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Analysis of Apsidal Motion and Light Curves of Three Eccentric Eclipsing Binaries: V398 Lac, V2544 Cyg and V785 Cas

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Abstract

In this study, the apsidal motion analyses of the eclipsing binary systems V398 Lac, V2544 Cyg, and V785 Cas are presented. The $(O-C)$ diagrams of the selected systems were constructed using all available and reliable times of minimum light compiled from the literature. As a result, the apsidal motion periods were determined to be 353 ± 50 yr, 43.3 ± 5.1 yr and 83.4 ± 8.5 yr, respectively. Furthermore, the TESS light curves of the three systems were analyzed to derive their photometric and absolute parameters. The component masses were determined as follows: for V398 Lac, $M_1 = 3.83 \pm 0.35\,M_\odot$ and $M_2 = 3.27 \pm 0.35\ M_\odot$; for V2544 Cyg, $M_1 = 1.75 \pm 0.38\,M_\odot$ and $M_2 = 1.40 \pm 0.38\,M_\odot$; for V785 Cas, $M_1 = 5.64 \pm 0.41\,M_\odot$ and $M_2 = 5.18 \pm 0.41\,M_\odot$. The parameters related to apsidal motion, as well as the observational and theoretical internal structure constants ($\bar{k}_{2,\mathrm{obs}}$ and $\bar{k}_{2,\mathrm{teo}}$), were calculated for studied binaries. The relativistic contribution to the observed apsidal motion rate was found to be relatively small, constituting less than approximately 5% of the total rate in each system. The theoretical internal structure constants were derived from the evolutionary models, assuming a standard chemical composition of $(X, Z) = (0.70, 0.01)$.

Key words: Stars – (stars:) binaries: eclipsing – stars: individual (V398 Lac, V2544 Cyg, V785 Cas)

1. Introduction

Eclipsing binary stars showing apsidal motion occupy a crucial position in stellar astrophysics, as their dynamic behavior provides unique insight into the internal mass distribution and structural properties of stars. Apsidal motion-the gradual rotation of the line of apsides in an elliptical orbit-is sensitive to the internal density concentration of the component stars and can be influenced by both classical (e.g., tidal and rotational distortions) and relativistic effects. Therefore, systems exhibiting apsidal motion serve as natural laboratories for testing stellar structure models and refining theoretical predictions of stellar evolution (e.g., Bulut et al. 2017; Claret et al. 2021). Furthermore, apsidal motion in the binary systems offers a valuable opportunity to test the predictions of General Relativity and alternative gravitational theories under strong-field conditions, beyond the solar system. Over the past decades, numerous studies have focused on comparing observed apsidal motion rates with theoretical values that include both classical contributions and general relativistic corrections (e.g., Gimenez 1985; Wolf et al. 2010; Baroch et al. 2021).

The primary method used to analyze apsidal motion is the $(O-C)$ diagram, which tracks the difference between observed and calculated eclipse timings. This technique allows the detection of long-term deviations from a linear ephemeris, revealing the presence and rate of apsidal advance. The accurate determination of the apsidal motion period, which covers tens or even hundreds of years, depends on the minimum time observations made on long time scales.

In this study, light curve and period variation analyses were performed for the eclipsing binary systems V398 Lac, V2544 Cyg, and V785 Cas, all of which exhibit eccentric orbital configurations. The light curves were extracted from photometric observations provided by the Transiting Exoplanet Survey Satellite (TESS). New times of minimum light, derived from TESS data using standard methods, were combined with previously published minima from the literature to conduct $(O-C)$ analyses, through which the apsidal motion parameters were determined. These systems were initially identified as eclipsing binary stars by the Hipparcos satellite ESA (1997). Their relatively short apsidal motion periods make them excellent candidates for detailed studies of orbital evolution and internal stellar structure.

The system V398 Lac (HD 210180 = HIP 109193 = BD + 51 3251; $V = 8.79$ mag) is a totally eclipsing binary system with an eccentric orbit. It was discovered by the Hipparcos satellite that the system is an eclipsing binary. Its orbital period is 5.4605 days. The system was studied using Hipparcos photometry by Bulut & Demircan (2008), who obtained initial results with the EBOP light curve analysis tool. They obtained

Table 1
New Minimum Times

Star	BJD	Error (day)	Type	Filter	Star	BJD	Error (day)	Type	Filter
V398 Lac	2453599.39247[a]	0.00043	P	*BVR*	V2544 Cyg	2460340.44362	0.00022	S	TESS
	2453666.36617[a]	0.00084	S	*BVR*	⋯	2460341.51818	0.00039	P	TESS
	2460586.21645	0.00007	S	TESS	⋯	2460342.53791	0.00041	S	TESS
	2460589.31242	0.00019	P	TESS	⋯	2460343.61153	0.00022	P	TESS
	2460591.62295	0.00009	S	TESS	⋯	2460344.63121	0.00020	S	TESS
	2460594.71860	0.00022	P	TESS	⋯	2460345.70572	0.00025	P	TESS
	2460597.02714	0.00124	S	TESS	⋯	2460346.72507	0.00015	S	TESS
	2460600.12437	0.00019	P	TESS	⋯	2460347.79928	0.00012	P	TESS
	2460602.43476	0.00013	S	TESS	⋯	2460348.81886	0.00020	S	TESS
	2460605.53051	0.00015	P	TESS	⋯	2460349.89276	0.00013	P	TESS
	2460607.84081	0.00031	S	TESS	⋯	2460350.91212	0.00028	S	TESS
V785 Cas	2459882.68936	0.00011	S	TESS	⋯	2460351.98672	0.00012	P	TESS
	2459883.98754	0.00006	P	TESS	⋯	2460353.00592	0.00028	S	TESS
	2459885.39250	0.00008	S	TESS	⋯	2460354.08079	0.00028	P	TESS
	2459886.69026	0.00004	P	TESS	⋯	2460355.09966	0.00036	S	TESS
	2459888.09463	0.00009	S	TESS	⋯	2460356.17469	0.00035	P	TESS
	2459889.39271	0.00013	P	TESS	⋯	2460357.19390	0.00037	S	TESS
	2459890.79779	0.00009	S	TESS	⋯	2460358.26811	0.00030	P	TESS
	2459892.09512	0.00005	P	TESS	⋯	2460359.28659	0.00043	S	TESS
	2459893.49905	0.00015	S	TESS	⋯	2460360.36253	0.00027	P	TESS
	2459894.79794	0.00007	P	TESS	⋯	2460361.38055	0.00022	S	TESS
	2459897.50027	0.00004	S	TESS	⋯	2460362.45610	0.00015	P	TESS
	2459898.90411	0.00011	P	TESS	⋯	2460363.47470	0.00024	S	TESS
	2459900.20249	0.00005	S	TESS	⋯	2460364.54959	0.00014	P	TESS
	2459901.60819	0.00014	P	TESS	⋯	2460365.56809	0.00023	S	TESS
	2459902.90525	0.00006	S	TESS	⋯	2460366.64363	0.00015	P	TESS
	2459904.30961	0.00006	P	TESS	⋯	⋯	⋯	⋯	⋯
	2459905.60777	0.00005	S	TESS	⋯	⋯	⋯	⋯	⋯
	2459907.01253	0.00013	P	TESS	⋯	⋯	⋯	⋯	⋯
	2459908.31055	0.00004	S	TESS	⋯	⋯	⋯	⋯	⋯
	2459909.71463	0.00008	P	TESS	⋯	⋯	⋯	⋯	⋯

Note.
[a] Observatory: COMUG.

the orbital eccentricity of the system as 0.28. Çakırlı et al. (2007) presented the first double-lined radial velocity curve of the system and determined the orbital and physical parameters of the two components. They classified the spectral type of the components as late B-type stars. Wolf et al. (2013) first determined the period of apsidal motion to be $U = 437 \pm 35$ yr.

The detached eclipsing binary system V2544 Cyg (GSC 3152-1202 = TIC 15396629; $V = 12.8$ mag) was identified as an eclipsing binary by Otero et al. (2006) through analysis of photometric data from the Northern Sky Variability Survey (NSVS), All Sky Automated Survey-3 (ASAS-3), and Hipparcos survey. The orbital period was calculated as $2^{d}.09372$. Kozyreva et al. (2009) contributed new times of minimum light for both primary and secondary eclipses and noted that the secondary minimum occurs at orbital phase 0.5475 suggesting a modest orbital eccentricity. They proposed the presence of relatively rapid apsidal motion in the system and estimated possible apsidal periods of 15 or 50 yr. The first dedicated light curve modeling and apsidal motion analysis were conducted by Bulut & Bulut (2015), who derived an apsidal motion period of $U = 36.6 \pm 9.8$ yr. Later, Volkov et al. (2017) performed a photometric study based on *V*-band light curves. In this study, they calculated an updated apsidal motion period of $U = 39.5 \pm 0.6$ yr. To date, no spectroscopic study has been published for this system.

V785 Cas (BD+64 302 = HIP 10173; $V = 9.28$ mag; Sp. B5V) is a detached eclipsing binary system characterized by a slightly eccentric orbit with an eccentricity of $e = 0.09$. The system was initially identified as a variable star based on photometric observations from the Hipparcos mission. It is listed among the candidate systems exhibiting apsidal motion in the catalog of eccentric eclipsing binaries compiled by Bulut & Demircan (2007). The first apsidal motion parameters for V785 Cas were determined by Wolf et al. (2013), who reported a relatively fast apsidal motion with a period of $U = 83.0 \pm 1.6$ yr.

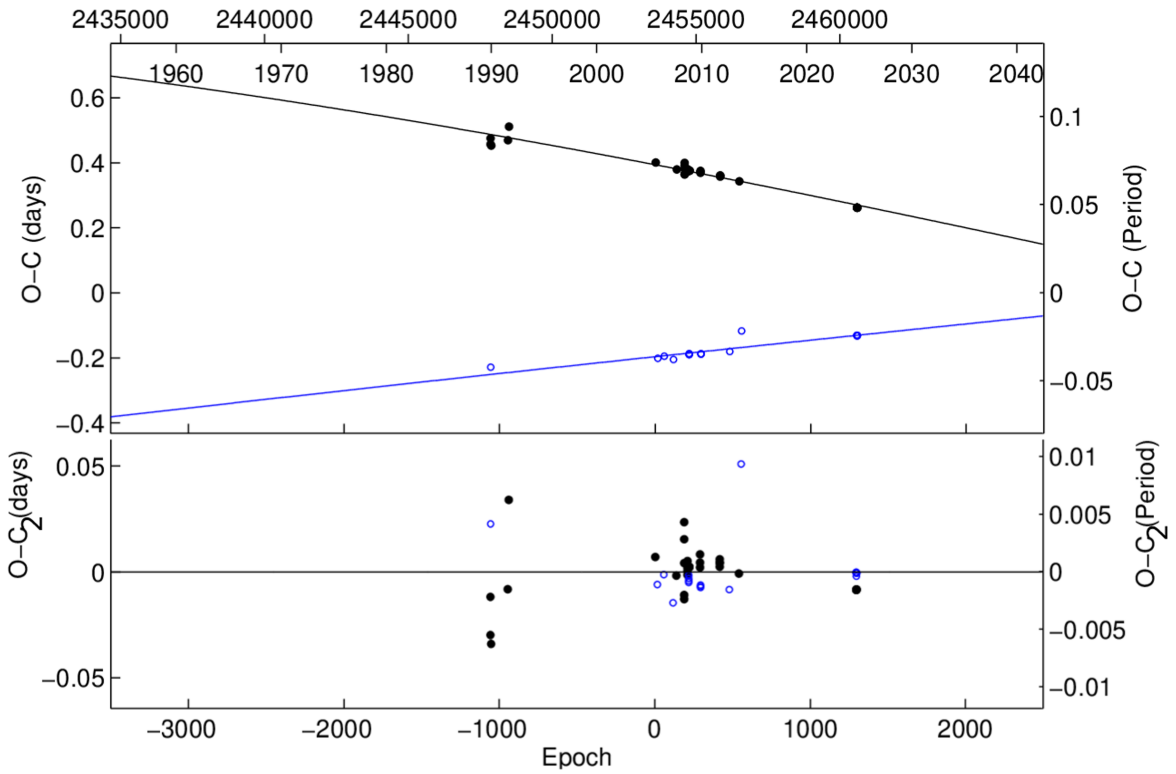


Figure 1. $(O - C)$ diagram of V398 Lac. The solid lines signify the apsidal motion. Filled and open symbols represent the individual primary and secondary minima, respectively. The bottom panel shows the $(O - C)_2$ residuals.

Table 2
Apsidal Motion Parameters of Selected Eclipsing Binaries

Parameter	Unit	V398 Lac	V2544 Cyg	V785 Cas
T_0	(BJD)	2453577.4346 ± 0.0462	2455004.4887 ± 0.0065	2452218.3304 ± 0.0085
P_s	(day)	5.40605 ± 0.00007	2.093768 ± 0.000004	2.702514 ± 0.000005
P_a	(day)	5.42142 ± 0.00007	2.143326 ± 0.000004	2.735429 ± 0.000005
e		0.228 ± 0.011	0.078 ± 0.015	0.089 ± 0.017
$\dot{\omega}$	(deg/cycle)	0.0151 ± 0.0032	0.0477 ± 0.0063	0.0321 ± 0.0013
ω_0	(deg)	220 ± 21	342.7 ± 2.6	159 ± 25
U	(yr)	353 ± 50	43.3 ± 5.1	83.4 ± 8.5

To date, no detailed spectroscopic or photometric light curve analysis has been conducted for the system.

2. Observational Data

TESS provides high-precision photometric data for a large number of eclipsing binary star systems, enabling the extraction of high-quality light curves and accurate times of minimum light. The three systems investigated in this study were observed by TESS, and the corresponding photometric data were retrieved from the Mikulski Archive for Space Telescopes (MAST).[3]

[3] https://archive.stsci.edu/missions-and-data/tess

Additional photometric observations for V398 Lac were conducted using the 30 cm Schmidt–Cassegrain telescope equipped with an ST10X-ME CCD camera at the Çanakkale Onsekiz Mart University Observatory (COMUG). Standard Johnson B, V, and R filters were employed during the observing sessions to ensure accurate multi-band photometric measurements. As a result of these observations, two new times of primary minima were obtained for V398 Lac.

New times of minimum light were determined from both ground-based observations and TESS data using the Kwee–van Woerden method (Kwee & van Woerden 1956), which is widely recognized for its reliability in determining eclipse timings, especially in detached eclipsing binary systems. The

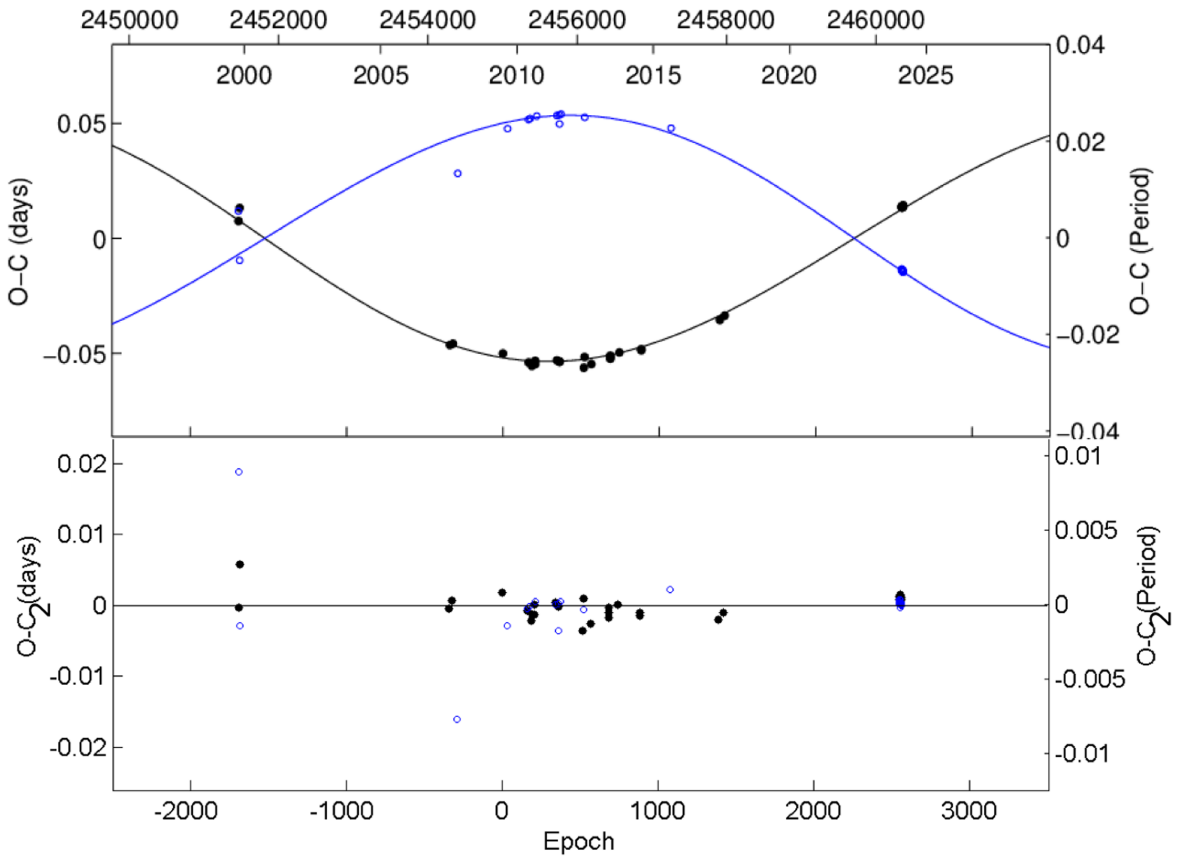


Figure 2. As per Figure 1 but for V2544 Cyg.

new calculated minimum times and uncertainties are presented in Table 1.

3. Apsidal Motion Analysis

The apsidal motion parameters of the selected systems were determined through the analysis of their $(O-C)$ diagrams. For this purpose, the method proposed by Gimenez & Garcia-Pelayo (1983) and refined by Giménez & Bastero (1995) was employed. This approach applies a weighted least-squares iterative procedure, which incorporates terms of orbital eccentricity up to the fifth order, ensuring a more accurate modeling of the apsidal motion in eccentric binary systems.

The method utilizes the following parameters as independent variables:

T_0: Reference epoch (HJD),

ω_0: Longitude of periastron at the reference epoch,

P_a: Anomalistic period (the time between successive periastron passages),

P_s: Sidereal period (the time between successive identical configurations in space),

e: Orbital eccentricity,

$\dot{\omega}$: Apsidal motion rate (in degrees per cycle),

U : The apsidal motion period ($U = 360\,P_a/\dot{\omega}$).

In the $(O-C)$ analyses, the times of minimum light determined in this study were combined with those obtained from the $(O-C)$ Gateway database[4] (Paschke & Brat 2006). As the TESS observation times are given in Barycentric Julian Date (BJD), all previously recorded minima times in Heliocentric Julian Date (HJD) were converted to BJD using a standard online conversion tool.[5] For the apsidal motion analysis, a total of 49, 64, and 111 photometric minima were used for V398 Lac, V2544 Cyg and V785 Cas respectively. The apsidal motion rates were calculated using a computational tool developed by Zasche et al. (2009).

For the apsidal motion analysis of V398 Lac, a total of 49 photometric/CCD times of minima were used, 32 of which correspond to primary minima. These observations span an interval of approximately 35 yr. The derived apsidal motion parameters are listed in Table 2. The apsidal motion rate was determined to be $\dot{\omega} = 0.0151 \pm 0.0032$ degrees per cycle, corresponding to an apsidal period of $U = 353 \pm 50$ yr. This updated apsidal period is shorter than the value previously published by Wolf et al. (2013). The observed minus

[4] https://var.astro.cz/en/

[5] https://astroutils.astronomy.osu.edu/time/

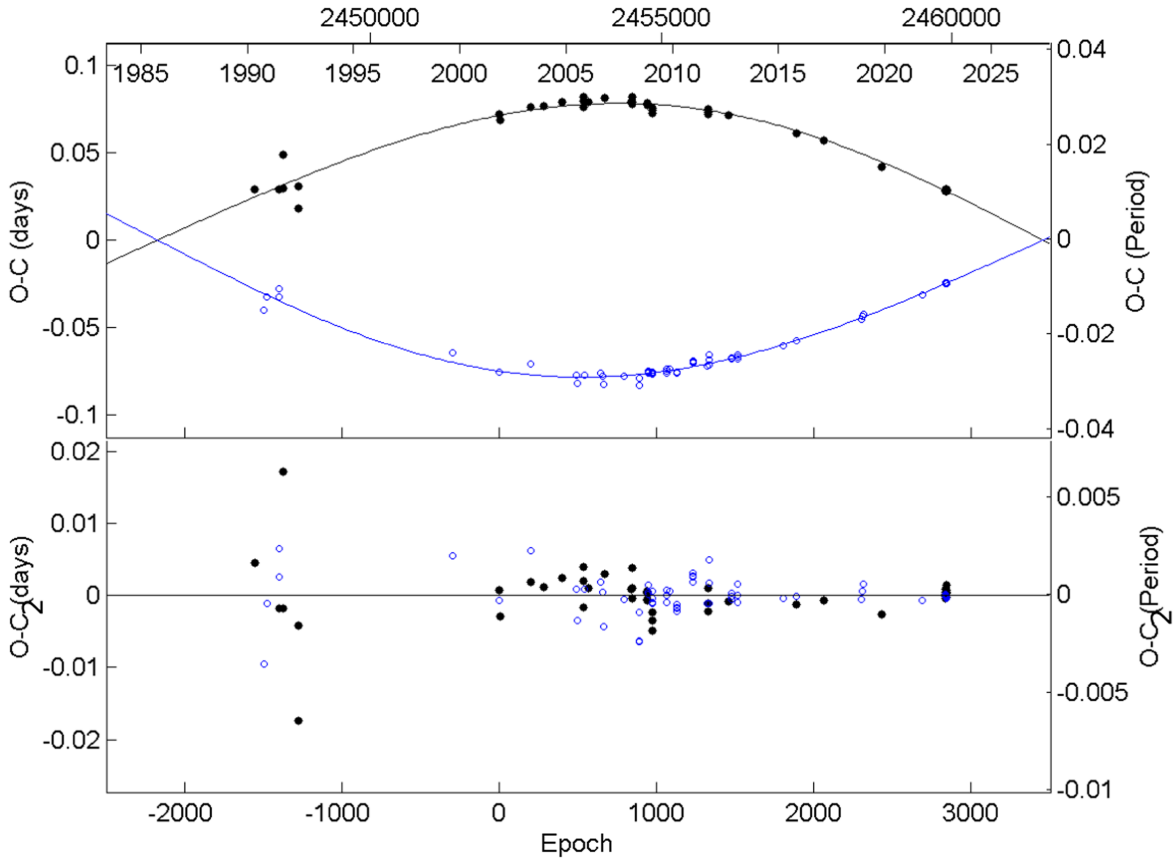


Figure 3. As per Figure 1 but for V785 Cas.

Table 3
Parameters of the TESS Light Curves for V398 Lac, V2544 Cyg and V785 Cas

Parameter	V398 Lac	V2544 Cyg	V785 Cas
P (day)	5.40605054 ± 0.00000001	2.093766998 ± 0.00000001	2.70251468 ± 0.0000001
i (deg)	83.947 ± 0.017	85.101 ± 0.031	78.377 ± 0.007
e	0.228	0.078	0.089
ω (deg)	240.8 ± 0.1	105.4 ± 0.4	68.51 ± 0.16
T_1 (K)	11694	7544	17425
T_2 (K)	10860 ± 300	7135 ± 100	17400 ± 150
Ω_1	5.950 ± 0.006	5.063 ± 0.023	5.553 ± 0.003
Ω_2	10.271 ± 0.016	5.451 ± 0.047	6.155 ± 0.003
q	0.8531 ± 0.0013	0.8070 ± 0.0045	0.9187 ± 0.0010
x_1	0.378	0.518	0.292
x_2	0.377	0.518	0.291
$A_1 = A_2$	1.0	1.0	1.0
$g_1 = g_2$	1.0	1.0	1.0
L_1 ($L_1/L_{\rm Total}$)	0.8375 ± 0.0015	0.6505 ± 0.0019	0.5966 ± 0.0006
L_2 ($L_2/L_{\rm Total}$)	0.1625	0.3495	0.4034
$r_{\rm mean,1}$	0.2086 ± 0.0002	0.2398 ± 0.0006	0.2207 ± 0.0001
$r_{\rm mean,2}$	0.0969 ± 0.0001	0.1891 ± 0.0011	0.1834 ± 0.0001

calculated $(O-C)$ diagram is shown in Figure 1, together with the $(O-C)_2$ residuals.

For V2544 Cyg, apsidal motion analysis was conducted using 64 times of eclipse minima, 26 of which correspond to primary eclipses. These data cover a baseline of approximately 24 yr, sufficient to trace a significant portion of the apsidal motion cycle. The derived apsidal motion parameters are provided in Table 2. The apsidal motion rate was found to be

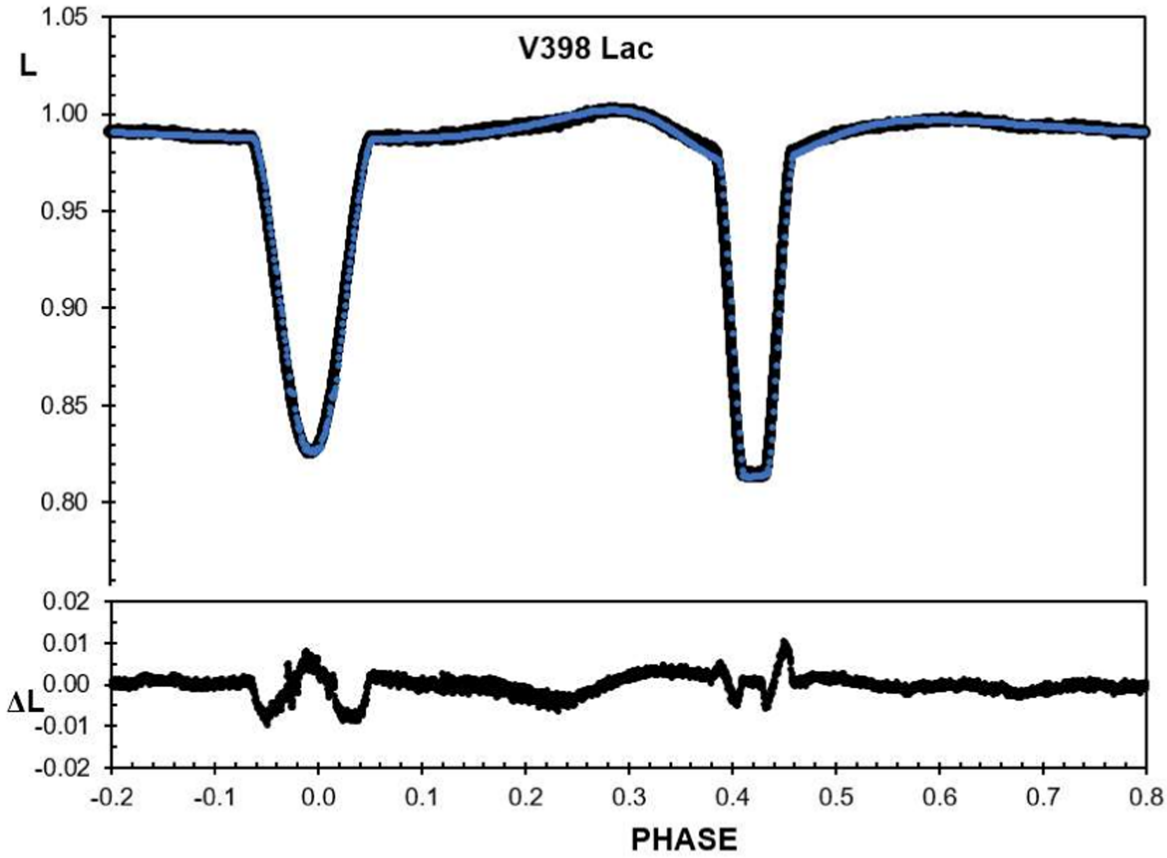


Figure 4. The light curves of V398 Lac are shown, with the black points representing the individual observations. The blue points correspond to the synthetic light curves generated based on the parameters provided in Table 3. The residuals between the observed and synthetic data are shown in the bottom panel.

$\dot{\omega} = 0.0477 \pm 0.0063$ degrees per cycle, resulting in an apsidal period of $U = 43.3 \pm 5.1$ yr. This value is longer than the previously reported period by Bulut & Bulut (2015) and Volkov et al. (2017). The corresponding $(O - C)$ diagram and $(O - C)_2$ residuals are presented in Figure 2.

For V785 Cas, the analysis was performed using 111 times of minimum light, with 76 representing primary minima. These timings span approximately 32 yr. The apsidal motion parameters are summarized in Table 2. An apsidal motion rate of $\dot{\omega} = 0.0321 \pm 0.0013$ degrees per cycle was derived, which corresponds to an apsidal period of $U = 83.4 \pm 8.5$ yr. This result is in excellent agreement with the period reported by Wolf et al. (2013). The $(O - C)$ diagram and the corresponding $(O - C)_2$ residuals are illustrated in Figure 3.

4. Light Curve Analysis

In order to compare the theoretical and observed values of the apsidal motion rate, it is crucial to determine the physical parameters of the stellar components. Therefore, the light curves obtained from the TESS observations of the selected systems were analyzed and the absolute parameters of the components were determined. The analysis was conducted using the PHOEBE software (Prša & Zwitter 2005), which is based on the Wilson–Devinney (WD) algorithm (Wilson & Devinney 1971). The solution was obtained in Mode 2, which is specifically designed for "detached binary" systems within the WD method.

The effective temperatures of the primary components for V398 Lac and V785 Cas are 11,694 K and 17,423 K, respectively, as reported in the Gaia DR3 catalog. In contrast, the literature provides a broad range of temperature estimates for V2544 Cyg. Gaia DR2 reports a temperature of 3994 K, while Bulut & Bulut (2015) derived an average value of approximately 4410 K based on color indices calculated from various photometric data. Volkov et al. (2017) estimated a significantly higher temperature of 21,300 K for the combined spectrum of the system using the $((U - B, B - V))$ two-color diagram. In the analysis, we adopted a temperature of 7544 K for the primary component of V2544 Cyg, based on data from the Large Sky Area Multi-Object Fiber Spectroscopic Telescope (LAMOST) archive.[6]

[6] https://www.lamost.org/lmusers/

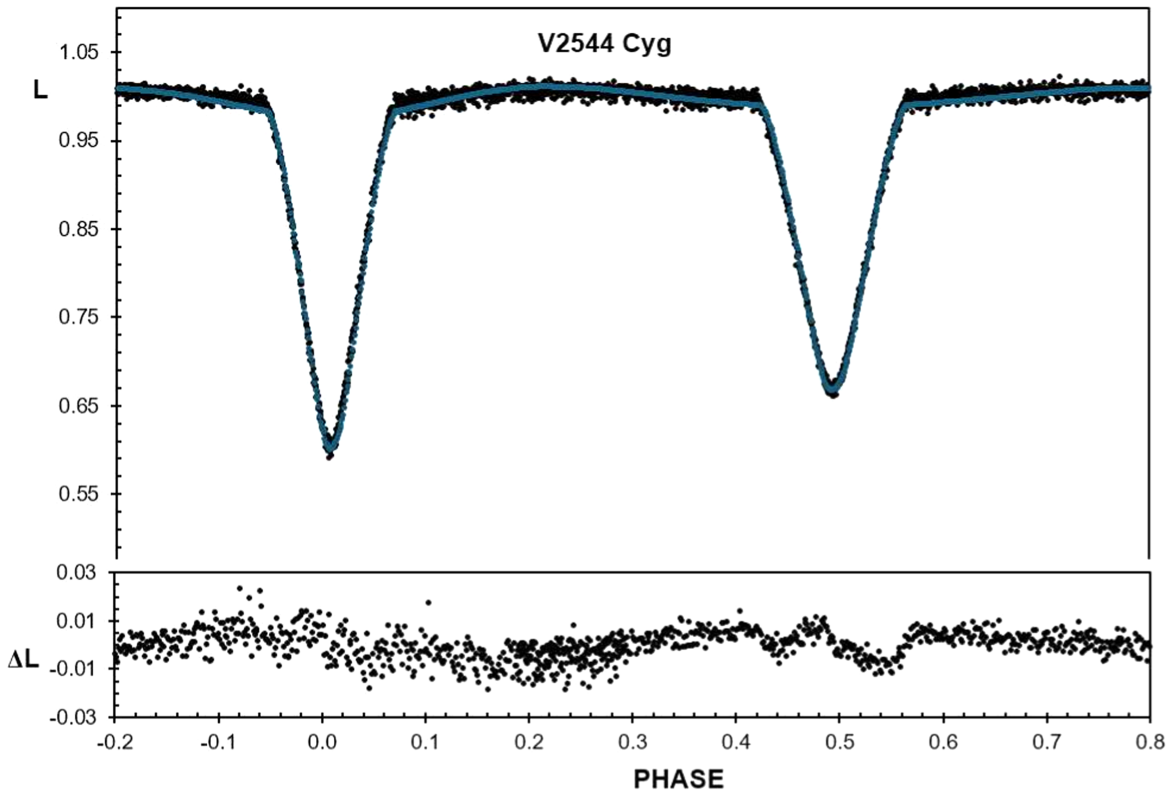


Figure 5. As per Figure 4 but for V2544 Cyg.

For V398 Lac, the spectroscopic mass ratio $q = 0.8580$ reported by Çakırlı et al. (2007) was adopted as the initial value in the light curve analysis. For V2544 Cyg and V785 Cas, no spectroscopically determined mass ratios are available. Therefore, the initial value of the mass ratio required for light curve modeling was estimated using the empirical mass–luminosity relation for main-sequence stars proposed by Graczyk (2003), given by

$$\log q = \frac{\log L_2 - \log L_1}{3.664}, \quad (1)$$

where L_1 and L_2 denote the luminosities of the primary and secondary components in units of the solar luminosity, respectively. These luminosities can be estimated from their bolometric magnitudes using the relation

$$\log L_i = \frac{4.75 - M_{i,\mathrm{bol}}}{2.5}, \quad (2)$$

where $M_{i,\mathrm{bol}}$ are the bolometric magnitudes of the individual components. The bolometric magnitudes $M_{i,\mathrm{bol}}$ were obtained from the PHOEBE software during the light curve analysis.

The gravity-darkening exponents and bolometric albedo coefficients were set to their theoretical values for radiative envelopes (Lucy 1968), i.e., $g_{1,2} = 1.0$ and $A_{1,2} = 1.0$. The limb-darkening coefficients were interpolated automatically by PHOEBE from van Hamme's tables (van Hamme 1993), using the linear cosine law. The eccentricity (e) values listed in Table 2 were adopted as fixed parameters in the light curve analyses.

The adjustable parameters in the light curve modeling included the secondary star's temperature (T_2), orbital inclination (i), longitude of periastron (ω), surface potentials ($\Omega_{1,2}$), the relative luminosity of the primary star (L_1), and the mass ratio (q). The final results from the light curve solution are summarized in Table 3. The uncertainties associated with the fitted parameters were estimated by calculating the standard deviations of the differential corrections provided by the WD method. The synthetic light curve, obtained from the model, is compared to the observed data in Figures 4–6, demonstrating the fit between the observed and theoretical light curves.

The light curve of V398 Lac (see Figure 4) exhibits a partial eclipse during the primary minimum and a total eclipse during the secondary minimum. The asymmetry observed in the maxima of the light curve is most likely attributable to the reflection effect, wherein radiation from one stellar component is reflected off the facing hemisphere of its companion.

The derived physical parameters for all studied systems are summarized in Table 4. For V398 Lac, the absolute parameters were calculated using the spectroscopic mass ratio (q) and

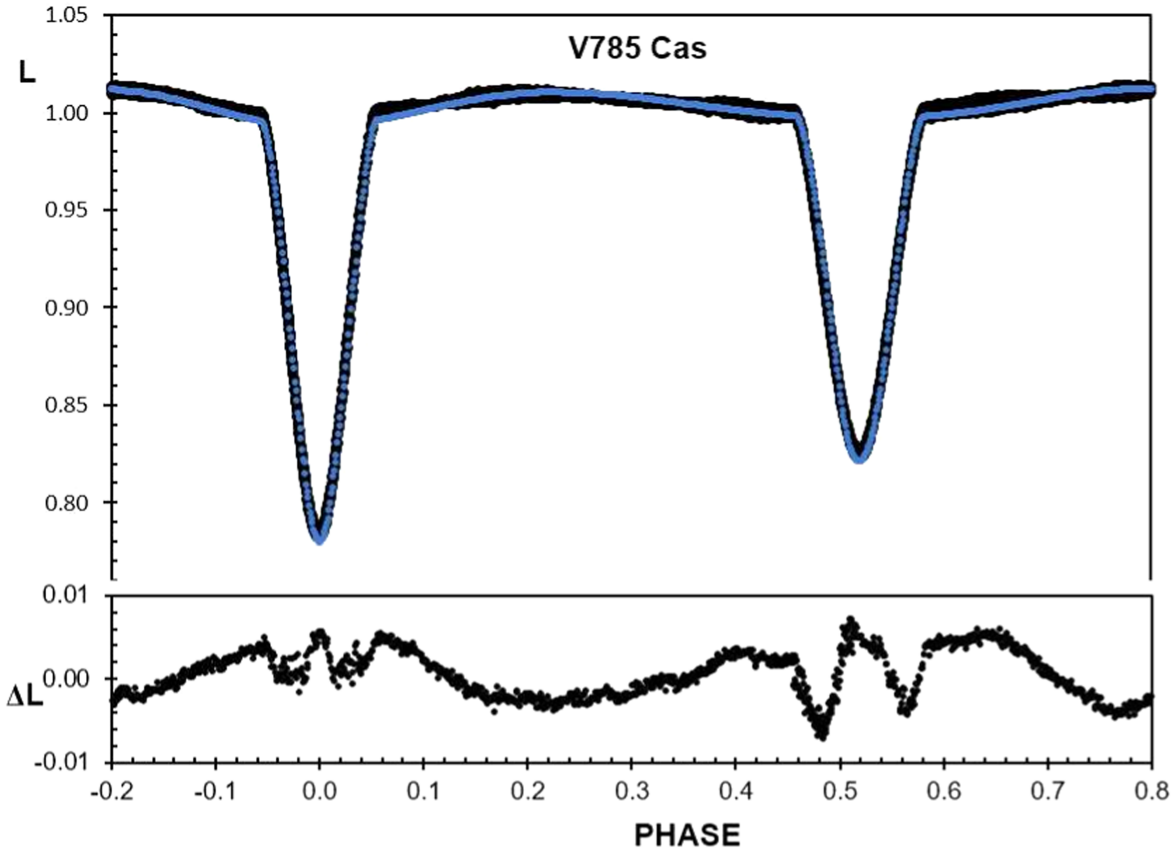


Figure 6. As per Figure 4 but for V785 Cas.

Table 4
Absolute Parameters for V398 Lac, V2544 Cyg and V785 Cas

Parameter	V398 Lac	V2544 Cyg	V785 Cas
M_1 ($M_\odot$)	3.83 ± 0.35	1.75 ± 0.38	5.64 ± 0.41
M_2 ($M_\odot$)	3.27 ± 0.35	1.40 ± 0.38	5.18 ± 0.41
R_1 ($R_\odot$)	5.20 ± 0.12	2.42 ± 0.14	3.99 ± 0.15
R_2 ($R_\odot$)	2.42 ± 0.12	1.91 ± 0.14	3.31 ± 0.15
log g (cgs)	3.59 ± 0.22	3.91 ± 0.28	3.99 ± 0.29
log g (cgs)	4.19 ± 0.21	4.02 ± 0.28	4.11 ± 0.29
T_1 (K)	11694 ± 300	7544 ± 100	17425 ± 150
T_2 (K)	10860 ± 300	7135 ± 100	17400 ± 150
log L ($L_\odot$)	2.66 ± 0.12	1.23 ± 0.11	3.12 ± 0.14
log L ($L_\odot$)	1.86 ± 0.12	0.92 ± 0.11	2.96 ± 0.14
a ($R_\odot$)	23.3 ± 0.8	10.1 ± 0.6	18.3 ± 0.6

semimajor axis (a) values provided by Çakırlı et al. (2007), in conjunction with the photometric parameters listed in Table 3. In contrast, due to the lack of detailed spectroscopic analyses for V2544 Cyg and V785 Cas, the physical parameters of these systems could not be determined directly. Instead, estimations were made under the assumption that both stellar components are located on the main sequence. The mass of the primary component in each system was estimated based on its effective temperature, using the mass-temperature calibration provided by Eker et al. (2018). By applying the photometric mass ratios listed in Table 3, the masses of the secondary components of V2544 Cyg and V785 Cas were calculated.

5. Internal Structure Constants

The observed apsidal motion rate ($\dot{\omega}_{\rm obs}$) is typically expressed as the sum of two contributions

$$\dot{\omega}_{\rm obs} = \dot{\omega}_N + \dot{\omega}_{\rm rel}, \tag{3}$$

where $\dot{\omega}_N$ is the Newtonian or classical contribution to the apsidal motion rate, arising from stellar rotation and tidal interactions, and $\dot{\omega}_{\rm rel}$ refers to the relativistic contribution to the apsidal motion rate due to relativistic effects. It is well known that the relativistic contribution to the observed apsidal motion rate is generally smaller than the Newtonian contribution.

The relativistic contribution to the observed apsidal motion rate is predicted by the General Theory of Relativity and is given by the following Einstein equation (Gimenez 1985)

$$\dot{\omega}_{\rm rel} = 5.45 \times 10^{-4} \frac{1}{1-e^2}\left(\frac{M_1+M_2}{P}\right)^{2/3}, \tag{4}$$

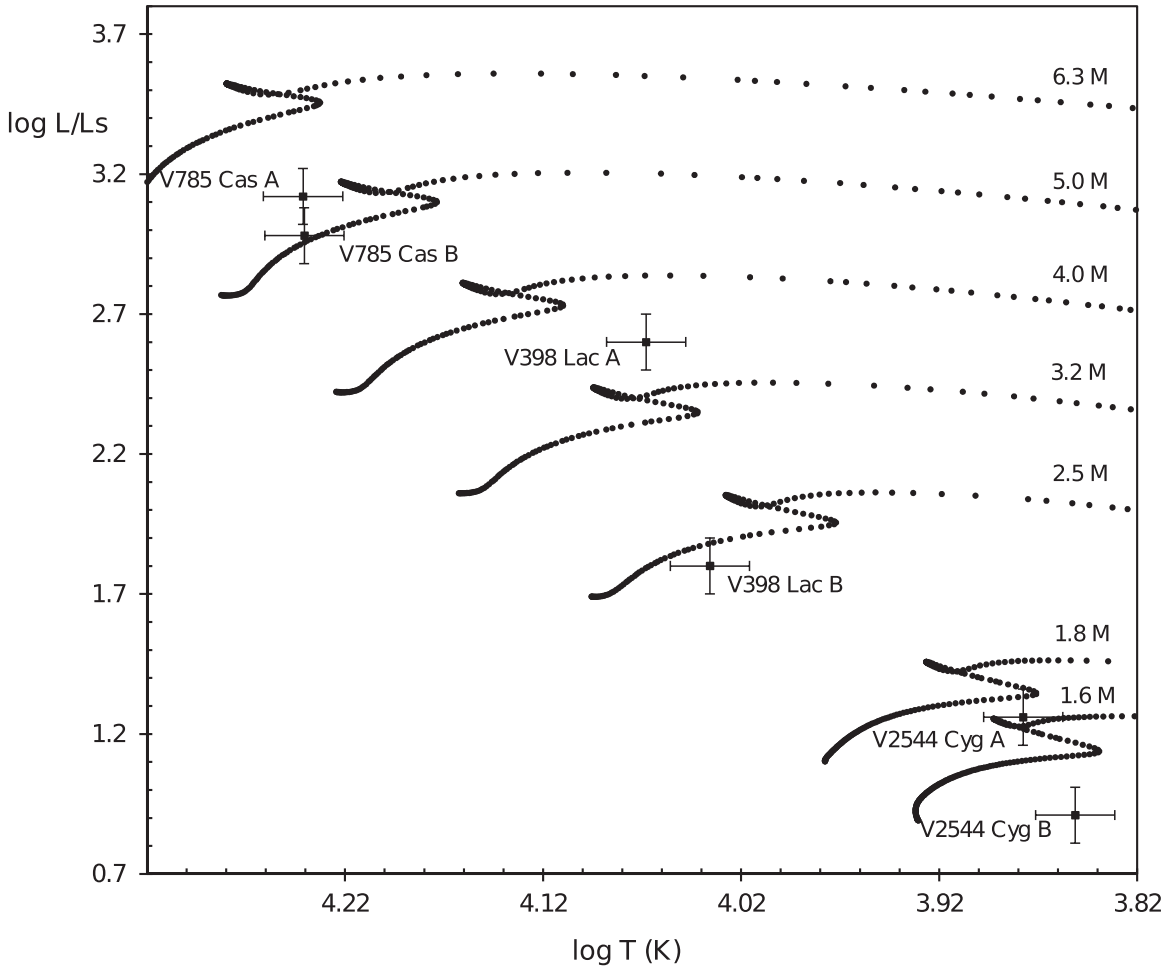


Figure 7. H–R diagram for the components of the studied binary systems. Stellar evolutionary tracks for different masses are plotted according to the models of Claret (2006).

where (M_i) denotes the masses of the components in solar mass.

The Newtonian contribution to apsidal motion is described by the formulation provided by Kopal (1978)

$$\frac{\dot{\omega}_N}{360} = \frac{P}{U} = c_{21}k_{21} + c_{22}k_{22}, \tag{5}$$

where (k_{2i}) represents the internal structure constant of the ith component. (c_{2i}) are functions related to the relative radii, masses and angular velocities of the axial rotation of the components, as well as the Keplerian angular velocity and the orbital eccentricity.

The observational mean value of the internal structure constant ($\bar{k}_{2,\mathrm{obs}}$) can be derived from the following expression

$$\bar{k}_{2,\mathrm{obs}} = \frac{1}{c_{21}+c_{22}}\frac{P}{U} = \frac{1}{c_{21}+c_{22}}\frac{\dot{\omega}_N}{360}. \tag{6}$$

The astrophysical parameters provided in Table 4 were utilized to calculate the average observational internal structure constant ($\bar{k}_{2,\mathrm{obs}}$) for the selected systems. Initially, the relative term in the apsidal motion rate ($\dot{\omega}_{\mathrm{rel}}$) was calculated using Equation (4). The ($\dot{\omega}_{\mathrm{rel}}$) value was then subtracted from the total observed apsidal motion rate ($\dot{\omega}_{\mathrm{obs}}$) derived from the ($O-C$) analysis, which yielded the classical contribution ($\dot{\omega}_N$) to the apsidal motion. Subsequently, using Equation (6), the average observational internal structure constant $\bar{k}_{2,\mathrm{obs}}$ was determined.

The theoretical internal structure constants for the components of the binary systems were obtained from the evolutionary model of Claret (2006), using the standard chemical composition (X, Z) = (0.70, 0.01). The values of $k_{21,\mathrm{theo}}$ and $k_{22,\mathrm{theo}}$ were determined by interpolation from the tables provided by Claret (2006), based on the masses and log g values of the component stars. The average theoretical values

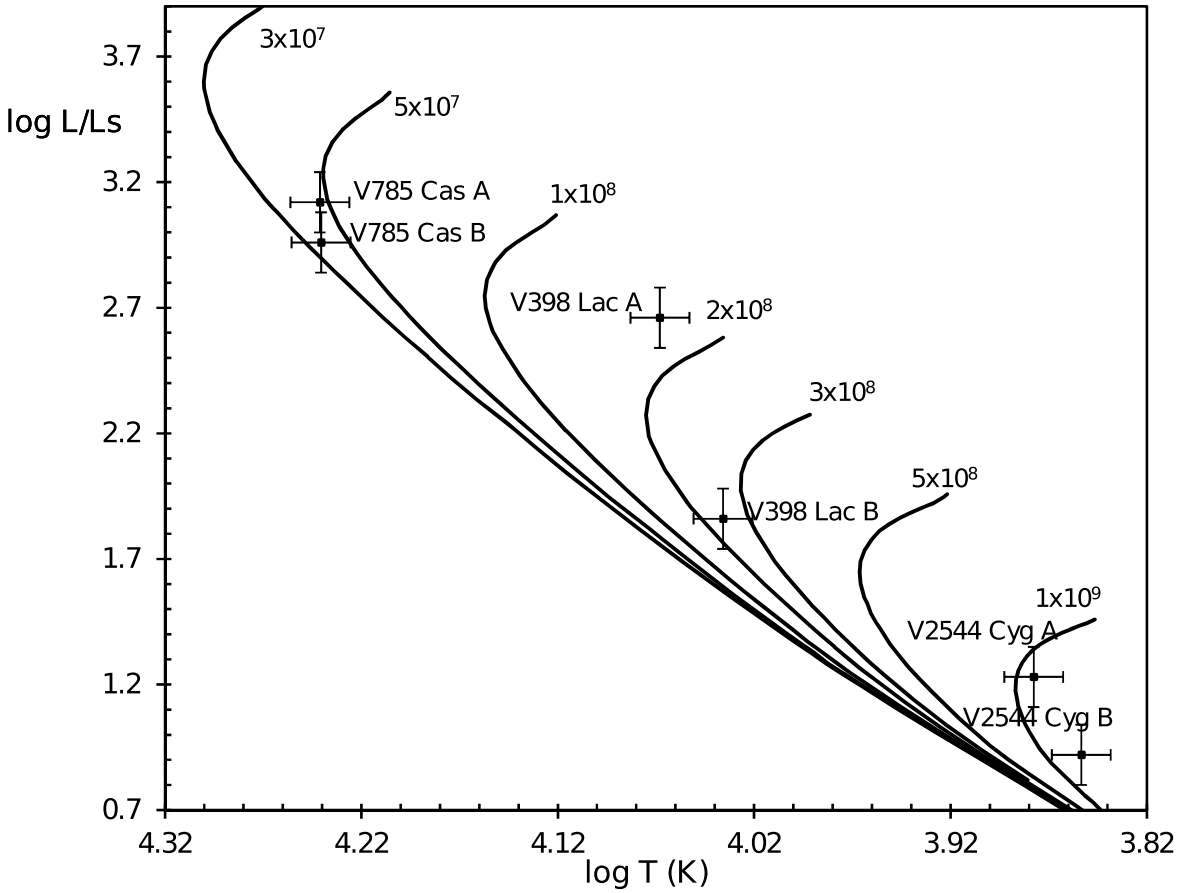


Figure 8. Isochrones corresponding to the components of the studied binary systems, generated using the CMD 3.8 web interface and the PARSEC 2.0 stellar evolutionary models.

for the internal structure constants ($\bar{k}_{2,\mathrm{theo}}$) were then computed using the following formula

$$\bar{k}_{2,\mathrm{theo}} = \frac{c_{21}k_{21,\mathrm{theo}} + c_{22}k_{22,\mathrm{theo}}}{c_{21} + c_{22}}. \quad (7)$$

The resulting parameters related to apsidal motion, along with the observational and theoretical internal structure constants, are presented in Table 5 for the systems under study.

6. Conclusions and Remarks

In this study, the apsidal motion analyses of V398 Lac, V2544 Cyg, and V785 Cas were re-examined using updated ($O-C$) diagrams constructed from all available times of minimum light. The apsidal motion periods were derived using the method proposed by Giménez & Bastero (1995), yielding values of 353 ± 50 yr for V398 Lac, 43.3 ± 5.1 yr for V2544 Cyg, and 83.4 ± 8.5 yr for V785 Cas. Comparison with previous studies reveals that the apsidal motion period of V398 Lac is shorter than that reported by Wolf et al. (2013), whereas the period for V2544 Cyg is longer than the value given by Bulut & Bulut (2015). The apsidal period obtained for V785 Cas is in good agreement with the result presented by Wolf et al. (2013). Moreover, no evidence of cyclic modulation that could indicate the presence of a tertiary component was found in the residuals of the $O-C$ diagrams (see Figures 1–3) for any of the systems analyzed.

Table 5
The Internal Structure Constants of the Selected Systems

Parameter	V398 Lac	V2544 Cyg	V785 Cas
M_1 ($M_\odot$)	3.83 ± 0.35	1.75 ± 0.38	5.64 ± 0.41
M_2 ($M_\odot$)	3.27 ± 0.35	1.40 ± 0.38	5.18 ± 0.41
r_1	0.2086 ± 0.0002	0.2398 ± 0.0006	0.2207 ± 0.0001
r_2	0.0969 ± 0.0001	0.1891 ± 0.0011	0.1834 ± 0.0001
$\dot{\omega}_{\mathrm{obs}}$ (deg/cycle)	0.0151 ± 0.0032	0.0477 ± 0.0063	0.0321 ± 0.0013
$\dot{\omega}_{\mathrm{rel}}$ (deg/cycle)	0.0007	0.0007	0.0014
$\dot{\omega}_{\mathrm{rel}}/\dot{\omega}$ (per cent)	5	1	4
$\log \bar{k}_{2,\mathrm{obs}}$	-2.431 ± 0.087	-2.124 ± 0.058	-2.192 ± 0.046
$\log \bar{k}_{2,\mathrm{teo}}$	-2.474	-2.458	-2.265

Additionally, the TESS light curves of the three systems were analyzed to derive their photometric and absolute parameters, which are summarized in Tables 3 and 4. As a result of these analyses, the component masses were determined as follows: for V398 Lac, $M_1 = 3.83 \pm 0.35\,M_\odot$ and $M_2 = 3.27 \pm 0.35\,M_\odot$; for V2544 Cyg, $M_1 = 1.75 \pm 0.38\,M_\odot$ and $M_2 = 1.40 \pm 0.38\,M_\odot$; for V785 Cas, $M_1 = 5.64 \pm 0.41\,M_\odot$ and $M_2 = 5.18 \pm 0.41\,M_\odot$. The parameters related to apsidal motion and the internal structure constants—both observational and theoretical—are listed in Table 5. The relativistic contribution to the observed apsidal motion rate is relatively small in all three systems, accounting for less than 5% of the total rate.

The theoretical internal structure constants were derived from the evolutionary models of Claret (2006), assuming a chemical composition of $(X, Z) = (0.70, 0.01)$. A comparison between the theoretical and observed internal structure constants reveals good agreement within the uncertainties for V398 Lac and V785 Cyg. However, the observed value for V2544 Cyg is slightly higher than the predicted theoretical value, suggesting a denser internal mass distribution than that estimated by the evolutionary models.

In Figure 7, the components of all three systems are plotted on the Hertzsprung–Russell (H–R) diagram along with evolutionary tracks from Claret (2006). Both components of V2544 Cyg and V785 Cas lie between the Zero-Age Main Sequence and the Terminal-Age Main Sequence (TAMS) on the H–R diagram, indicating that they are in the main-sequence evolutionary stage. The primary component of V398 Lac is located near the TAMS, suggesting that it is approaching the end of its main-sequence lifetime and may be evolving toward the giant phase. In contrast, the secondary component of V398 Lac appears to be in the early main sequence but follows the evolutionary track of a lower-mass star ($2.5\,M_\odot$), indicating that its luminosity is lower than expected for its calculated mass. This discrepancy may be attributed to differences in evolutionary stage or the different physical mechanisms. In Figure 8, the components are compared with a series of isochrones generated for different stellar ages using the CMD 3.8 web interface and the PARSEC v2.0 evolutionary models (Bressan et al. 2012), provided by the Osservatorio Astronomico di Padova.[7] The best-fitting isochrones suggest stellar ages of approximately 2×10^8 yr for V398 Lac, 1×10^9 yr for V2544 Cyg, and 5×10^7 yr for V785 Cas.

Future high-resolution spectroscopic observations and detailed analyses, particularly for V2544 Cyg and V785 Cas, will be essential for refining their absolute parameters and for gaining deeper insights into the physical nature and evolutionary development of these binary systems.

[7] http://stev.oapd.inaf.it/cgi-bin/cmd

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