm obs}<\nu_{\rm a}$, one has $F_{\nu,{\rm s32}} = \left(\frac{\nu_{\rm obs}}{\nu_{\rm a}}\right)^{\frac{5}{2}}\left(\frac{\nu_{\rm a}}{\nu_{\rm c}}\right)^{-\frac{p}{2}}\left(\frac{\nu_{\rm c}}{\nu_{\rm m}}\right)^{-\frac{p-1}{2}} F_{\nu,{\rm max}}$.

For the analytical expression, one has

$$F_{\nu,{\rm s32}} = 3^{-2}2^{\frac{25k-84}{4k-16}}(p-1)^{-1}(2-q_{\rm E})^{\frac{k}{2(k-4)}-\frac{5}{4}}D_{\rm L}^{-2}c_2^{-1} c^{\frac{5k-12}{4k-16}}(4-k)^{\frac{10-3k}{k-4}+\frac{15}{4}}\pi^{\frac{5k-16}{2k-8}}\phi_{\rm p}x_{\rm p}^{-\frac{5}{2}}q^{-\frac{1}{2}} (17-4k)^{\frac{k-6}{k-4}-\frac{5}{4}}(1+z)^{\frac{(k+4)q_{\rm E}-12k+32}{16-4k}-\frac{1}{4}q_{\rm B}} m_{\rm e}^{\frac{3}{2}}m_{\rm p}^{\frac{2}{k-4}}\nu_{\rm obs}^{\frac{5}{2}}A^{\frac{2}{k-4}}\epsilon_{\rm B,0}^{-\frac{1}{4}}E_0^{\frac{k+4}{16-4k}}\tau_{\rm E}^{\frac{(1-q_{\rm E})(k+4)}{4k-16}} \tau_{\rm B}^{-\frac{1}{4}q_{\rm B}}t^{\frac{(k+4)q_{\rm E}+2k-24}{4k-16}+\frac{1}{4}q_{\rm B}}. \quad (97)$$

Separately, one has

$$F_{\nu,{\rm s32}} = 5.78\times10^{9}\eta^{\frac{1}{4}q_{\rm B}-\frac{1}{4}q_{\rm E}}(2-q_{\rm E})^{-\frac{5}{4}} (1+z)^{2+\frac{1}{4}q_{\rm E}-\frac{1}{4}q_{\rm B}}D_{28}^{-2}\nu_{14.5}^{\frac{5}{2}}\epsilon_{\rm B,-1}^{-\frac{1}{4}} n_0^{-\frac{1}{2}}E_{53}^{\frac{1}{4}}\tau_{\rm E,4}^{\frac{1}{4}(q_{\rm E}-1)}\tau_{\rm B,4}^{-\frac{1}{4}q_{\rm B}}t_{\rm d}^{\frac{3}{2}+\frac{1}{4}q_{\rm B}-\frac{1}{4}q_{\rm E}}\ {\rm Jy}, \quad (98)$$

for ISM, and

$$F_{\nu,\rm s32} = 1.24\times10^{11}\eta^{\frac{1}{4}q_{\rm B}-\frac{3}{4}q_{\rm E}}(2-q_{\rm E})^{-\frac{7}{4}} (1+z)^{1+\frac{3}{4}q_{\rm E}-\frac{1}{4}q_{\rm B}} D_{28}^{-2}\nu_{14.5}^{\frac{5}{2}}\epsilon_{\rm B,-1}^{-\frac{1}{4}} E_{53}^{\frac{3}{4}}\tau_{\rm E,4}^{\frac{3}{4}(q_{\rm E}-1)}\tau_{\rm B,4}^{-\frac{1}{4}q_{\rm B}} t_{\rm d}^{\frac{5}{2}+\frac{1}{4}q_{\rm B}-\frac{3}{4}q_{\rm E}}\ {\rm Jy}, \quad (99)$$

for wind medium.

For the expression of $F_{\nu,\rm s33}$, namely in the condition $\nu_{\rm m} < \nu_{\rm c} < \nu_{\rm a} < \nu_{\rm obs}$, one has $F_{\nu,\rm s33} = \left(\frac{\nu_{\rm obs}}{\nu_{\rm c}}\right)^{-\frac{p}{2}}\left(\frac{\nu_{\rm c}}{\nu_{\rm m}}\right)^{-\frac{p-1}{2}} F_{\nu,\rm max}$.

For the analytical expression, one has

$$F_{\nu,\rm s33} = 3^{\frac{3+p}{2}}2^{-\frac{9p}{4}-\frac{1}{2}}\pi^{-\frac{p}{2}}c^{\frac{5p}{4}-\frac{3}{2}}m_{\rm p}^{p-2}m_{\rm e}^{1-\frac{3p}{2}}\sigma_{\rm T}^{-1} (3-k)^{-1}(1+Y)^{-1}D_{\rm L}^{-2}\phi_{\rm p}x_{\rm p}^{\frac{p}{2}}q^{\frac{p}{2}+3} (4-k)^{\frac{6-3p}{4}}(2-q_{\rm E})^{\frac{3p-6}{4}}(p-1)^{1-p} (p-2)^{p-1}(17-4k)^{\frac{2+p}{4}} (1+z)^{\frac{p+2}{4}q_{\rm E}+\frac{p-2}{4}q_{\rm B}+(p-1)q_{\rm e}} \epsilon_{\rm e}^{p-1}\epsilon_{\rm B,0}^{\frac{p-2}{4}}\nu_{\rm obs}^{-\frac{p}{2}}E_0^{\frac{p+2}{4}}\tau_{\rm E}^{\frac{p+2}{4}(q_{\rm E}-1)}\tau_{\rm B}^{\frac{p-2}{4}q_{\rm B}}\tau_{\rm e}^{(p-1)q_{\rm e}} t^{1-\frac{p+2}{4}q_{\rm E}-\frac{p}{2}-(p-1)q_{\rm e}-\frac{p-2}{4}q_{\rm B}}. \quad (100)$$

Separately, one has

$$F_{\nu,\rm s33} = 4.53\times10^{2}\eta^{-\frac{1}{8}q_{\rm B}-\frac{9}{8}q_{\rm E}-\frac{3}{2}q_{\rm E}}(2-q_{\rm E})^{\frac{3}{8}} (1+z)^{\frac{9}{8}q_{\rm E}+\frac{1}{8}q_{\rm B}+\frac{3}{2}q_{\rm e}}(1+Y)^{-1}D_{28}^{-2} \nu_{14.5}^{-\frac{5}{4}}\epsilon_{\rm e,-0.5}^{\frac{3}{2}}\epsilon_{\rm B,-1}^{\frac{1}{8}}E_{53}^{\frac{9}{8}}\tau_{\rm E,4}^{\frac{9}{8}(q_{\rm E}-1)}\tau_{\rm B,4}^{\frac{1}{8}q_{\rm B}} \tau_{\rm e,4}^{\frac{3}{2}q_{\rm e}}t_{\rm d}^{-\frac{1}{4}-\frac{1}{8}q_{\rm B}-\frac{9}{8}q_{\rm E}-\frac{3}{2}q_{\rm e}}\ {\rm Jy}, \quad (101)$$

for ISM, and

$$F_{\nu,\rm s33} = 8.61\times10^{2}\eta^{-\frac{1}{8}q_{\rm B}-\frac{9}{8}q_{\rm E}-\frac{3}{2}q_{\rm e}}(2-q_{\rm E})^{\frac{3}{8}} (1+z)^{\frac{9}{8}q_{\rm E}+\frac{1}{8}q_{\rm B}+\frac{3}{2}q_{\rm e}}(1+Y)^{-1}D_{28}^{-2} \nu_{14.5}^{-\frac{5}{4}}\epsilon_{\rm e,-0.5}^{\frac{3}{2}}\epsilon_{\rm B,-1}^{\frac{1}{8}}E_{53}^{\frac{9}{8}}\tau_{\rm E,4}^{\frac{9}{8}(q_{\rm E}-1)}\tau_{\rm B,4}^{\frac{1}{8}q_{\rm B}} \tau_{\rm e,4}^{\frac{3}{2}q_{\rm e}}t_{\rm d}^{-\frac{1}{4}-\frac{1}{8}q_{\rm B}-\frac{9}{8}q_{\rm E}-\frac{3}{2}q_{\rm e}}\ {\rm Jy}, \quad (102)$$

for wind medium.

4 RESULTS AND APPLICATION

4.1 Results

With the formulas given above, one can get the dynamics, typical frequencies and observed flux densities. Notice that considering the large amount of parameters, the results are quite different with different sets of parameters. To show the results, we take a set of typical values for parameters, namely $k = 0, n = 1\,{\rm cm}^{-3}, \epsilon_{\rm e,0} = 0.1, \epsilon_{\rm B,0} = 0.01, q_{\rm E} = q_{\rm e} = q_{\rm B} = 0.1, \tau_{\rm e} = \tau_{\rm B} = \tau_{\rm E} = 1000\,{\rm s}, p = 2.3, E_0 = 4\times10^{52}$ erg and $z = 0.1$. With these parameters, we can plot the evolution of the radiation and the spectra. It is easier to show the parameter dependence by changing one parameter while keeping others fixed. As an example, we plot them by varying the magnetic parameter $\epsilon_{\rm B,0}$.

Figure 1 shows the light curves in the optical band. One can clearly see flux density increasing with the increase of $\epsilon_{\rm B,0}$, the magnetic field of which assumes an important role in the synchrotron radiation.

Figure 2 shows the spectra at 10 days with different $\epsilon_{\rm B,0}$, and Figure 3 shows the time evolution of the typical frequencies $\nu_{\rm c}$, $\nu_{\rm m}$ and $\nu_{\rm a}$. With more parameters being changed, the behavior of the dynamics and the radiation will be more complicated. One may understand the underlying parameters (and also the underlying physics) while fitting unusual GRB afterglows.

4.2 Modeling the Afterglow of GRB 060607A

To show the ability of the extended model, we apply the whole set of formulas to some certain GRBs, which are not applicable with the 'standard model.' Using the external forward shock models, Wang et al. (2015) have fit 85 GRBs up to March 2014 with well-monitored X-ray and optical light curves. They found that a 'Gold' sample (fitted well) includes $45/85$ GRBs, and a 'Silver' sample includes $37/85$ GRBs, while a 'bad' sample (fitted badly) includes $3/85$ GRBs. Their results showed that external shock models work very well for at least ~ 53 of the GRB afterglows samples. If post standard models (e.g., structured jet) are carried out, up to $\sim$ 96 percent of GRBs can be accounted for within the external shock models.

Only three GRB afterglows (namely GRB 060607A, GRB 070208 and GRB 070420) violate the expectations of the external shock models, so they argued it demands another emission component (e.g., the central engine afterglow) to account for emission on at least one band (e.g., the X-ray band). Recently, De Pasquale et al. (2016) argued that the late X-ray afterglow of GRB 130427A challenged external shock models because it required extreme values of the physical parameters. Therefore, we can change the way of asking: to what extent can we use the time-dependent parameter model to fit these 'exceptional' GRBs under reasonable physical parameter values? In the following, as an example, we try to fit one of their 'bad' samples, the GRB 060607A afterglow, with our modified standard model.

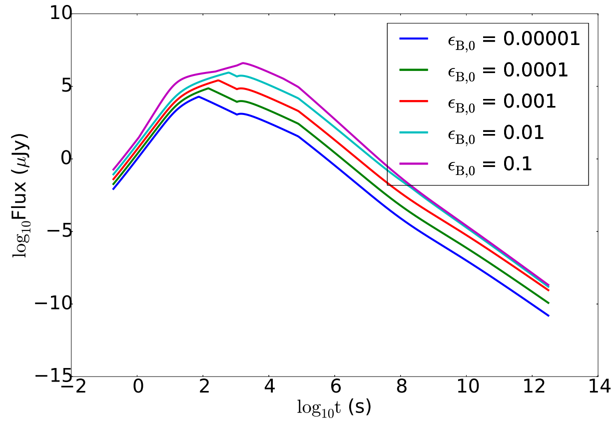

Fig. 1 The optical light curves (at 10^{14} Hz) with different magnetic parameter $\epsilon_{\rm B,0}$. Values for the parameter $\epsilon_{\rm B,0}$ for different curves are $10^{-5}, 10^{-4}, 10^{-3}, 10^{-2}$ and 10^{-1} from bottom to top, respectively. The other parameters are fixed to $k = 0, n = 1\,{\rm cm}^{-3}, \epsilon_{\rm e,0} = 0.1, q_{\rm E} = q_{\rm e} = q_{\rm B} = 0.1, \tau_{\rm e} = \tau_{\rm B} = \tau_{\rm E} = 1000\,{\rm s}, p = 2.3, E_0 = 4 \times 10^{52}$ erg and $z = 0.1$.

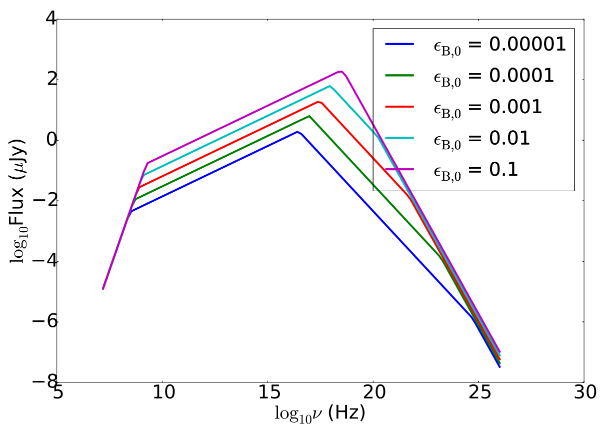

Fig. 2 The spectra at time 10 days. Parameters are the same as in Fig. 1.

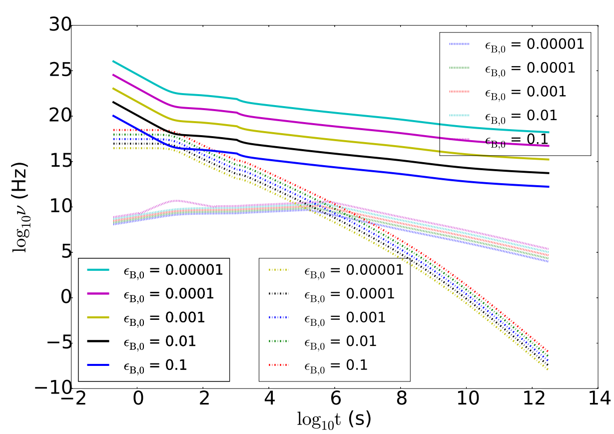

Fig. 3 The evolution of typical frequencies $\nu_{\rm c}$ (*solid lines*), $\nu_{\rm m}$ (*dashed lines*) and $\nu_{\rm a}$ (*dotted lines*). Parameters are the same as in Fig. 1.

GRB 060607A was detected by *Swift* (Ziaeepour et al. 2006). *Swift* slewed immediately to the burst. The *Swift* Burst Alert Telescope (BAT) light curve showed a double-peak structure with a duration of about $40\,\mathrm{s}$. The peak count rate was $\sim 3000\,\mathrm{counts\,s^{-1}}$ $(15-350\,\mathrm{keV})$, at $\sim 0\,\mathrm{s}$ after the trigger. The XRT began observing the field at $05:13:18$ (UT), $65\,\mathrm{s}$ after the BAT trigger. XRT found a bright, variable X-ray source. The initial flux in the $2.5\,\mathrm{s}$ image was $3.3\times10^{-9}\,\mathrm{erg\,cm^{-2}\,s^{-1}}$ $(0.2-10\,\mathrm{keV})$. UVOT took a $100\,\mathrm{s}$ finding chart exposure with the White $(160-650\,\mathrm{nm})$ filter starting $75\,\mathrm{s}$ after the BAT trigger. There is a candidate afterglow in the rapidly available sub-image with a 1-sigma error radius of about $0.5\,\mathrm{arcsec}$. The redshift is $z=3.082$ for GRB 060607A (Ledoux et al. 2006). Panchromatic Robotic Optical Monitoring and Polarimetry Telescopes (PROMPT) observed the early-time optical afterglow of GRB 060607A and obtained a densely sampled multi-wavelength light curve that begins only tens of seconds after the GRB (Nysewander et al. 2009).

Some authors have studied the afterglow of GRB 060607A (Malesani et al. 2007; Nysewander et al. 2009; Covino et al. 2008; Staff et al. 2008; Ziaeepour et al. 2008; Zhang 2009). Molinari et al. (2007) and Jin & Fan (2007) constrained the circumburst medium profile. They found a constant and low-density medium profile for GRB 060607A. In addition, they showed that the presence of infrared flashes in these two afterglows is consistent with the standard hydrodynamical external reverse shock model. Although a highly magnetized model can explain the data, it is no longer demanded. A weak reverse shock in the standard hydrodynamical model is achievable if the typical synchrotron frequency is already below the band at the shock-crossing time. However, as demonstrated in Wang et al. (2015), the afterglow cannot be fitted within the standard afterglow model. We modeled the afterglow in X-ray ($2.4\times10^{18}\,\mathrm{Hz}$) and optical bands ($4\times10^{14}\,\mathrm{Hz}$), with the number of parameters released being 12. The numerical code we adopted is a revised version of the one developed in Wang et al. (2014). The values of parameters we obtained are $E_0=2.0\times10^{53}\,\mathrm{erg}$, $k=2.1$, $n_0=1$, $L_0=5.0\times10^{51}\,\mathrm{erg}$, $\epsilon_{\mathrm{B},0}=3\times10^{-7}$, $\epsilon_{\mathrm{e},0}=0.3$, $q_\mathrm{E}=0.9$, $q_\mathrm{B}=1$, $q_\mathrm{e}=1.5$, $\tau_\mathrm{E}=1\,\mathrm{s}$, $\tau_\mathrm{B}=10^4\,\mathrm{s}$ and $\tau_\mathrm{e}=10^4\,\mathrm{s}$ (see Fig. 4). Before the injected energy becomes dominant, the total energy is roughly E_0, which is the only parameter. After it becomes dominant, the parameters are L_0, q_E and τ_E, which are three. One can see the modeling of this afterglow is qualitatively acceptable, but the details are not fitted well. One of the reasons is that the energy may not be injected continuously. As shown in the X-ray light curve in the figure, there are several X-ray flares, which means the energy may also be injected abruptly several times. Another reason might be that the afterglow is out of the range of parameters that are being changed, but it is morphologically different, such as being electron-positron pair dominated or long extended reverse shock dominated.

5 CONCLUSIONS AND DISCUSSION

In this work, we extended the 'standard afterglow' model with five parameters to 12 parameters by making the 'standard' parameters time dependent, i.e. changing the total kinetic energy E_k into E_0, q_E and τ_E, the number density of the environment n into A and k, the equipartition factor of the energy density for electrons ϵ_e into $\epsilon_{\mathrm{e},0}, q_\mathrm{e}$ and τ_e, the equipartition factor of the energy density for the magnetic field ϵ_B into $\epsilon_{\mathrm{B},0}, q_\mathrm{B}$ and τ_B, while the power law index of the electrons' distribution p was unchanged. The full dynamics and radiation have been derived with these 12 new parameters. We also derived the full set of scaling laws with fixed environment styles being $k=0$ (ISM) and $k=2$ (wind), and gave typical values for all the parameters.

This study can be used as a complete reference for modeling GRB afterglows, by releasing several parameters that are not constant while others are constant. For example, one can set the afterglows to be energy injected and in an ISM with constant ϵ_e and ϵ_B. Consequently, one can obtain the formulas from the general ones above by setting E_0, q_E, τ_E, A, $\epsilon_{\mathrm{B},0}$ and $\epsilon_{\mathrm{e},0}$ to be free parameters while letting k, q_e and q_B be 0. For the most general case, one should release all 12 parameters to be free, i.e., E_0, q_E, τ_E, k, A, p, $\epsilon_{\mathrm{e},0}$, τ_e, q_e, $\epsilon_{\mathrm{B},0}$, τ_B and q_B. Given these analytical expressions, the GRB afterglow model is more convenient to use and to compare with each other.

When $\nu_\mathrm{a}<\nu_\mathrm{c}$, there are twelve cases, namely $F_{\nu,\mathrm{f11}}$, $F_{\nu,\mathrm{f12}}$, $F_{\nu,\mathrm{f13}}$,$F_{\nu,\mathrm{f14}}$, $F_{\nu,\mathrm{s11}}$, $F_{\nu,\mathrm{s12}}$, $F_{\nu,\mathrm{s13}}$, $F_{\nu,\mathrm{s14}}$, $F_{\nu,\mathrm{s21}}$, $F_{\nu,\mathrm{s22}}$, $F_{\nu,\mathrm{s23}}$ and $F_{\nu,\mathrm{s24}}$. In these cases, SSA becomes important as a heating source for the low-energy tail. Consequently, the electrons are dominated by a quasi-thermal component until to a transition Lorentz factor, above which the electrons are no longer affected by the self absorption heating and keep the normal power law distribution. For these strong absorption cases, a thermal peak due to pile-up electrons would appear around ν_a in the synchrotron spectrum, which would also result in some new features in the SSA spectrum (e.g., Gao et al. 2013).

In these formulas, we found some interesting results. In eight cases, the index of time is independent of the value of parameter k. In other

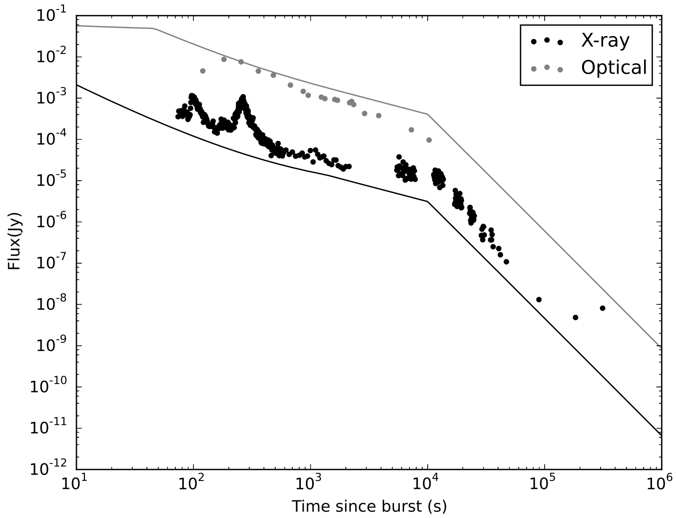

Fig. 4 Modeling the afterglow of GRB 060607A. The grey dots are the optical R band light curve, and the black dots are the X-ray light curve. The grey solid curve and the black solid curve are the modeled light curves for the optical band and X-ray band respectively. The corresponding values of the parameters are $E_0 = 2.0 \times 10^{53}$ erg, $k = 2.1$, $n_0 = 1$, $L_0 = 5.0 \times 10^{51}$ erg s^{-1}, $\epsilon_{\rm B,0} = 3 \times 10^{-7}$, $\epsilon_{\rm e,0} = 0.3$, $q_{\rm E} = 0.9$, $q_{\rm B} = 1$, $q_{\rm e} = 1.5$, $\tau_{\rm E} = 1$ s, $\tau_{\rm B} = 10^4$ s and $\tau_{\rm e} = 10^4$ s. Data are taken from *Swift* (Ziaeepour et al. 2006) and Wang et al. (2015).

words, radiation evolution is not related to the density of the external medium. These cases are $F_{\nu,\rm f13}, F_{\nu,\rm f14}, F_{\nu,\rm f23}, F_{\nu,\rm f24}, F_{\nu,\rm f33}$, $F_{\nu,\rm s14}$, $F_{\nu,\rm s24}$ and $F_{\nu,\rm s33}$.

In most cases, the peak of flux increases as k increases, but in three cases, namely $F_{\nu,\rm f12}$, $F_{\nu,\rm s12}$ and $F_{\nu,\rm f13}$, the peak of flux decreases as k increases, while in seven cases, namely $F_{\nu,\rm f13}$, $F_{\nu,\rm f14}$, $F_{\nu,\rm f23}$, $F_{\nu,\rm f34}$, $F_{\nu,\rm s14}$, $F_{\nu,\rm s24}$ and $F_{\nu,\rm s33}$, their peak of flux is almost independent of k. This is not obvious from numerical calculations. From this example, we can see the advantages of the analytic formulas. Although exceptional samples can be fitted if appropriate parameters are selected, degeneracy occurs.

This possibility was pointed out by Eichler & Waxman (2005). Therefore one has to fix several 'reasonable' parameters for fitting other parameters. Or, one can set the time-dependent parameters changing together with time in the same profile, such as keeping $\epsilon_{\rm B}/\epsilon_{\rm e}$ constant. Therefore, one can get some more reasonable fitting parameters in some special cases.

Through fitting the afterglow of GRB 060607A, we found that it can be interpreted in the framework of our model within a reasonable range of parameters. Recently, Warren et al. (2017) claimed that they found convincing evidence for energy injection into the afterglow of GRB 150424A, and their analysis might shed light on understanding afterglow plateau emission, the nature of which is still under debate. In future work, we can fit more of the unique afterglows which are similar to it with our model, to strengthen the conclusion further.

Notice that formulas in which parameters are released cannot cover all the cases of the afterglow. The real afterglow of the GRBs can be even more complicated. For the environment, it could be a density jump like meeting a cloud in a star forming region (Dai & Lu 1998b). For the component, it could be electron-positron pair dominated (Beloborodov 2002). For the radiation mechanism, the IC scattering may contribute (e.g. Sari & Esin 2001). For the energy injection, in which not only pure energy is injected into the afterglow (Cohen & Piran 1999), it can be matter dominated (Zhang & Mészáros 2002b), or electron-positron pair dominated (Dai 2004). For the late radiation that can last weeks or even years, the supernova may contribute (e.g. Galama et al. 1998) and the blast wave will eventually go into its non-relativistic phase (Huang et al. 1998). Therefore, understanding the late afterglow should consider variation of the 'standard' parameters as well as extension of the 'standard' model itself.

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