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Computable extensions of generalized fractional kinetic equations in astrophysics

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Abstract Fractional calculus and special functions have contributed a lot to mathematical physics and its various branches. The great use of mathematical physics in distinguished astrophysical problems has attracted astronomers and physicists to pay more attention to available mathematical tools that can be widely used in solving several problems of astrophysics/physics. In view of the great importance and usefulness of kinetic equations in certain astrophysical problems, the authors derive a generalized fractional kinetic equation involving the Lorenzo-Hartley function, a generalized function for fractional calculus. The fractional kinetic equation discussed here can be used to investigate a wide class of known (and possibly also new) fractional kinetic equations, hitherto scattered in the literature. A compact and easily computable solution is established in terms of the Lorenzo-Hartley function. Special cases, involving the generalized Mittag-Leffler function and the R-function, are considered. The obtained results imply the known results more precisely.

Key words: fractional differential equations — Mittag-Leffler functions — reaction-diffusion problems — Lorenzo-Hartley function

1 INTRODUCTION

Studies related to the structure and evolution of stars are of great importance in astrophysics as the bulk of matter present in the universe is concentrated in stars. A star (like the Sun) can be taken as a symmetric gas sphere in thermal and hydrostatic equilibrium with negligible rotation and magnetic fields. The star is characterized by its mass, luminosity, effective surface temperature, radius, central density and central temperature. The stellar structures and their mathematical models are investigated on the basis of the above characteristics in addition to some additional information related to the equation of state, nuclear energy generation rate, and opacity. Such stellar models describe how mass, pressure, temperature, and luminosity vary outward from the center of the star. The assumptions of thermal equilibrium and hydrostatic equilibrium indicate that there is no time dependence in the mathematical model, which involve mathematical equations describing the internal structure of the star (Kourganoff 1973; Perdang 1976; Clayton 1983). Energy in such stellar structures is produced by the process of chemical reactions (thermonuclear reactions). Computation of such chemical reactions is of prime importance as it plays a central role in the evolution of such stellar structures. The two most important nuclear reactions (cycles) in stars, during their evolution, are the pp chain (proton-proton chain) and the CNO cycle (involving nuclei of carbon, nitrogen, and oxygen). The total energy production and luminosity of the star is based on the pp chain and the composition of stellar plasma is described by the CNO cycle. The production and destruction of nuclei in such thermonuclear reactions can be described by reaction-type (kinetic) equations. Solutions of such reaction-type (linear/nonlinear) equations determine distribution functions of the dynamical states of a single particle. The linear reaction-type equation, $dy/dx = y$, can be used to describe the fundamental principles of standard Boltzmann-Gibbs statistical mechanics. The nonlinear generalization of the reaction-type equation $dy/dx = y^q$ leads to new insights into generalized Boltzmann-Gibbs statistical mechanics, which is also known as nonextensive statistical mechanics. The generalization of the entropic function of Boltzmann-Gibbs statistical mechanics to nonextensive statistical mechanics leads to a q-exponential distribution. This q-exponential distribution can be reduced into a Gaussian distribution, a Cauchy-Lorentz distribution or into a Lévy distribution, for different values of the parameter q (Tsallis 2002; Gell-Mann & Tsallis 2004). In a recent investigation, Ferro et al. (2004) showed that a very small deviation from the Maxwell-Boltzmann particle distribution and the use of nonextensive statistical mechanics can be applied to describe the modified nuclear reaction rates in stellar plasmas, which is consistent with the need for modification to nuclear reaction rates of stellar plasmas and their chemical composition. Few exact solutions of nonlinear reaction-type (kinetic) equations are known. The solution of a linear kinetic equation, which describes small deviations from the equilibrium solution of the nonlinear kinetic equation, is more thoroughly investigated (see, Kourganoff 1973; Haubold & Mathai 2000).

If an arbitrary reaction is characterized by a time dependent quantity $N = N(t)$ then it is possible to calculate the rate of change of dN/dt by the mathematical equation

$$\frac{dN}{dt} = -d + p, \tag{1}$$

where d is the destruction rate and p is the production rate of N. In general, through feedback or other interaction mechanisms, destruction and production depend on the quantity N itself: $d = d(N)$ or $p = p(N)$, which is a complicated dependence since the destruction or production at time t depends not only on $N(t)$ but also on the past history $N(\tau), \tau < t$, of the variable N. This may be formally given by the following equation (Haubold & Mathai 2000):

$$\frac{dN}{dt} = -d(N_t) + p(N_t), \tag{2}$$

where the function N_t is defined by $N_t(t^*) = N(t - t^*),\ \ t^* > 0$.

Haubold & Mathai (2000) studied a special case of this equation, when spatial fluctuations or inhomogeneities in the quantity $N(t)$ are neglected, given by the equation

$$\frac{dN_i}{dt} = -c_i N_i(t) \tag{3}$$

with the initial condition that $N_i(t = 0) = N_0$ is the number density of species i at time $t = 0$ and constant $c_i > 0$, known as the standard kinetic equation. A detailed discussion of the above equation is given in Kourganoff (1973). The solution of the above standard kinetic equation can be put into the following form:

$$N_i(t) = N_0 e^{-c_i t}. \tag{4}$$

An alternative form of the same equation can be obtained on integration:

$$N(t) - N_0 = c\ {}_0D_t^{-1} N(t), \tag{5}$$

where ${}_0D_t^{-1}$ is the standard integral operator. Also, the fractional generalization of the above standard kinetic Equation (5) is given by Haubold & Mathai (2000), in the following form:

$$N(t) - N_0 = c^{\nu} \ {}_0D_t^{-\nu} N(t), \tag{6}$$

where ${}_0D_t^{-\nu}$ is the well-known standard Riemann-Liouville fractional integral operator (Oldham & Spanier 1974; Samko et al. 1993; Miller & Ross 1993) defined by

$${}_0D_t^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1} f(u) du, \Re(\nu) > 0 \tag{7}$$

with ${}_0D_t^0 f(t) = f(t)$. In particular, when $\nu = 1$ the above fractional integral operator ${}_0D_t^{-\nu}$ reduces to the standard integral operator ${}_0D_t^{-1}$.

The solution of the above standard fractional kinetic Equation (6) in the computable series representation is given by (see Haubold & Mathai 2000)

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(\nu k + 1)} (ct)^{\nu k}, \tag{8}$$

which involves the reaction rate (thermonuclear function). The exponential solution (4) of the standard kinetic equation can be obtained by putting $\nu = 1$ in the above series solution.

To further investigate possible modifications of the reaction rate through a kinetic equation, Saxena, Mathai & Haubold (2002) studied three generalizations of the fractional kinetic equation in terms of the Mittag-Leffler functions which extended the work of Haubold & Mathai (2000). In another paper, Saxena, Mathai & Haubold (2004a) developed solutions for generalized fractional kinetic equations associated with generalized Mittag-Leffler functions and the R-function. Further, some general solutions for generalized fractional kinetic equations have also been investigated by Saxena, Mathai & Haubold (2004b, 2007).

The fractional kinetic equation is also discussed by Zaslavsky (1994) to describe particle kinetics in the case of incomplete Hamiltonian chaos. The Lévy process and Lévy flights (Lévy 1937; Kahane 1995; Montroll & Shlesinger 1984), as well as fractal time and fractal Brownian motion (Shlesinger 1989) are some of the limiting cases of the fractional kinetic equation. Saichev & Zaslavsky (1997) explored the possible generalization of the normal diffusion equation with help given by the formulation of a fractional kinetic equation.

In a recent paper, Lorenzo & Hartley (1999) investigated two generalized functions, the R-function and the Lorenzo-Hartley function (denoted by the $G_{\nu,\mu,\delta}$ function), and determined the time domain dynamic response of a thermocouple consisting of two dissimilar metals with a common junction point in terms of the R-function. The F-function (Hartley & Lorenzo 1998), the Mittag-Leffler function (Mittag-Leffler 1903, 1905), the new generalized Mittag-Leffler function (Prabhakar 1971), the R-function, and the Lorenzo-Hartley function (Lorenzo & Hartley 1999) are some generalized functions for fractional calculus. Such functions provide direct solutions and understanding for fundamental linear fractional order differential equations and related initial value problems. The R-function and the Lorenzo-Hartley function may have some other useful applications in analysis.

In a recent article (Chaurasia & Pandey 2008), we have established a computable fractional generalization of the fractional kinetic equation and derived a solution for the same, which can be used to compute (particle reaction rate) the change of chemical composition in stars like the Sun. Such a fractional kinetic equation consists of a large number of kinetic equations as its special cases. In the present article, we introduce and investigate further computable extensions of the generalized fractional kinetic equation. The fractional kinetic equation and its solution, discussed in terms of the Lorenzo-Hartley function, are written in a compact and easily computable form. The paper is organized as follows. Section 2 contains the definition of the Lorenzo-Hartley function and its relationships with some other functions. Some of the differintegral properties of the Lorenzo-Hartley function are analyzed in Section 3. Extensions of the generalized fractional kinetic equation and their mathematical solutions are studied in Section 4 while some special cases are established in subsection 4.1. Finally, some concluding remarks and observations are given in Section 5.

2 THE LORENZO-HARTLEY FUNCTION AND ITS RELATIONSHIPS WITH SOME OTHER FUNCTIONS

The Lorenzo-Hartley function $G_{\nu,\mu,\delta}(a,c,t)$ was introduced by Lorenzo & Hartley (1999), and is defined as

$$G_{\nu,\mu,\delta}(a,c,t) = \sum_{k=0}^{\infty} \frac{(\delta)_k a^k (t-c)^{(k+\delta)\nu-\mu-1}}{k!\Gamma((k+\delta)\nu-\mu)},$$

$$\Re(\nu\delta-\mu) > 0, \tag{9}$$

where $(\delta)_k$ is Pochhammer's symbol defined by

$$(\delta)_k = \begin{cases} 1, & k=0 \\ \delta(\delta+1)...(\delta+k-1), & \delta \neq 0, k \in \mathcal{N}, \end{cases} \tag{10}$$

where $\mathcal{N}$ is a set of natural numbers.

In particular, at $c = 0$, the above Lorenzo-Hartley function reduces to the following form

$$G_{\nu,\mu,\delta}(a,0,t) = G_{\nu,\mu,\delta}(a,t)$$

$$= \sum_{k=0}^{\infty} \frac{(\delta)_k a^k (t)^{(k+\delta)\nu-\mu-1}}{k!\Gamma((k+\delta)\nu-\mu)},$$

$$\Re(\nu\delta-\mu) > 0. \tag{11}$$

The Lorenzo-Hartley function yields the following relationships with various classical special functions:

Mittag-Leffler function (Mittag-Leffler 1903, 1905)

$$G_{\nu,(\nu-1),1}(-a,t) = E_\nu[-at^\nu] = \sum_{k=0}^{\infty} \frac{(-a)^k t^{k\nu}}{\Gamma(k\nu+1)}. \tag{12}$$

Agarwal's function (Agarwal 1953)

$$G_{\nu,(\nu-\mu),1}(1,t) = E_{\nu,\mu}[t^\nu] = \sum_{k=0}^{\infty} \frac{t^{k\nu-1+\mu}}{\Gamma(k\nu+\mu)}. \tag{13}$$

Erdélyi's function

$$G_{\nu,(\nu-\mu),1}(1,t) = t^{\mu-1} E_{\nu,\mu}[t^\nu] = t^{\mu-1} \sum_{k=0}^{\infty} \frac{t^{k\nu}}{\Gamma(k\nu+\mu)}, \tag{14}$$

where $E_{\nu,\mu}[t^\nu]$ is the well known Erdélyi's (Erdélyi, Magnus & Oberhettinger 1954) function defined as

$$E_{\nu,\mu}[t^{\nu}] = \sum_{k=0}^{\infty} \frac{t^{k\nu}}{\Gamma(k\nu+\mu)}, \quad \nu > 0, \quad \mu > 0. \tag{15}$$

Robotnov & Hartley function (Hartley & Lorenzo 1998)

$$G_{\nu,0,1}(-a,t) = F_{\nu}[-a,t] = \sum_{k=0}^{\infty} \frac{(-a)^k t^{(k+1)\nu-1}}{\Gamma((k+1)\nu)}. \tag{16}$$

Miller & Ross's function (Miller & Ross 1993)

$$G_{1,-\mu,1}(a,t) = E_t(\mu,a) = \sum_{k=0}^{\infty} \frac{a^k t^{(k+\mu)}}{\Gamma(k+\mu+1)}. \tag{17}$$

Generalized Mittag-Leffler function (Wiman 1905)

$$\begin{aligned} G_{\nu,\mu,1}(a,t) &= t^{\nu-\mu-1} E_{\nu,\nu-\mu}[at^{\nu}] \\ &= \sum_{k=0}^{\infty} \frac{a^k t^{(k+1)\nu-\mu-1}}{\Gamma((k+1)\nu-\mu)}, \end{aligned} \tag{18}$$

where $E_{\nu,(\nu-\mu)}[at^{\nu}]$ is the well known generalized Mittag-Leffler function (Wiman 1905) defined as

$$E_{\nu,\mu}[t] = \sum_{k=0}^{\infty} \frac{t^k}{\Gamma(k\nu+\mu)}, \quad \nu > 0, \quad \mu > 0. \tag{19}$$

New generalized Mittag-Leffler function

$$\begin{aligned} G_{\nu,\mu,\delta}(a,t) &= t^{\nu\delta-\mu-1} E^{\delta}{}_{\nu,(\nu\delta-\mu)}[at^{\nu}] \\ &= t^{\nu\delta-\mu-1} \sum_{k=0}^{\infty} \frac{(\delta)_k a^k t^{\nu k}}{k!\Gamma((k+\delta)\nu-\mu)}, \end{aligned} \tag{20}$$

where $E^{\delta}{}_{\nu,(\nu\delta-\mu)}[at^{\nu}]$ is another generalized form of the following Mittag-Leffler function introduced by Prabhakar (1971):

$$E^{\delta}{}_{\nu,\mu}[t] = \sum_{k=0}^{\infty} \frac{(\delta)_k t^k}{k!\Gamma(k\nu+\mu)}, \quad \nu > 0, \quad \mu > 0, \quad \delta > 0. \tag{21}$$

R-function (Lorenzo & Hartley 1999)

$$\begin{aligned} &G_{\nu,\mu,1}(a,t) = R_{\nu,\mu}(a,t) = \sum_{k=0}^{\infty} \frac{a^k t^{(k+1)\nu-\mu-1}}{\Gamma((k+1)\nu-\mu)}, \\ &\nu > 0, \quad \mu > 0, \quad (\nu-\mu) > 0. \end{aligned} \tag{22}$$

3 SOME DIFFERINTEGRAL PROPERTIES OF THE LORENZO-HARTLEY FUNCTION

Some differintegral properties of the Lorenzo-Hartley function are as follows:

Eigen-property: The Lorenzo-Hartley function has an eigenfunction character under ν $^{\text{th}}$ order differintegration with $\mu = 0$ and $\delta = 1$.
The ν^{th} order differintegration (denoted by ${}_c d_t^{\nu}$) is given by (see Oldham & Spainer 1974)

$$ {}_c d_t^{\nu} [t-c]^p = \frac{\Gamma(p+1)[t-c]^{p-\nu}}{\Gamma(p-\nu+1)}, \quad p > -1. \tag{23}$$

The well-known standard Riemann-Liouville integral operator (Oldham & Spanier 1974) ${}_0 D_t^{-\nu}$ is a special case of the same with $c = 0$.

Now ν^{th} order differintegration of the Lorenzo-Hartley function, at $\mu = 0$ and $\delta = 1$, is given by the use of the above definition (23)
i.e.

$$ {}_c d_t^{\nu} G_{\nu,0,1}(a,c,t) = \sum_{k=0}^{\infty} \frac{(a)^k (1)_k {}_c d_t^{\nu} (t-c)^{(k+1)\nu-1}}{k!\Gamma((k+1)\nu)}, \tag{24}$$

or

$$ {}_c d_t^{\nu} G_{\nu,0,1}(a,c,t) = \sum_{k=1}^{\infty} \frac{(a)^k (t-c)^{k\nu-1}}{\Gamma(k\nu)} + \lim_{k\to 0} \frac{(a)^k (t-c)^{k\nu-1}}{\Gamma(k\nu)}, \quad \nu > 0. \tag{25}$$

Now let $k = m + 1$, then

$$ {}_c d_t^{\nu} G_{\nu,0,1}(a,c,t) = \sum_{m=0}^{\infty} \frac{(a)^{(m+1)} (t-c)^{(m+1)\nu-1}}{\Gamma((m+1)\nu)} + \lim_{m\to -1} \frac{(a)^{(m+1)} (t-c)^{(m+1)\nu-1}}{\Gamma((m+1)\nu)}, \quad \nu > 0 \tag{26}$$

or

$$ {}_c d_t^{\nu} G_{\nu,0,1}(a,c,t) = a\ G_{\nu,0,1}(a,c,t) + a \lim_{m\to -1} \frac{(a)^{m} (t-c)^{(m+1)\nu-1}}{\Gamma((m+1)\nu)}, \quad \nu > 0. \tag{27}$$

The term on the extreme right in the above Equation (27) will be zero for $t \neq c$; thus, for $t > c$, we have

$$ {}_c d_t^{\nu} G_{\nu,0,1}(a,c,t) = aG_{\nu,0,1}(a,c,t), \quad \nu > 0. \tag{28}$$

If we take $a = 1$ in the above Equation (28), the Lorenzo-Hartley function returns to itself under ν^{th} order differintegration, i.e.

$$ {}_c d_t^{\nu} G_{\nu,0,1}(1,c,t) = G_{\nu,0,1}(1,c,t), t > c, \nu > 0. \tag{29}$$

Differintegration of the Lorenzo-Hartley function: Considering the u $^{\text{th}}$ order (i.e. any arbitrary order) *differintegration* (denoted by ${}_c d_t^{u}$ and defined in the same manner as in (23) when ν is replaced by u) of the Lorenzo-Hartley function

$$ {}_c d_t^{u} G_{\nu,\mu,\delta}(a,c,t) = {}_c d_t^{u} \Big[\sum_{k=0}^{\infty} \frac{(\delta)_k (a)^k (t-c)^{(k+\delta)\nu-\mu-1}}{k!\Gamma((k+\delta)\nu-\mu)} \Big], \Re(\nu\delta - \mu) > 0 \tag{30}$$

or

$$ {}_c d_t^u G_{\nu,\mu,\delta}(a,c,t) = \sum_{k=0}^{\infty} \frac{(\delta)_k (a)^k {}_c d_t^u (t-c)^{(k+\delta)\nu-\mu-1}}{k!\Gamma((k+\delta)\nu-\mu)}, \Re(\nu\delta-\mu) > 0. \tag{31} $$

On applying the known result (23) to the above Equation (31), we obtain

$$ {}_c d_t^u G_{\nu,\mu,\delta}(a,c,t) = \sum_{k=0}^{\infty} \frac{(\delta)_k (a)^k (t-c)^{(k+\delta)\nu-(\mu+u)-1}}{k!\Gamma((k+\delta)\nu-(\mu+u))}, \tag{32} $$

or

$$ {}_c d_t^u G_{\nu,\mu,\delta}(a,c,t) = G_{\nu,(\mu+u),\delta}(a,c,t), \Re(\nu\delta-\mu-u) > 0. \tag{33} $$

This shows that any arbitrary order (u^{th} order) *differintegration* of the Lorenzo-Hartley function returns another Lorenzo-Hartley function .

4 EXTENSIONS OF GENERALIZED FRACTIONAL KINETIC EQUATIONS

Theorem 1 If $c > 0, b \geq 0, \delta > 0, \nu > 0, \mu > 0$, and $(\delta\nu - \mu) > 0$, then for the solution of

$$ N(t) - N_0 G_{\nu,\mu,\delta}(-c^\nu, b, t) = -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu} {}_0 D_t^{-r\nu} N(t), \tag{34} $$

where ${}_0 D_t^{-r\nu}, r \in \mathcal{N}$ is the well-known standard Riemann-Liouville integral operator (Oldham & Spanier 1974; Samkoet al. 1993; Miller & Ross 1993) defined by (7). There holds the formula

$$ N(t) = N_0 G_{\nu,(\mu+\nu n),(\delta+n)}(-c^\nu, b, t), \tag{35} $$

provided that each member of (35) exists.

Proof Taking a Laplace transform of both sides of Equation (34), we have

$$ L\{N(t);p\} - L\{N_0 G_{\nu,\mu,\delta}(-c^\nu,b,t);p\} = L\Big\{ -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu} {}_0 D_t^{-r\nu} N(t); p \Big\}. \tag{36} $$

By the application of the convolution theorem of the Laplace transform (Erdélyi et al. 1954) in Equation (7), we have

$$ L\{{}_0 D_t^{-\nu} f(t); p\} = L\Big\{ \frac{t^{\nu-1}}{\Gamma(\nu)} \Big\} L\{f(t)\} = p^{-\nu} f^{\sim}(p), \tag{37} $$

where $f^{\sim}(p) = \int_0^\infty e^{-pt} f(t) dt, p \in C, \Re(p) > 0$. Now using (37) in (36) and by the use of a Laplace transform of the Lorenzo-Hartley function (Lorenzo & Hartley 1999) with the shifting theorem for the Laplace transform (Wylie 1975), we obtain

$$ N(p) - N_0 \frac{p^\mu e^{-bp}}{(p^\nu + c^\nu)^\delta} = -[{}^n C_1 c^\nu p^{-\nu} + {}^n C_2 c^{2\nu} p^{-2\nu} + \cdots + {}^n C_n c^{n\nu} p^{-n\nu}] N(p) \tag{38} $$

or

$$ N(p) = N_0 \frac{p^{\mu-\nu\delta} e^{-bp}}{(1 + c^\nu p^{-\nu})^{\delta+n}}. \tag{39} $$

Now, taking the inverse Laplace transform of both the sides of Equation (39), we obtain

$$N(t) = L^{-1}[N(p)] = L^{-1}\Big[N_0 \frac{p^{\mu-\nu(\delta+n)+\nu n} e^{-bp}}{(1+c^\nu p^{-\nu})^{\delta+n}}\Big] \tag{40}$$

$$= N_0 L^{-1}\Big[e^{-bp} \sum_{k=0}^{\infty} (\frac{-c^\nu}{p^\nu})^k \frac{(\delta+n)_k p^{\mu-\nu\delta}}{k!}\Big] \tag{41}$$

$$= N_0 \sum_{k=0}^{\infty} \frac{(\delta+n)_k}{k!} (-c^\nu)^k L^{-1}\Big[e^{-bp} p^{\mu-\nu\delta-\nu k}\Big] \tag{42}$$

$$= N_0 (t-b)^{\nu\delta-\mu-1} \sum_{k=0}^{\infty} \frac{(\delta+n)_k}{k!} (-c^\nu)^k \frac{(t-b)^{k\nu}}{\Gamma((k+\delta)\nu-\mu)}, \tag{43}$$

or

$$N(t) = N_0 G_{\nu,(\mu+\nu n),(\delta+n)}(-c^\nu, b, t),$$

which is Equation (35). This completes the proof of Theorem 1.

The following corollaries can be deduced from Theorem 1.

Corollary 1.1 If $c > 0, b = 0, \delta > 0, \nu > 0, \mu > 0$, and $(\delta\nu - \mu) > 0$, then for the solution of

$$N(t) - N_0 G_{\nu,\mu,\delta}(-c^\nu, 0, t) = -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu} {}_0D_t^{-r\nu} N(t), \tag{44}$$

there holds the formula

$$N(t) = N_0 G_{\nu,(\mu+\nu n),(\delta+n)}(-c^\nu, 0, t), \tag{45}$$

provided that each side of (45) exists.

Corollary 1.2 If $c > 0, b \geq 0, \delta > 0, \nu > 0, \mu > 0$, and $(\delta\nu - \mu) > 0$, then for the solution of

$$N(t) - N_0 G_{\nu,(\nu\delta-\mu),\delta}(-c^\nu, b, t) = -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu} {}_0D_t^{-r\nu} N(t), \tag{46}$$

there holds the formula

$$N(t) = N_0 G_{\nu,[(\delta+n)\nu-\mu],(\delta+n)}(-c^\nu, b, t), \tag{47}$$

provided that each side of (47) exists.

Corollary 1.3 If $c > 0, b = 0, \delta > 0, \nu > 0, \mu > 0$, and $(\delta\nu - \mu) > 0$, then for the solution of

$$N(t) - N_0 G_{\nu,(\nu\delta-\mu),\delta}(-c^\nu, 0, t) = -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu} {}_0D_t^{-r\nu} N(t), \tag{48}$$

there holds the formula

$$N(t) = N_0 G_{\nu,[(\delta+n)\nu-\mu],(\delta+n)}(-c^\nu, 0, t), \tag{49}$$

provided that each side of (49) exists.

Corollary 1.4 If $c > 0, b \geq 0, \delta > 0, \nu > 0, \mu > 0$, and $(\nu - \mu) > 0$, then for the solution of

$$N(t) - N_0 G_{\nu,\mu,1}(-c^\nu, b, t) = -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu}\, {}_0D_t^{-r\nu} N(t), \tag{50}$$

there holds the formula

$$N(t) = N_0 G_{\nu,(\mu+\nu n),(1+n)}(-c^\nu, b, t), \tag{51}$$

provided that each side of (51) exists.

4.1 Special Cases

A number of known results (Saxena, Mathai & Haubold 2002, 2004a, 2007; Chaurasia & Pandey 2008) can be obtained as some of the special cases of Theorem 1. For $n = 1$, Theorem 1 reduces to the following known result recently given by Chaurasia & Pandey (2008).

Corollary 1.5 If $c > 0, b \geq 0, \delta > 0, \nu > 0, \mu > 0$, and $(\delta\nu - \mu) > 0$ then for the solution of the equation

$$N(t) - N_0 G_{\nu,\mu,\delta}(-c^\nu, b, t) = -c^\nu\, {}_0D_t^{-\nu} N(t), \tag{52}$$

there holds the formula

$$N(t) = N_0 G_{\nu,(\nu+\mu),(\delta+1)}(-c^\nu, b, t), \tag{53}$$

provided that each side of (53) exists.

On interpreting the Lorenzo-Hartley function involved in Equations (48) and (49) as the new generalization of the Mittag-Leffler function (Prabhakar 1971), we find the following corollary.

Corollary 1.6 If $c > 0, \delta > 0, \nu > 0, \mu > 0$, and $(\nu\delta - \mu) > 0$, then for the solution of the equation

$$N(t) - N_0 t^{\mu-1} E_{\nu,\mu}^{\delta}(-c^\nu t^\nu) = -\sum_{r=1}^{n} \binom{n}{r} c^{r\nu}\, {}_0D_t^{-r\nu} N(t), n \in \mathcal{N}, \tag{54}$$

there holds the formula

$$N(t) = N_0 t^{\mu-1} E_{\nu,\mu}^{\delta+n}(-c^\nu t^\nu) \qquad n \in \mathcal{N}, \tag{55}$$

which is the known result recently given by Saxena, Mathai & Haubold (2007). In particular, if we take $n = 1$ in the above Equations (54) and (55), we obtain the following known result given by Saxena, Mathai & Haubold (2004a).

If $c > 0, \delta > 0, \nu > 0$, and $\mu > 0$, then for the solution of the equation

$$N(t) - N_0 t^{\mu-1} E_{\nu,\mu}^{\delta}(-c^\nu t^\nu) = -c^\nu\, {}_0D_t^{-\nu} N(t), \tag{56}$$

there holds the formula

$$N(t) = N_0 t^{\mu-1} E_{\nu,\mu}^{\delta+1}(-c^\nu t^\nu). \tag{57}$$

Moreover, on taking $\delta = 1$ in the above Equations (56) and (57), we find another known result given by Saxena, Mathai & Haubold (2002).

Also, on interpreting the Lorenzo-Hartley function involved in the Equation (50) as the R-function and taking $n = 1$, we obtain the following known result given by Saxena, Mathai & Haubold (2004a).

Corollary 1.7 If $c > 0, b \geq 0, \nu > 0, \mu > 0$, and $\nu > (\mu + 1)$, then for the solution of the equation

$$N(t) - N_0 R_{\nu,\mu}(-c^\nu, b, t) = -c^\nu {}_0D_t^{-\nu} N(t), \tag{58}$$

there holds the formula

$$N(t) = \frac{N_0}{\nu}(t-b)^{\nu-\mu-1}[E_{\nu,(\nu-\mu-1)}(-c^\nu(t-b)^\nu) + (\mu+1)E_{\nu,(\nu-\mu)}(-c^\nu(t-b)^\nu)]. \tag{59}$$

In particular, on putting b=0 in the above Equations (58) and (59), we arrive at the following known result given by Saxena, Mathai & Haubold (2004a).

If $c > 0, b = 0, \nu > 0, \mu > 0$, and $\nu > (\mu + 1)$, then for the solution of the equation

$$N(t) - N_0 R_{\nu,\mu}(-c^\nu, 0, t) = -c^\nu {}_0D_t^{-\nu} N(t), \tag{60}$$

there holds the formula

$$N(t) = \frac{N_0}{\nu} t^{\nu-\mu-1}[E_{\nu,(\nu-\mu-1)}(-c^\nu t^\nu) + (\mu+1)E_{\nu,(\nu-\mu)}(-c^\nu t^\nu)]. \tag{61}$$

5 CONCLUDING REMARKS AND OBSERVATIONS

The deviation of the particle distribution function from the Maxwell-Boltzmann distribution implies the need for relevant modification in the computation of the particle reaction rate for evolution of stars and galaxies. Kinetic equations are used in computation of chemical changes in stars (like the Sun). Haubold & Mathai (2000) investigated the standard kinetic equation which is suitable for incorporating changes in the Maxwell-Boltzmann distribution function. In their following work, to provide analytical techniques and further investigation of the possible modification of thermonuclear reaction rates, Saxena, Mathai & Haubold (2002, 2004a,b, 2007) investigated some other fractional generalizations of the standard kinetic equation which contain the thermonuclear function as a time constant.

In this article, we have introduced an extended fractional generalization of the standard kinetic equation and established a solution for the same. Fractional kinetic equations can be used to compute the particle reaction rate and describe the statistical mechanics associated with the particle distribution function. The fractional kinetic equation discussed in this article involves the Lorenzo-Hartley function $G_{\nu,\mu,\delta}(-c^\nu, b, t)$, the generalized function for fractional calculus. The Lorenzo-Hartley function has close relationships with the R-function, the generalized Mittag-Leffler functions, the Mittag-Leffler function, the Robotnov & Hartley function, etc. It can be seen that the fractional kinetic equation, discussed in this article, contains a number of the known (and possibly also new) fractional kinetic equations involving various other special functions (the R-function, the generalized Mittag-Leffler functions, the Mittag-Leffler function, the Robotnov & Hartley function, etc.). Several further fractional kinetic equations and their solutions analogous to those exhibited here by Theorem 1 and its Corollaries (1.1), (1.2), (1.3), and (1.4), involving the generalized function $G_{\nu,\mu,\delta}(-c^\nu, b, t)$, can be determined by specifying the parameters ν, μ, δ and b. It is interesting to note that the solutions of all the fractional kinetic equations discussed here are in the series forms of the Lorenzo-Hartley function, which can be considered as a compact and elegant expression used for the computation.

The results obtained in this paper have been achieved by application of the theory of the generalized function for fractional calculus, particularly the Lorenzo-Hartley function, and by the use of Laplace transform techniques.

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