

# Scattering Law Analysis Based on Hapke and Lommel-Seeliger Models for Asteroidal Taxonomy

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**Abstract** In deriving the physical properties of asteroids from their photometric data, the scattering law plays an important role, although the shape variations of asteroids result in the main variations in lightcurves. By following the physical behaviors of light reflections, Hapke et al. deduced complex functions to represent the scattering process, however, it is very hard to accurately simulate the surface scattering law in reality. For simplicity, other numerical scattering models are presented for efficiently calculating the physical properties of asteroids, such as the Lommel-Seeliger (LS) model. In this article, these two models are compared numerically. It is found that in some numerical applications the LS model in simple form with four parameters can be exploited to replace the Hapke model in complex form with five parameters. Furthermore, the generated synthetic lightcurves by the Cellinoid shape model also show that the LS model can perform as well as the Hapke model in the inversion process. Finally, by applying the Principal Component Analysis (PCA) technique to the parameters of the LS model, we present an efficient method to classify C and S type asteroids, instead of the conventional method using the parameters of the Hapke model.

**Key words:** scattering — techniques: photometric — minor planets — asteroids: general

## 1 INTRODUCTION

Being composed of primitive matter from the early solar system, an asteroid can reveal the evolution mechanism of planets and provide information on the formation of our solar system (DeMeo & Carry 2014). By using ground-based observations, the physical properties of asteroids can be derived from their photometric data such as rotational periods, pole orientations and overall shapes (Lu et al. 2013, 2014; Lu & Ip 2015; Lu et al. 2016).

In deriving the physical properties of asteroids from their photometry data, the scattering law plays an important role, although the shape variations of asteroids result in the main variations in their lightcurves (Karttunen 1989; Karttunen & Bowell 1989). As shown in Figure 1, the scattering geometry can be defined in terms of the angle of incidence  $i$  between the light source and local

surface normal, the angle of emergence  $e$  between the observer and the local surface normal, and the phase angle  $\alpha$  between the light source and observer, as seen from the object, in addition to the azimuthal angle  $\phi$  between the planes of incidence and emergence. By following the physical behaviors of light reflections, Hapke (1981, 1984, 1986) and Hapke & Wells (1981) deduced complex functions to represent the scattering process. Their evolved model has been applied extensively to spacecraft and telescopic observations for a lot of objects throughout the solar system to derive their physical and surface characteristics (Li et al. 2004, 2006, 2007, 2010, 2013).

However, application of the Hapke model is very complicated when applying a numerical inversion process to investigate the physical properties of asteroids. There are many equivalent methods that have attempted to numerically replace it such as the joint linear expo-

ponential model published by Muinonen *et al.* (2009), the Minnaert model proposed by Minnaert (1941), and the Lommel-Seeliger (LS) model constructed by Seeliger (1884). In particular, the LS model is very efficient in numerical calculations of synthetic brightnesses of asteroids. Recently, many applications have been developed to derive the physical parameters for asteroids with the LS model (Cellino *et al.* 2015; Muinonen & Lumme 2015).

Parameters in the Hapke model have corresponding physical meanings such as albedo, which is a measure of the amount of reflected light from the surface, and others as described in Table 1. Moreover, the LS model can be applied efficiently to calculate the physical parameters for asteroids (Muinonen *et al.* 2015; Wang *et al.* 2016). In this article, we make an attempt to combine two models, by presenting an equivalent method to connect the physical meanings of the Hapke model to the parameters of the LS model. Following the method, the parameters of the LS model can be efficiently derived in fitting the photometric data and subsequently they can be analyzed by the Principal Component Analysis (PCA) technique to obtain a classification for asteroids (Abdi & Williams 2010). Ultimately, to better identify asteroid types, the least absolute shrinkage and selection operator (LASSO) technique is introduced to efficiently classify C and S type asteroids based on LS model parameters (Tibshirani 1996; Mei & Ling 2009; Musoro *et al.* 2014).

The paper will be arranged as follows. First of all, in Section 2, the Hapke model and LS model will be concisely presented. In Section 3, numerical simulations are conducted to confirm that the LS model can perform well as a replacement for the Hapke model in the inversion application. Furthermore, equivalent to the Hapke model, the parameters of LS are analyzed by the PCA method for classifying C and S type asteroids. Subsequently, the LASSO method is employed for the taxonomy of C and S type asteroids from the LS model parameters. In the end, we sum up the primary work of this article and discuss future work in Section 4.

## 2 SCATTERING MODELS

### 2.1 Hapke Model

The Hapke model is based on a semi-physical model that uses the analytical solutions of radiative transfer on an asteroidal surface with simple assumptions, coupled with empirical models that describe the scattering behavior of the particulate surface (Hapke 2012).

There are at least five parameters in the Hapke model, including the single scattering albedo ( $\omega$ ), photometric roughness ( $\bar{\theta}$ ), amplitude and width of the op-

position effect ( $B_{s_0}$  and  $h_s$  respectively for the shadow-hiding opposition effect (SHOE), and  $B_{c_0}$  and  $h_c$  respectively for the coherent backscattering opposition effect (CBOE)), as well as one or more parameters to describe the single-particle phase function. The model exploited throughout this article is a five-parameter version, which only considers the SHOE, and adopts a one-term Henyey-Greenstein (HG) phase function (Buratti & Veverka 1983; Li *et al.* 2015) involving one asymmetry parameter  $g$  to describe the single-particle phase function. The five parameters are summarized in Table 1 and the radiance factor (RADF), the ratio of bidirectional reflectance of a surface to that of a perfectly diffuse surface illuminated at  $i = 0^\circ$  (Hapke 2012), is shown in Equation (1), where  $B(\alpha)$  is the shadow hiding opposition surge function,  $p(\alpha)$  is the average particle single-scattering phase function,  $H(x)$  is the Ambartsumian-Chandrasekhar function for multiple scattering, and  $S$  is the Hapke roughness shadowing function (Hapke 2012).

$$\begin{aligned} \text{RADF}(i, e, \alpha) &= \frac{\omega}{4} \frac{\mu_0}{\mu_0 + \mu} \\ &\times \{ [1 + B(\alpha)] p(\alpha) + H(\mu_0)H(\mu) - 1 \} \\ &\times S(i, e, \alpha, \bar{\theta}), \end{aligned} \quad (1)$$

$$B(\alpha) = \frac{B_{s_0}}{1 + \frac{1}{h_s} \tan \frac{\alpha}{2}}, \quad (2)$$

$$p(\alpha) = \frac{1 - g^2}{(1 + 2g \cos \alpha + g^2)^{\frac{3}{2}}}, \quad (3)$$

$$H(x) = \frac{1 + 2x}{1 + 2\sqrt{1 - \omega x}}. \quad (4)$$

### 2.2 LS Model

The LS model is a widely-used scattering model (Besse *et al.* 2013), developed by Seeliger (1884) and improved by Hapke (2012). The RADF (Takir *et al.* 2015) of the LS model with four free parameters is shown in Equation (5), where  $f(\alpha) = e^{\beta\alpha + \gamma\alpha^2 + \delta\alpha^3}$  with three polynomial coefficients  $\beta$ ,  $\gamma$  and  $\delta$  that describe the variation in surface reflectance with respect to phase angle  $\alpha$  (in degrees) and  $A_{1s} = \frac{\omega}{4\pi}$  is the LS albedo with the average particle single scattering albedo,  $\omega$ .

Here in both Equation (1) and Equation (5),  $\mu_0$  and  $\mu$  are defined as the projections of the illuminating direction and emitting direction on the normal direction of the surface, respectively.

$$\text{RADF}(i, e, \alpha) = A_{1s} \frac{\mu_0}{\mu_0 + \mu} f(\alpha). \quad (5)$$

**Table 1** Physical Meanings of Five Parameters in the Hapke Model

Parameter	Symbol	Meaning
Single scattering albedo	$\omega$	Fraction of total incident energy that is scattered by a single particle towards all directions
Asymmetry factor	$g$	Spatial energy distribution in a single particle scattering phase function
Opposition surge amplitude	$B_{so}$	Amplitude of opposition effect (SHOE only)
Opposition surge width	$h_s$	Width of opposition effect (SHOE only)
Roughness parameter	$\bar{\theta}$	Average deviation from local normal with respect to average

### 3 SCATTERING LAW ANALYSIS FOR HAPKE AND LS MODELS

Both the Hapke and LS models are explained in the previous section. Intuitively, the LS model has a much simpler form than the Hapke model. In this section, first, the equivalent parameters of the LS model are numerically fitted to the given parameters of the Hapke model. By applying the two equivalent parameters to generate synthetic lightcurves based on a Cellinoid shape model, their similar morphologies of synthetic lightcurves confirm that the LS model can provide the same result as the Hapke model. Then the PCA technique is applied to analyze equivalent parameters of the LS model, derived from the Hapke parameters of C and S type asteroids. Then the result shows that different types of asteroids can be identified from parameters of the LS model.

#### 3.1 Equivalent Parameters

To better simulate the real circumstances, the derived parameters of different scattering models for (101955) Bennu by Takir et al. (2015) are exploited for comparison of the equivalent parameters.

The phase curves of (101955) Bennu for the Hapke model with  $\omega = 0.031$ ,  $B_{so} = 3.9$ ,  $h_s = 0.11$ ,  $g = -0.32$  and  $\bar{\theta} = 20^\circ$  and the fitted LS model are shown in Figure 2. From Figure 2, it is clear that the LS model can fit the Hapke model very well and the opposition effect in the phase curve is also fitted well. The fitted value of the Sum of Squared Errors (SSE), defined as

$$SSE = \sum_i (R_{\text{Hapke}}^i - R_{\text{LS}}^i)^2 \quad (6)$$

is 0.0001312, where  $R_{\text{Hapke}}^i$  and  $R_{\text{LS}}^i$  respectively denote the reflectances of the Hapke and LS models for the corresponding phase angle  $i$ .

Additionally, the fitted equivalent parameters of the LS model are listed in Table 2. Meanwhile, the parameters of the LS model derived by Takir et al. (2015) are also listed for comparison and confirmation.

Furthermore, the parameters of the Hapke model and the fitted equivalent parameters of the LS model listed

**Table 2** Equivalent Parameters from the LS Model

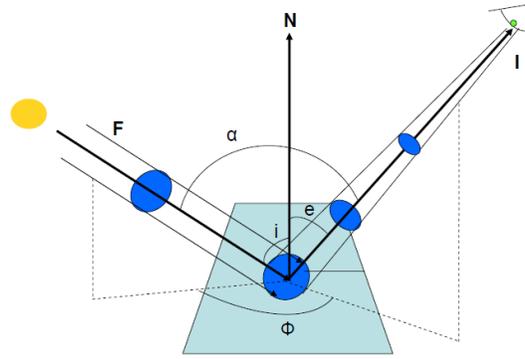
	Our study	Takir et al. (2015)
$A_{1s}$	0.0286	0.030
$\beta$	$-4.20 \times 10^{-2}$	$-4.36 \times 10^{-2}$
$\gamma$	$2.63 \times 10^{-4}$	$2.69 \times 10^{-4}$
$\delta$	$-1.09 \times 10^{-6}$	$-9.90 \times 10^{-7}$

in Table 2 are applied to generate synthetic lightcurves based on a Cellinoid shape model presented by Lu et al. (2017), which are shown in Figure 3. Both of the synthetic lightcurves have a similar morphology with the same positions of maximum and minimum. Although deviations between the maximum and minimum of the two synthetic lightcurves exist, the equivalent parameters of the LS model can fit the Hapke model well as the relative brightness is commonly applied in real applications. The slightly larger amplitude of the LS model than the Hapke model will not affect results of the inverse process (Lu et al. 2017).

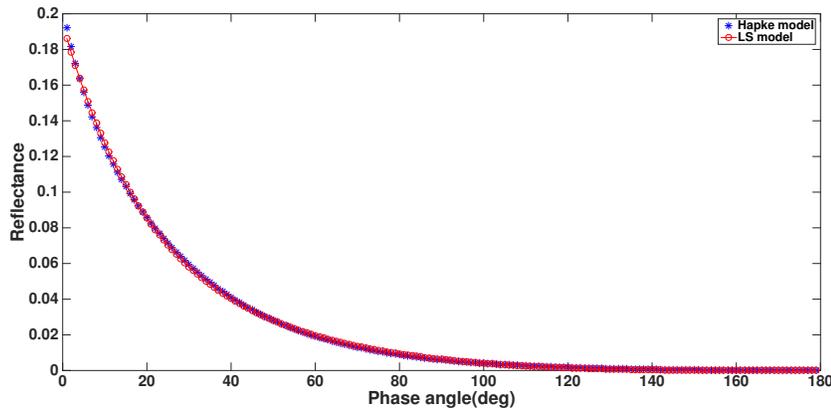
#### 3.2 Asteroidal Taxonomy Based on the LS Model

As the parameters in the Hapke model can reveal information about the surface of asteroids, Helfenstein & Veeverka (1989) introduced the taxonomy of C and S type asteroids according to the five parameters. The mean parameters are listed in Table 3, where the equivalent parameters of the LS model are also listed. The corresponding phase curves for C and S type asteroids with these mean parameters of Hapke and LS models are shown in Figure 4. Apparently, for both types of asteroids, the LS model can fit the Hapke model well in simulating the phase curves.

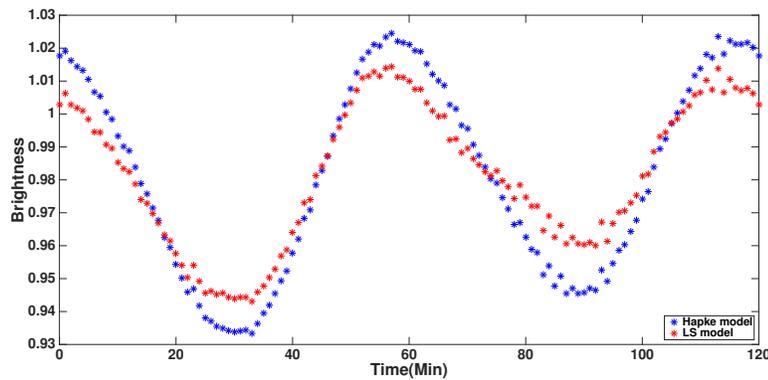
In order to find taxonomic relations in the LS model for the two types of asteroids, we build test sets containing 1200 sample points for each type, which are selected respectively from the C and S type asteroids with the five corresponding Hapke parameters. Subsequently, equivalent parameters of the LS model for two types of asteroids are derived.



**Fig. 1** Schematic diagram illustrating bidirectional reflectance with solar phase angle ( $\alpha$ ), angle of incidence ( $i$ ) and angle of emergence ( $e$ ).



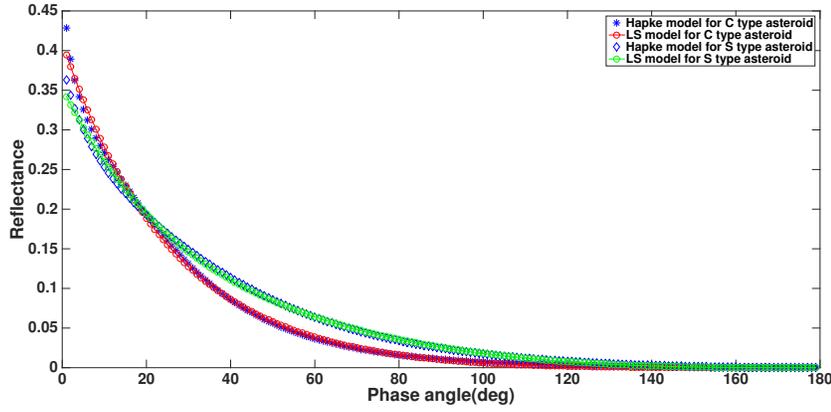
**Fig. 2** Phase curves of (101955) Bennu for the Hapke (shown as *blue* ‘\*’) and LS (shown as *red* ‘o’) models.



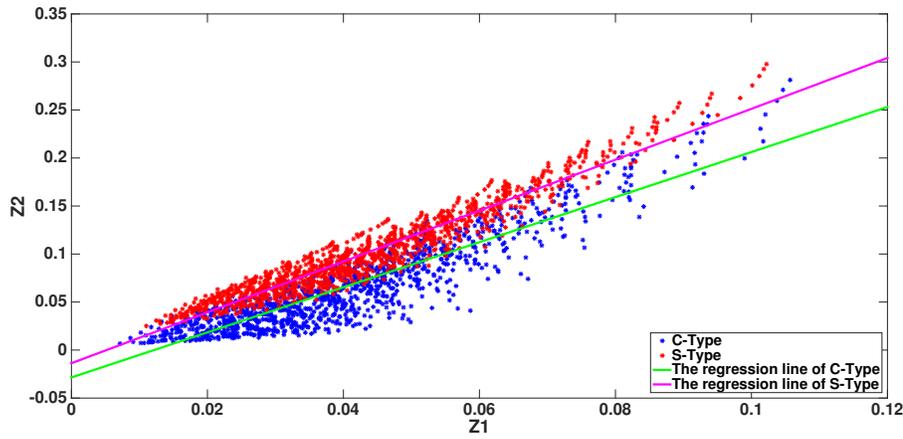
**Fig. 3** Synthetic lightcurves based on a Cellinoid shape with scattering parameters of the Hapke (*blue* ‘\*’) and LS (*red* ‘\*’) models.

Then the PCA technique (Abdi & Williams 2010) is applied to determine the correlation of the four variables in the LS model and the results are presented in Table 4. The cumulative probability of previous variables of  $Z_1$  and  $Z_2$  is more than 95% and their dependence relations on the four variables of the LS model are formulated in

Equation (7) and Equation (8), respectively. Eventually, the distributions of the two principal components for C and S type asteroids are shown in Figure 5 with a clear clustered feature. The regression lines for C and S type asteroids are shown in the same figure and their formulas



**Fig. 4** Phase curves for both C and S type asteroids derived from the Hapke and LS models.



**Fig. 5** Cluster distribution map of C (blue '\*') and S (red '\*') type asteroids. The regression line of C type is represented as a green line and the one of S type is shown as a pink line.

**Table 3** Mean Parameters of Hapke and LS Models for C and S Type Asteroids

		Mean of C type		Mean of S type	
Hapke model	$\omega$	0.037		$\omega$	0.23
	$B_{so}$	1.03		$B_{so}$	1.6
	$h_s$	0.025		$h_s$	0.08
	$g$	-0.47		$g$	-0.27
	$\bar{\theta}$	20°		$\bar{\theta}$	20°
LS model	$A_{1s}$	0.0218		$A_{1s}$	0.0938
	$\beta$	$-3.77 \times 10^{-2}$		$\beta$	$-2.99 \times 10^{-2}$
	$\gamma$	$1.51 \times 10^{-4}$		$\gamma$	$2.12 \times 10^{-4}$
	$\delta$	$-7.68 \times 10^{-7}$		$\delta$	$-9.12 \times 10^{-7}$

are given in Equation (9) and Equation (10), respectively.

$$Z_1 = 0.2461A_{1s} - 0.5197\beta + 0.5921\gamma - 0.5647\delta, \quad (7)$$

$$Z_2 = 0.9366A_{1s} - 0.0773\beta - 0.1798\gamma + 0.2907\delta, \quad (8)$$

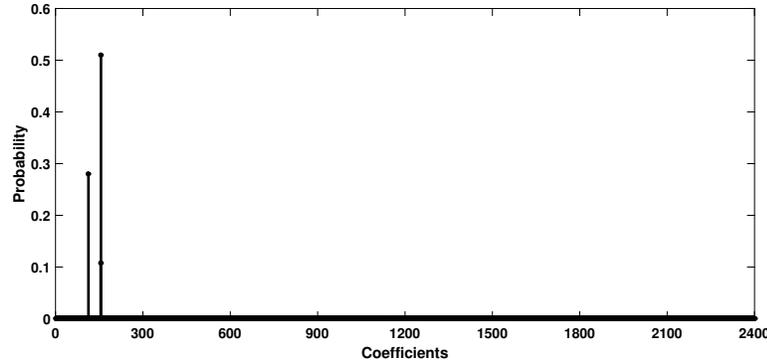
$$Z_2 = 2.3467Z_1 - 0.0285, \quad (9)$$

$$Z_2 = 2.6490Z_1 - 0.0137. \quad (10)$$

The PCA results show that the parameters of the LS model for C and S type asteroids appear clustered, which can be applied in asteroid taxonomy. Following this, we try to find an efficient way to classify the C and S type asteroids based on the LS model parameters in the subsequent subsection.

**Table 4** PCA Results

Principal Component	$Z_1$	$Z_2$	$Z_3$	$Z_4$
Percent Variance (%)	70.74	25.24	3.59	0.43
Cumulative Percent (%)	70.74	95.98	99.57	100

**Fig. 6** Successfully derived indicators for C type asteroids by the LASSO method.

### 3.3 Identifying C and S types Based on the LS Model

As previously described, the principal components of four parameters in the LS model show clear clustering. Here we present an efficient technique to identify the taxonomy of an asteroid from its LS parameters. The LASSO method (Tibshirani 1996; Mei & Ling 2009; Musoro *et al.* 2014) in machine learning is commonly applied to search the best-fit solution with sparse non-zero variables.

Asteroid taxonomy from the LS parameters can be expressed in the LASSO format as a linear system of equations,

$$\mathbf{y} = [\mathbf{T}_c \ \mathbf{T}_s] \begin{bmatrix} \mathbf{d}_c \\ \mathbf{d}_s \end{bmatrix} = \mathbf{T}\mathbf{d}, \quad (11)$$

where the coefficient matrix  $\mathbf{T} = [\mathbf{T}_c, \mathbf{T}_s]$  consists of known LS parameters for both C and S type asteroids. In the context of machine learning, the coefficient matrix  $\mathbf{T}$  is the training set. For any given LS parameters  $\mathbf{y}$ , by solving the linear system (11) the vector  $\mathbf{d} = [\mathbf{d}_c, \mathbf{d}_s]^T$  will indicate the possible asteroid type.

As the LASSO method minimizes the following optimization problem with the  $\ell_1$  regularization term,

$$\mathbf{d}^* = \arg \min_{\mathbf{d}} \frac{1}{2} \|\mathbf{y} - \mathbf{T}\mathbf{d}\|_2^2 + \lambda \|\mathbf{d}\|_1, \quad (12)$$

a probability distribution can be derived to confirm that the asteroid with this group of LS parameters can be classified as C or S type. As a result of the employed  $\ell_1$  regularization, which encourages small components of  $\mathbf{d}$  to become exactly zero, the derived result  $\mathbf{d}^*$  presents

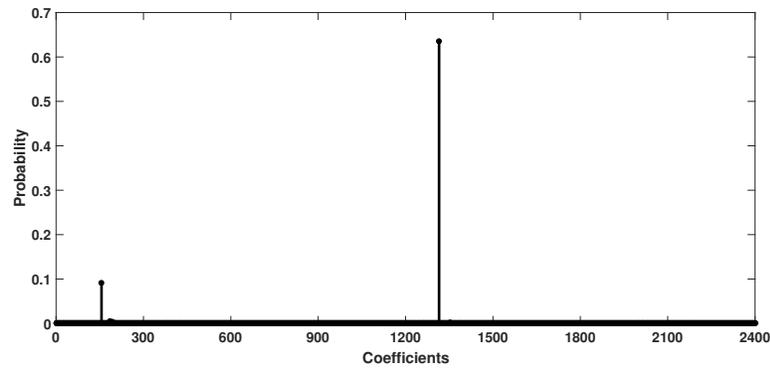
sparse solutions with several non zero components, indicating the asteroid type.

First, the training set including a total of 2400 sample points is built by merging the LS model parameters for known C and S type asteroids with the respective 1200 sample points, where the LS model parameters are calculated equivalently from Hapke model parameters for known C and S type asteroids. Then we also generate a test set containing 200 points for known C and S type asteroids. Finally, by applying the gradient projection method (Figueiredo *et al.* 2007) to solve the problem in Equation (12), the taxonomy for the test set is derived.

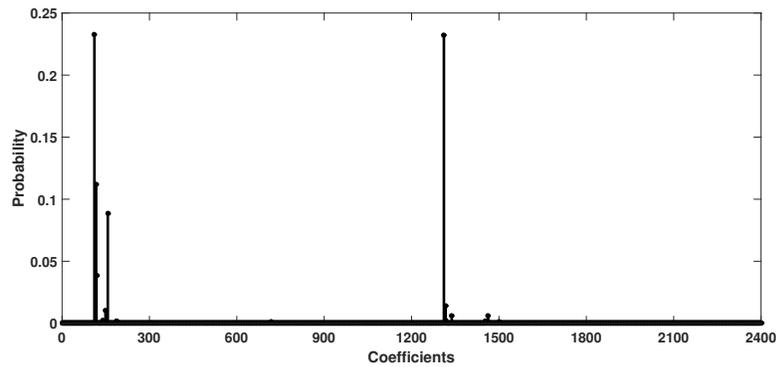
As shown in Figures 6 and 7, the LASSO method can successfully derive the asteroid type. However, there are also some exceptional situations as shown in Figure 8. The probabilities for both C and S type are too close to discriminate them successfully. In all, there are 9 and 13 indeterminate cases for C and S type asteroids, respectively. Compared to the total test set with 200 sample points, the LASSO technique can identify C and S type asteroids from LS model parameters with an approximate probability of 90%. Furthermore, as shown in Figures 6 and 7, this method can also identify the similarity of two asteroids based on their LS model parameters.

## 4 CONCLUSIONS

In this article, we analyze the two scattering models, Hapke and LS. Considering that the Hapke model is deduced with physical meanings and the LS model is efficient in numerical calculation, we numerically compare the two scattering models and confirm that the LS model can fit the Hapke model well in terms of phase curves.



**Fig. 7** Successfully derived indicators for S type asteroids by the LASSO method.



**Fig. 8** Indeterminate indicators for C and S type asteroids by the LASSO method.

Additionally, for the shape determination of asteroids, the LS model can generate similar lightcurves with the same morphology as the Hapke model. Moreover, we primarily investigate two types asteroids, C and S type, classified based on the parameters of the Hapke model. By calculating the equivalent parameters of the LS model, we apply the PCA technique to show clear clustered features of two principal components of the LS model for C and S type asteroids. Moreover, we also introduce the method based on LASSO to classify an asteroid from LS scattering parameters to C or S type. This is very useful in real application. The shape, pole, rotational period, as well as the four LS scattering parameters can be derived from photometric data by an inversion process such as the Kaasalainen inversion method or the Cellinoid inversion process. Then by applying the technique presented in this article, an asteroid can be classified as C or S type in a fast way.

As now we only test C and S type asteroids, we want to explore more general taxonomy of asteroids and try to apply our technique to classify them in the future. We also expect that a new taxonomy method based on the LS model can be developed in the future.

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