# Review Article: Resonant Families of Periodic Orbits in the Restricted Threebody Problem\*

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#### Abstract

The restricted three-body problem (RTBP) is a fundamental model in celestial mechanics. Periodic orbits in the synodic frame play a very important role in understanding the dynamics of the RTBP model. Most of these periodic orbits, when interpreted in the sidereal frame, are actually resonant periodic orbits. As a result, numerical computation of the periodic orbits is also one approach for researchers to understand the orbital resonances of the three-body problem. Extensive studies have been carried out on this topic, concerning either the circular case or the elliptic case of this model. In this paper, we make a brief review of the history and current status of the studies on resonant periodic orbits in the RTBP model. Starting from the unperturbed two-body problem, we organize the review paper by the two cases of this model—the circular restricted three-body problem and the elliptic restricted three-body problem.

Key words: celestial mechanics - proper motions - planets and satellites: dynamical evolution and stability

# 1. Introduction

A resonance arises when there is a simple numerical relationship between frequencies or periods (Murray & Dermott 1999). Mean motion resonance (MMR) or orbital resonance is of great importance in the dynamics of planets and satellites. Orbital resonance is believed to be the source of many interesting dynamical phenomena related to long-term stability and instability. In the solar system, there are a great number of MMRs involving asteroids, planetary rings, moonlets, and smaller Kuiper Belt objects (a few involve planets, dwarf planets, or larger satellites). Meanwhile, resonant orbits have been widely employed in mission design for planetary flyby trajectories (NASA/ESA 2009) and as nominal orbits, including the mission extension phase of IBEX (Mccomas et al. 2011; Dichmann et al. 2014) and the TESS mission (Gangestad et al. 2013). Besides, investigations on transferring from a low Earth orbit to the vicinity of the Earth-Moon libration points via resonant arcs (Vaquero & Howell 2014a) and between different resonant orbits (Vaquero & Howell 2014b; Lei & Xu 2018) are also carried out.

Generally, there are two approaches to study resonances in celestial mechanics. One is the Hamiltonian approach. By focusing on the specific resonant term and eliminating other periodic terms (the so-called averaging process), the Hamiltonian is reduced to a  $1^{\circ}$  of freedom system with the resonance angle as the action variable (Henrard & Lemaitre 1983b;

Feng et al. 2016; Tan et al. 2020). The other is the approach of periodic orbits. It is proved by Birkhoff (1927), Arenstorf (1963), Guillaume (1969), Schmidt (1972) that families of symmetric periodic orbits exist in the rotating frame, which can be generated by continuation w.r.t. the parameter  $\mu$  of the periodic orbits in the corresponding degenerate ( $\mu = 0$ ) model. The periodic orbits can be computed rather easily with good accuracy and they refer to the complete system, i.e., no approximation is involved (Hadjidemetriou 1992). Resonant motion is associated with the periodic motion. Therefore, the computation of families of periodic orbits provides useful information on the resonant structure of the phase space (Berry 1978; Hadjidemetriou 1993b; Benet et al. 1999). Furthermore, the stability of resonant periodic orbits has been extensively studied in (Broucke 1968; Hadjidemetriou 1975; Hadjidemetriou & Ichtiaroglou 1984) to better understand the dynamics of the real solar system. However, there exist some limitations of computing the periodic orbits: (a) time-consuming computations of mappings; (b) the problem of accuracy in the exact initial conditions and, more importantly, their stability (Hadjidemetriou 1993b). Therefore, researchers sometimes use a combination of the two methods above to provide a comprehensive understanding of the phase space. Anyhow, the periodic orbit is a powerful tool for studying resonant problems. A review of the existence proofs for various types of periodic orbits in connection with the study of problems in celestial mechanics and stellar dynamics is given by Hadjidemetriou (1984). Meanwhile, a systematical review of



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periodic orbits and their stability in different planetary systems is presented by Hadjidemetriou (1988). In this paper, we briefly review the efforts on the studies of resonant periodic families in the circular restricted three-body problem (CRTBP) and elliptic restricted three-body problem (ERTBP) models carried out by previous researchers.

The remainder of the paper is structured as follows. Families of two kinds of periodic orbits (circular and elliptic cases) in the unperturbed model are introduced in Section 2. Families of resonant periodic orbits in the CRTBP and ERTBP are introduced in Sections 3 and 4, respectively. In Section 5, we conclude this review.

# 2. Families of Periodic Orbits in the Unperturbed Model

We start from the unperturbed two-body problem to obtain a clear view of the resonant structure in the restricted three-body problem (RTBP).  $P_1$  and  $P_2$  are denoted as the two primaries with masses  $m_1$  and  $m_2$  of the system, respectively. The second primary  $P_2$  revolves around the first primary  $P_1$  in a circular orbit. We study the motion of an infinitesimal particle  $P_3$  ( $m_3 = 0$ ) in a uniformly rotating frame *O*-*xy* (the synodic frame), whose initial phase is on the positive *x*-axis which points from  $P_1$  to  $P_2$ . The parameter  $\mu$  is defined as  $\mu = m_2/(m_1 + m_2)$ . Take the units of the system as  $[M] = M_1$ , and  $[L] = \overline{P_1P_2}$ . With these units, the period of the second primary is equal to  $2\pi$  and the angular velocity of the *O*-*xy* frame (and of the second primary) n' is equal to one. In the unperturbed case, we have  $\mu = 0$ , i.e., the second primary has a zero mass and moves on a Keplerian orbit around the first primary.

Theory of periodic orbits in the unperturbed model has been presented by many researchers (Hadjidemetriou & Ichtiaroglou 1984; Hadjidemetriou 1988; Ichtiaroglou et al. 1989; Hadjidemetriou 1993b). Moreover, investigations on the limits of the periodic solutions in the restricted problem as  $\mu \rightarrow 0$  and families of periodic solutions which are called generating families have been investigated by researchers (Hénon 1997; Bruno & Varin 2006, 2007; Zaborsky 2020). Generally, two kinds of periodic orbit families exist in the unperturbed model ( $\mu = 0$ ). One is the family of circular periodic orbits, and the other is the family of elliptic periodic orbits. We make a brief introduction to them in this section.

For the planar case, we denote the initial condition of the periodic orbit as  $(x_0, y_0, \dot{x_0}, \dot{y_0})$ , its Jacobi constant as *C*, and its orbital period as *T*. Usually, a symmetric periodic orbit can be represented by a point in the plane  $x_0-\dot{y_0}$ , or  $x_0-C$ , or equivalently, in the plane *T*–*C*. A monoparametric family of symmetric periodic orbits is represented by a smooth curve in the plane  $x_0-\dot{y_0}$ , which is called the "characteristic curve". The term "*characteristic curve*" has been indicated by many researchers to study the resonant periodic families (Hadjidemetriou & Ichtiaroglou 1984; Ichtiaroglou et al. 1989;

Hadjidemetriou 1992; Kotoulas & Hadjidemetriou 2002; Hou et al. 2018). In this review, we present the first order resonant periodic families in the T-C plane and discuss their properties from a different perspective.

# 2.1. Families of Circular Orbits

In the inertial frame, a periodic orbit of  $P_3$  is called direct (retrograde) orbit if  $P_3$  revolves around  $P_1$  in the same (opposite) direction as that of  $P_2$  (Ichtiaroglou et al. 1989). Therefore, there exist two families of circular periodic orbits: the direct family Dand the retrograde family R. In this review, we pay our attention to the direct family D. Due to the existence of  $P_2$ , family D has two branches (denoted as  $D_{int}$  and  $D_{ext}$ ), which correspond to the inside and outside of the orbit of  $P_2$ , respectively. In Figure 1, we present the circular families of the unperturbed model in the T-Cplane. Both of the families  $D_{int}$  and  $D_{ext}$  have one thing in common: the Jacobi constant C decreases from infinity and gradually converges to a constant as the period T continues to increase. Taking one orbit in family  $D_{int}$  as an example, the orbit is circular in both the inertial frame and synodic frame (see orbit-1 in the first column of Figure 2).

# 2.2. Families of Resonant Periodic Orbits

There is an infinite number of resonant periodic families which bifurcate from the circular family D at the resonance n: n' = p: q, where n is the mean motion of the small body, n'is that of the second primary, and p, q are integers. For each of these resonant periodic families, the semimajor axis a, and the value n: n' remains unchanged. The eccentricity, however, increases with decreasing value of the Jacobi constant C. For each resonance, there exist two different branches of the resonant family, differing in phase only (Hadjidemetriou 1993b). In Figure 1, characteristic curves of some example resonant periodic families related to the first, second, and third order resonances are shown. They appear as vertical lines in the T-C plane. Along each vertical line, from top to bottom, the orbital eccentricity gradually increases. As we have mentioned, actually there are two branches of the n: n' resonant periodic family. Taking the 2:1 resonant periodic family as an example, one branch has the periapsis on the positive x-axis (denoted as family  $I_{2,1}$ ) and the other has the apoapsis on the positive x-axis (denoted as family  $II_{2:1}$ ). Example orbits of each branch are presented in the middle and the right column of Figure 2, respectively. In the unperturbed model, the T-C curves of the two branches exactly coincide with each other, so they appear as a single vertical line (actually two identical vertical lines) in Figure 1. In the following perturbed model, due to the perturbation from the secondary, the two branches separate from each other in the T-C plane.

One remark is that characteristic curves of the unperturbed model displayed in Figure 1 look different from those displayed in the  $x_0$ -C plane (Hadjidemetriou 1988; Hadjidemetriou & Ichtiaroglou 1984) or the  $x_0$ - $\dot{y}_0$  plane (Ichtiaroglou et al. 1989;



Figure 1. Characteristic curves of families of periodic orbits in the unperturbed model. Upper left: the interior circular periodic family  $D_{int}$  and the first order interior resonant periodic families of 2:1, 3:2, 4:3, and 5:4 resonances. Upper right: the exterior circular periodic family  $D_{ext}$  and the first order exterior resonant periodic families of 1:2, 2:3, 3:4, and 4:5 resonances. The abscissa is half the orbital period in the synodic frame and the ordinate is the Jacobi constant. Lower left: family  $D_{int}$  and the 3:2, 5:3 and 7:4 resonances. Lower right: family  $D_{ext}$  and the 2:3, 3:5 and 4:7 resonances.

Hadjidemetriou 1993a; Kotoulas & Hadjidemetriou 2002), but they all represent the same resonant periodic families. All the periodic orbits in the unperturbed model are orbitally stable (Hadjidemetriou & Ichtiaroglou 1984).

# 3. Families of Periodic Orbits in the CRTBP Model

Periodic orbits in the unperturbed model can be continued from  $\mu = 0$  to  $\mu > 0$ . In this section, we focus on the resonant periodic families in the CRTBP model, including the symmetric and asymmetric periodic families.

# 3.1. The Symmetric Resonant Periodic Families

There are two different kinds of symmetric periodic orbits when  $\mu > 0$ : (a) *periodic orbits of the first kind*: the periodic orbits correspond to the nearly circular orbits of the small body, which are continued from the families  $D_{int}$  and  $D_{ext}$  in Section 2; (b) *periodic orbits of the second kind*: the periodic orbits correspond to elliptic orbits of the small body, which are continued from the resonant periodic orbits in Section 2. In what follows, we mainly focus on periodic orbits of the second kind and review families near the first, second, and even higher order resonances in different systems separately. The results obtained by different researchers are presented.

# 3.1.1. The First Order Resonances

Continuation of circular orbits in family D of the small body is possible in all cases except at the first order interior resonance n: (n + 1) or exterior resonance (n + 1): n, (n = 1, 2, 3...). A review of the existence proof is given in Hadjidemetriou (1984). Analytic studies of behaviors near the



**Figure 2.** Example orbits in the unperturbed model: the upper three orbits are in the inertial frame and the lower three are in the synodic frame. The first column illustrates an example orbit (orbit-1:  $C_1 = 3.095526261903474$ ) in the family  $D_{int}$ . The second column shows an example orbit (orbit-2:  $C_2 = 2.500304912949211$ ) in the family  $I_{2:1}$  and the last column displays an example orbit (orbit-3:  $C_3 = 2.500039570216568$ ) in the family  $II_{2:1}$ . The larger and smaller blue dots represent the positions of  $P_1$  and  $P_2$ , respectively. The green dots indicate the initial phases of the example orbits.



Figure 3. Genealogy of the first order interior resonant periodic families in the *T*–*C* plane, continued from the upper left frame of Figure 1 to the Sun-Earth (upper frame), Sun-Jupiter (middle frame) and Earth-Moon (lower frame) system. The abscissa is half the orbital period in the synodic frame and the ordinate is the Jacobi constant.

first order resonances have been made by Guillaume (1969); Hadrava & Kadrnoska (1986).

Numerical investigations of symmetric periodic families near the first order resonance have been made by many researchers. (Colombo et al. 1968; Hadjidemetriou & Ichtiaroglou 1984) perform a detailed analysis of families near the interior first order resonance 2:1, 3:2, and 4:3 with the mass parameter  $\mu = 0.001$ . Each of the continued families consists of a part with a circular branch (continued from family  $D_{int}$ ) while the remaining parts consist of elliptic branches. For each first order resonance, there exist two branches of periodic orbits of the second kind, one with periapsis on the positive *x*-axis (denoted as family  $I_{(n+1):n}$ ) and the other with apoapsis on the positive *x*-axis (denoted as family  $II_{(n+1):n}$ ). One family is stable and the other is unstable, but along the family, the stability type changes. Families of periodic orbits in the 2:1 resonance in the Sun-Jupiter system are investigated by many researchers (Hadjidemetriou 1993b; Hadjidemetriou & Voyatzis 2000; Celletti et al. 2002; Voyatzis et al. 2009). The results show that family  $I_{2:1}$  is stable. Family  $II_{2:1}$  consists of two parts. One part is stable and the other is unstable, separated by a collision orbit (Hadjidemetriou 1993b; Hadjidemetriou & Voyatzis 2000). Meanwhile, families of resonant periodic orbits at the 3:2 resonance are also investigated in Hadjidemetriou & Voyatzis (2000). Details of the 2:1 and 3:2 resonant periodic families in the space of initial conditions  $x_0$ -h are shown in Figure 1 of



Figure 4. Genealogy of the first order exterior resonant periodic families in the T-C plane, continued from the upper right frame of Figure 1 to the Sun-Earth (upper frame), Sun-Jupiter (middle frame), and Earth-Moon (lower frame) system. The abscissa is half the orbital period in the synodic frame and the ordinate is the Jacobi constant.

Hadjidemetriou & Voyatzis (2000). Besides, families of periodic orbits in the 3:2 resonance are also studied by Antoniadou et al. (2011), Antoniadou & Libert (2018a) in the CRTBP model with  $\mu = 0.001$ . More details about the 3:2 resonant periodic families are given by those authors. Two families are formed near the 3:2 resonance. Family  $I_{3:2}$  is stable but could not be continued to high eccentricities of the small body, due to close encounters with the primaries. Family  $I_{3:2}$  consists of two parts, one is stable and the other is unstable, divided by the region where collisions occur (see figure 4 of Antoniadou & Libert 2018a).

Besides, studies about the asteroid motion in the 2:1 and 3:2 resonances in the Sun-Jupiter system are made by many

researchers to have a better understanding of their different behaviors (gap in the 2:1 resonance, group in the 3:2 resonance) (Moons & Morbidelli 1993; Michtchenko & Ferraz-Mello 1995, 1996; Morbidelli 1996; Hadjidemetriou & Lemaître 1997; Hadjidemetriou 1999; Hadjidemetriou & Voyatzis 2000; Tsiganis et al. 2002).

Exterior first order resonant periodic families n: (n + 1), (n = 1, 2, 3...) evolve in a similar way as the interior first order resonant periodic families. The symmetric resonant families join smoothly the circular family (continued from family  $D_{ext}$ ), leaving a gap on the characteristic curves at the position of the 1st order resonance when  $\mu \neq 0$ . Similarly, there are two families of periodic orbits of the second kind for each exterior

1.5

1.5



**Figure 5.** Example orbits of first order resonance in the Sun-Jupiter system: orbit-4 (family  $I_{2:1}$ ): C = 2.500232027717493; orbit-7 (family  $I_{2:1}$ ): C = 2.500104038547575; orbit-5 (family  $I_{3:2}$ ): C = 2.501824165160431; orbit-8 (family  $I_{3:2}$ ): C = 2.500138565225013; orbit-6 (family  $I_{4:3}$ ): C = 2.803722842001424; orbit-9 (family  $I_{4:3}$ ): C = 2.813137090752240.



**Figure 6.** The family of periodic orbits of the first kind and the families of the second kind near the 3:1 and 2:1 resonance (schematically). Left: shown in  $x_0$ –*C* plane. The stable family  $II_{3:1}$  cannot be represented on the present diagram ( $x_0 > 0$ ) because its vertical intersection point with the *x* axis is at the negative *x* axis (see orbit-13 in Figure 8). Right: shown in *T*–*C* plane. The near circular family is represented by the dotted line, and the unstable segment is denoted as (*A*,*B*) in the frame. The abscissa is half of the orbital period *T*/2. The families  $I_{3:1}$  and  $II_{3:1}$  are shown by dashed lines. Periodic orbits in these two families actually make two loops in the synodic frame (see Figure 8), so the abscissa of the two families shown in the right frame is actually *T*/4.

first order resonance n: (n + 1), denoted as family  $I_{n:(n+1)}$  (with periapsis on the positive x-axis) and family  $II_{n:(n+1)}$  (with apoapsis on the positive x-axis), respectively. A systematic investigation of symmetric periodic orbits at the exterior 1:2, 2:3, and 3:4 resonances in the Sun-Neptune system is given by Kotoulas & Hadjidemetriou (2002). (Hadjifotinou & Hadjidemetriou 2002; Kotoulas & Hadjidemetriou 2003) calculated the periodic orbits and their stability for the 2:3 and the 3:4 resonant motion, respectively. A comparative study of the 2:3 and the 3:4 resonant motion in the Sun-Neptune system is also performed by Kotoulas & Voyatzis (2004). The results show that the stable and unstable region of the family  $I_{2:3}$  and  $I_{3:4}$  in the CRTBP model are separated by a collision orbit. Family  $II_{2:3}$  and family  $II_{3:4}$  are stable for all values of the eccentricities, with the exception of a small region where collision orbits exist (Kotoulas & Hadjidemetriou 2002). Besides, systematic studies of families at the exterior first order resonance n: (n+1) with  $2 \le n \le 7$  are conducted by Kotoulas & Voyatzis (2005), Voyatzis & Kotoulas (2005) in the Sun-Neptune system. Family  $I_{n:(n+1)}$  joins smoothly with the family  $II_{(n+1):(n+2)}$  through a segment of near circular orbits. A closed characteristic curve is formed by three families at the exterior 6:7 and the 7:8 resonance (Voyatzis & Kotoulas 2005). Besides, the stability and bifurcation analysis of the exterior 1:2 and 2:3 resonant families in the Earth-Moon system are studied by Li et al. (2021).

As an example, we continue these first order resonant periodic families from the unperturbed two-body system (see the upper left and upper right of Figure 1) to the Sun-Earth ( $\mu_{\rm SE}$ ), Sun-Jupiter ( $\mu_{\rm SJ}$ ) and Earth-Moon ( $\mu_{\rm EM}$ ) system and

present them in the T-C plane. Interior as well as exterior first order resonant periodic families, including the near circular family are displayed in Figures 3 and 4, respectively. Stable family members are highlighted with dark blue color in Figures 3 and 4. Example orbits corresponding to different branches of the 2:1, 3:2, and 4:3 resonant families in the Sun-Jupiter system are displayed in Figure 5. Comparing Figure 3 as well as Figure 4 with the upper frames of Figure 1, we notice that characteristic curves of the two branches of each resonant periodic family no longer coincide with each other. Also, we find: (1) starting from higher values of n, the interior (n + 1): n and the exterior n: (n + 1) resonances are gradually influenced by the 1:1 resonance, in the form that the genealogy of first order resonant periodic families becomes more complex and more distorted from the unperturbed two-body system with increasing  $\mu$ , especially for those close to the 1:1 resonance. (2) With increasing  $\mu$ , stable orbits in the first order resonances are becoming fewer and fewer. This is due to the gradual increasing strength of the 1:1 resonance, which leads to its gradual overlap with nearby interior and exterior resonances. The resonances being overlapped generally become chaotic. For example, judging from Figure 3, for the Sun-Jupiter system, we cannot expect stable first order resonance periodic orbits beyond the 4:3 one, and for the Earth-Moon system, we cannot expect stable first order resonance periodic orbits beyond the 3:2 one.

#### 3.1.2. The Second Order Resonances

Continuation of the circular family *D* from  $\mu = 0$  to  $\mu > 0$  is possible at the second order resonances (2n + 1): (2n - 1) and



Figure 7. The symmetric and asymmetric periodic families near 1:2 and 1:3 resonances in the Sun-Jupiter system. Family AI and AII are symmetric about the *y* axis. The red color indicates the unstable orbits, while the blue color indicates the stable orbits.

(2n-1): (2n+1), (n = 1, 2, 3...) (Hadjidemetriou 1988). It is proved by Hadjidemetriou (1982) that at the resonant circular orbits n: n' = 3: 1, 5: 3, 7: 5, ..., (2n + 1): (2n - 1), ... there always exists a Hamiltonian perturbation which generates instability. According to Hadjidemetriou & Ichtiaroglou (1984), the increase of the mass of the second primary  $(\mu > 0)$  is such a perturbation. Asteroid motions near the 3:1 resonance in the Sun-Jupiter system have been investigated by researchers (Henrard & Caranicolas 1990; Hadjidemetriou & Ichtiaroglou 1984; Grau 1995; Hadjidemetriou 1986, 1992, 1993a, 1993b). It is proved by Hadjidemetriou (1982, 1985, 1993b); Hadjidemetriou & Ichtiaroglou (1984) that a small unstable region (A, B) is generated in the characteristic curve near the 3:1 resonance, where A and B represent critical orbits. A similar property holds for the resonance 5:3 and the resonance 7:5 (Hadjidemetriou & Ichtiaroglou 1984). From the two critical orbits that define the unstable region, two families of symmetric periodic orbits of the second kind bifurcate, which are denoted as family  $I_{3:1}$  and family  $I_{3:1}$ , differing in phase only (Hadjidemetriou & Ichtiaroglou 1984; Hadjidemetriou 1993a, 1993b). The family  $I_{3:1}$  is unstable and the family  $II_{3:1}$  is stable (Hadjidemetriou 1992; Celletti et al. 2002). The two 3:1 resonant periodic families are displayed in Figure 6. The families of periodic orbits of the first kind (near circular family) and the families of the second kind near 3:1 and 2:1 resonances are presented in the  $x_0$ -C plane (see the left frame of Figure 6). In the right frame of Figure 6, we show the near circular family and families near 3:1 resonance in the T-Cplane. The unstable segment (A, B) in the near circular family and families  $I_{3:1}$  as well as  $II_{3:1}$  bifurcating from them are vividly shown. We also present one example orbit in each of the two 3:1 families in Figure 8 (see orbit-10 and orbit-13). From the orbit shapes, we know that periodic orbits in the family  $I_{3:1}$  have two perpendicular crossings with the *x*-axis at x > 0, while those in the family  $II_{3:1}$  have two perpendicular crossings with the negative *x*-axis. In addition, the unstable 3:1 family for greater eccentricity value of the small body is further studied by Antoniadou & Libert (2018a) and another bifurcation point is found.

Furthermore, a systematic study of periodic families at the 3:5, 5:7, and 7:9 exterior mean motion resonances is performed by Kotoulas & Voyatzis (2005); Voyatzis & Kotoulas (2005) in the Sun-Neptune system. Similar to the interior second order resonance, for each resonance two families bifurcate: one has two perpendicular crossings with the *x* axis at x < 0, and the other has two perpendicular crossings with the *x* axis at x > 0. Collision orbits and close encounter orbits occur along the families, leading to more complex characteristics of the phase space.

#### 3.1.3. Other Higher Order Resonances

A systematic investigation of the interior resonance in the Sun-Jupiter-Asteroid system, with frequency 4:  $3 \le n$ :  $n' \le 5$ : 1 has been carried out by Celletti et al. (2002). For each resonance, the stability of the two families of periodic orbits (family *I* and family *II*) as a function of the averaged eccentricity  $e \in (0, 1)$  is indicated in the work. The 4:1 resonance of the Sun-Jupiter asteroid system is investigated by Hadjidemetriou (1993b). Continuation of the family  $D_{int}$  near the 4:1 resonance is possible, but no instability is generated during the continuation process. Hadjidemetriou indicated that there is only one critical point named *A*, from which two resonant families of periodic orbits (family  $I_{4:1}$  and  $II_{4:1}$ ) bifurcate. The family  $I_{4:1}$  is stable and the family  $II_{4:1}$  is unstable. Besides, families at the exterior third order resonances



**Figure 8.** Example orbits in the CRTBP model: (1) Column 1: the unstable 3:1 resonant orbit (orbit-10: C = 3.018610188218501) and stable 3:1 resonant orbit (orbit-13: C = 3.062181617495304). (2) Column 2: the 1:2 symmetric (orbit-11: C = 2.415187005979774) and asymmetric orbit (orbit-14: C = 2.4679972205675615). (3) Column 3: the 1:3 symmetric (orbit-12: C = 2.166233870424831) and asymmetric orbit (orbit-15: C = 2.624973271833538). The larger and smaller blue dots represent the positions of  $P_1$  and  $P_2$ , respectively.

(4:7, 5:8, 7:10) in the Sun-Neptune system are studied in (Kotoulas & Voyatzis 2005; Voyatzis & Kotoulas 2005). Two families of elliptic resonant periodic orbits bifurcate from the circular family at the 4:7, 5:8, and 7:10 resonances. Detailed bifurcation and stability properties of the families can be found in Voyatzis & Kotoulas (2005). Furthermore, periodic orbits near the 2:5 resonance, typically corresponding to the Sun-Jupiter-Saturn system are investigated by Kwok & Nacozy (1982, 1985), Michtchenko & Ferraz-Mello (2001).

#### 3.2. The Asymmetric Resonant Periodic Orbits

In the context of the planar CRTBP model, asymmetric periodic orbits have been identified for the exterior MMRs near 1:2, 1:3, 1:4,... as well as for co-orbital motion (1:1 resonance). For the 1:1 resonance, (Taylor 1981) studied the problem of horseshoe periodic orbits in the 1:1 resonance for the Sun-Jupiter system and the asymmetric periodic orbits were found. The 1:1 asymmetric periodic families also exist and emanate from  $L_4$  in the planar CRTBP model (Zagouras et al. 1996) and was continued to general planetary system (Giuppone et al. 2010; Hadjidemetriou & Voyatzis 2011).

Message (1958) indicated the existence of asymmetric periodic orbits near the 1:2 resonances in the Sun-Jupiter system. Schubart (1964) also detected the existence of asymmetric periodic orbits near the 1:2 and 1:3 resonances for the Sun-Jupiter system while investigating the long-period effects in nearly commensurable motion. Asymmetric periodic orbits related to the 1:2 and 1:3 resonances in the Sun-Jupiter system have been verified using numerical integrations by Frangakis (1968, 1973a). By analyzing many cases of interior and exterior commensurabilities, Frangakis (1973b) found that only for  $\pm 1: n$  (-1: *n* means that the orbit is retrograde) would it be possible to have an asymmetric periodic solution in the averaged problem. Beaugé (1994) used an analytical approach to show the existence of asymmetric periodic orbits in the 1: n exterior MMRs and their absence in other exterior MMRs as e.g., the 2:3 and 3:4. Furthermore, the existence of asymmetric periodic orbits associated with the 1: n resonances was clearly revealed in the Poincare surfaces of the section by Winter & Murray (1997); Voyatzis et al. (2005). Besides, Bruno (1994) also confirmed the existence of asymmetric periodic orbits only for the exterior 1: n resonances when studying the generating orbits.

The family of asymmetric periodic orbits near the 1:2 resonances was numerically determined by Message (1970) using a method that detects bifurcations of asymmetric periodic solutions with the symmetric periodic solutions. Using the theory derived by Message (1970), Message & Taylor (1978) indicate that bifurcations of a family of asymmetric periodic solutions with a family of symmetric periodic solutions exist near the 1: (n + 1), (n = 1, 2, ..., 7) resonances and the complete family near 1:4 resonance is determined by numerical



**Figure 9.** The bifurcations near 2:1 resonance of the ERTBP model in the Sun-Jupiter system ( $\mu = \mu_{SJ}$ ), shown in the  $x_0$ -e' plane. Blue: Families  $I_{2:1e}$ . Red: Family  $II_{2:1e}$ .

integration. Values of eccentricity for which the bifurcation occurs for 1: (n + 1), (n = 1, 2, ..., 7) resonances are also given in (Message & Taylor 1978). Extensive numerical studies of asymmetric periodic orbits in the planar CRTBP have been conducted by Taylor (1983) for the Sun-Jupiter mass ratio. Taylor showed the existence of bifurcations of families of asymmetric periodic orbits near the exterior 1:(n+1) resonances with n up to 50 and computed the whole families for  $(1 \le n \le 5)$  as well as particular segments for  $(6 \le n \le 12)$ . Families of symmetric and asymmetric periodic orbits and their stability type for the 1:2, 1:3, and 1:4 exterior resonances in the Sun-Neptune system are studied in Voyatzis et al. (2005). Asymmetric orbits bifurcating from the corresponding symmetric ones are found. We present the 1:2 and 1:3 asymmetric periodic families in the Sun-Jupiter system (see Figure 7). As can be seen from the left of Figure 7, an unstable segment exists in family  $II_{1:2}$ , which starts from a low initial eccentricity value and is extended up to high eccentricity values (small value of C). A similar pattern of the characteristic curve exists for the 1:3 resonance (see the right of Figure 7). The characteristic curve of stable asymmetric periodic orbits bifurcates from the ends of the unstable segment (critical points), but along the asymmetric family, the stability type changes (see the red dashed line in the right of Figure 7). Each characteristic curve of the asymmetric periodic orbits in Figure 7 contains two families (family AI and AII), Along the family AI  $\dot{y}_0 > 0$ , while along the family AII  $\dot{y}_0 < 0$ . The representative 1:2 and 1:3 symmetric and asymmetric orbits are displayed in the second and third columns of Figure 8, respectively.

Besides, studies indicate that the asymmetric resonant solutions also exist in the extrasolar planetary systems (Lee & Peale 2003; Beaugé et al. 2003; Voyatzis & Hadjidemetriou 2005). Research concerning the capture particles into exterior resonances when considering dissipative effects are conducted by Patterson (1987), Sicardy et al. (1993), Lazzaro et al. (1994), Beaugé et al. (2003), Ferraz-Mello et al. (2003).

#### 3.3. Resonant Periodic Families in 3D-CRTBP model

The three-dimensional families of symmetric periodic orbits in the circular problem bifurcate from the vertical critical orbits of the corresponding circular problem (Hénon 1973). Numerical computation of resonant families of three-dimensional periodic orbits are presented in Ichtiaroglou et al. (1989), at the resonances 3:1, 5:3, 1:3 and 3:5 for  $\mu = 0.001$ . The threedimensional orbits are found by continuation to the third dimension of the corresponding planar problem. For the Sun-Neptune system, spatial resonant families are studied in the 3D-CRTBP, namely for the 1:2 resonances by Kotoulas (2005), for the 1:2, 2:3 and 3:4 resonances by Kotoulas & Hadjidemetriou (2002) and for the first order resonances ( $n: (n + 1), 2 \le n \le 6$ ), second order resonances (3:5, 5:7, 7:9) and the third order resonances (4:7, 5:8, 7:10) by Kotoulas & Voyatzis (2005).

Furthermore, Markellos (1978) indicates that asymmetric vertical critical orbits contain bifurcation points for the generation of families of spatial asymmetric periodic orbits. He investigated the exterior 1:2 resonances by computing the asymmetric vertical orbits of the planar CRTBP for the mass parameter  $\mu \in [0.001, 0.5]$  and some samples of spatial asymmetric periodic orbits are provided in the 3D-CRTBP model. Following the approach of Markellos, Voyatzis et al. (2018) computed the 1: n, n = 2, 3, 4, 5 resonant families of asymmetric periodic orbits in the 3D-CRTBP model, which can be related to the dynamics of trans-Neptunian objects.

#### 4. Families of Periodic Orbits in the ERTBP Model

The CRTBP is a simple model to explain the resonant structure. However, the role of the secondary's orbit eccentricity cannot be revealed by this model. Therefore, many attempts were made to have a better explanation of the resonant structure, using the model of ERTBP. A short review on resonant periodic families in the elliptic restricted problem was given in Hadjidemetriou (1988). In the ERTBP model, the eccentricity of the secondary  $P_2$  is not zero  $(e' \neq 0)$ . The periodic orbits are considered in the rotating frame O-xy with the origin O at the center of mass of the two primaries and the positive x-axis along the line  $P_1$ - $P_2$ . This is a non-autonomous Hamiltonian system with two degrees of freedom, which depends periodically on time, with a period equal to  $2\pi$  (the period of the secondary's orbit) (Hadjidemetriou 1993b; Kotoulas & Hadjidemetriou 2002). The energy integral no longer exists, which means that the stability depends on two pairs of eigenvalues, instead of one in the planar circular case (Kotoulas & Hadjidemetriou 2002).

Generally, there exist isolated periodic orbits in the ERTBP model for a fixed value of e'. A family of periodic orbits can be obtained by varying the eccentricity of the second primary e'. Details on the method to compute these families of periodic orbits and their stability have been studied by Broucke (1969), which was dedicated to the Earth-Moon system. Families of the periodic orbits in the ERTBP model bifurcate from the families in the CRTBP model mentioned in Section 3, at those periodic orbits whose period is always equal to  $2\pi$ , or a multiple of it. This means that periodic orbits in the elliptic problem in the rotating frame are also periodic in the inertial frame (Broucke 1969; Hadjidemetriou 1992; Kotoulas 2005). Similarly, a family of resonant periodic orbits in the ERTBP model can be represented by a *characteristic curve* in the three-dimensional space  $x_0$ – $\dot{y}_0$ -e' (Kotoulas & Hadjidemetriou 2002).

#### 4.1. The Symmetric Resonant Periodic Families

#### 4.1.1. The First Order Resonances

For the first order resonances, researchers (Colombo et al. 1968; Hadjidemetriou & Ichtiaroglou 1984; Bien 1980; Ferraz-Mello 1988) have shown that families of the elliptic problem bifurcate from orbits of the circular problem which lie on the resonant elliptic branches 2:1, 3:2,... Besides, Bien (1980) indicated that for the Sun-Jupiter system the bifurcation takes place at the orbit with its eccentricity  $e \simeq 0.73$  ( $T = 2\pi$ ) in the 2:1 resonance and  $e \simeq 0.45$  ( $T = 4\pi$ ) in the 3:2 resonance. Both of these orbits lie on the stable branch of the corresponding resonance (Hadjidemetriou 1988). For the interior 2:1 resonance in the Sun-Jupiter system, Hadjidemetriou (1993b) indicated that there exists only one bifurcation periodic orbit on the family of periodic orbits of the CRTBP model near the 2:1 resonance, with a period exactly equal to  $2\pi$ . We present the bifurcation of the families near the 2:1 resonance from CRTBP to ERTBP in the Sun-Jupiter system in the  $x_0$ -e' space (see Figure 9). Families in the ERTBP model near the 2:1 resonance are denoted as family  $I_{2:1e}$  and  $II_{2:1e}$ , respectively. The two families bifurcating from the family  $I_{2:1}$  are also schematically shown in the  $x_0 - \dot{y}_0 - e'$  space by Hadjidemetriou (1993b); Hadjidemetriou & Voyatzis (2000). As an example, we select the eccentricity of Jupiter e' = 0.2, two isolated periodic orbits of the family  $I_{2:1e}$  and  $II_{2:1e}$  are displayed in the first column of Figure 11 (see orbit-16 and orbit-19). Similarly, the 3:2 resonance has a stable family  $I_{3:2}$  of symmetric periodic orbits which shows a bifurcation point on the characteristic curve to the ERTBP model (Hadjidemetriou & Voyatzis 2000; Antoniadou et al. 2011). Antoniadou indicated that two families of symmetric periodic orbits originate from this point and both of them start stable. Antoniadou and Libert justify the existence of another bifurcation point (denoted as  $B_{II_{s,1}}^{3:2}$  in (Antoniadou & Libert 2018a)), which lies on the stable parts of the characteristic curve of family  $II_{3:2}$ . From the bifurcation point, two new families are computed, one being stable and the

other being unstable. Furthermore, periodic orbits in the 3:2 resonance and their stability of the extrasolar planetary systems are studied in Varadi (1999).

Ferraz-Mello (1988) found that the eccentricity of the asteroid becomes larger when we go deeper into the 2:1 and the 3:2 resonances in the ERTBP, which is verified in (Hadjidemetriou 1993b; Hadjidemetriou & Voyatzis 2000). Averaged Hamiltonian valid for high values of eccentricity of the asteroid is investigated by researchers (Morbidelli & Giorgilli 1990a, 1990b; Ferraz-Mello et al. 1992; Hadjidemetriou & Voyatzis 2000). Furthermore, many periodic orbits for the ERTBP model associated with the dynamics of asteroids and Kuiper Belt objects have been computed by Hadjidemetriou (1999), Voyatzis & Kotoulas (2005), Henrard & Lemaitre (1983a, 1987), Lemaitre & Henrard (1990), Morbidelli & Giorgilli (1990b).

For exterior resonances, Kotoulas & Hadjidemetriou (2002) performed a systematic study of periodic orbit families of the elliptic problem at the 1:2, 2:3, and 3:4 resonance with Neptune. For the exterior 1:2 mean motion resonance in the Sun-Neptune system, there are two pairs of families of periodic orbits bifurcating from the family  $II_{1:2}$  at different eccentricities of  $e_N$  (Kotoulas & Hadjidemetriou 2002). Voyatzis et al. (2009) studied the interior 2:1 resonance and the exterior 1:2 resonance in the Sun-Jupiter system. There are bifurcation points on the characteristic curve, three of them belonging to symmetric families and the other two belonging to asymmetric families. Moreover, an extensive numerical study of the periodic orbits in the ERTBP model consisting of a star, an inner massive planet, and an outer massless body in the 1:2 mean motion is made by Haghighipour et al. (2003), Antoniadou et al. (2011). For the 2:3 and 3:4 resonances in the Sun-Neptune system, there is only one pair of families of periodic orbits bifurcating from family  $II_{2:3}$  and  $II_{3:4}$ , respectively (Kotoulas & Hadjidemetriou 2002). The dynamical mechanisms in the 2:3 as well as the 3:4 resonance have been investigated in several studies (Levison & Stern 1995; Duncan et al. 1995; Hadjifotinou & Hadjidemetriou 2002; Kotoulas & Hadjidemetriou 2003). Furthermore, a comparative study of the 2:3 and the 3:4 resonances is performed in Kotoulas & Voyatzis (2004). Besides, systematical studies of bifurcations and stability of exterior first order resonance  $p: q = n: (n + 1), n = 1, 2, \dots 6$  are conducted in (Kotoulas & Voyatzis 2005; Voyatzis & Kotoulas 2005). Bifurcations and stability of the resonant families at the 4:5, 5:6, and 5:7 resonances in the ERTBP model are presented clearly by Kotoulas & Voyatzis (2005).

# 4.1.2. The Second Order Resonances

The numerical computations performed by Hadjidemetriou & Ichtiaroglou (1984) reveal that the periodic orbits of the first kind with a period equal to  $\pi$ ,  $3\pi$ ,  $5\pi$ ... exist on the circular

branch of the families of periodic orbits of the circular problem. All these orbits lie on the unstable zone at the resonance 3:1, 5:3, 7:5..., respectively (Hadjidemetriou 1988), from which families of periodic orbits of the elliptic problem bifurcate. Numerical computations of Hadjidemetriou (1992) indicate that there are two pairs of 3:1 resonant families of periodic orbits of the planar ERTBP model, for the Sun-Jupiter system. Two of them, (denoted as family  $I_{3:1c}$ ,  $II_{3:1c}$ ), bifurcate from a periodic orbit  $E_1$  of the CRTBP model, which lie in the unstable region AB (see Figure 6) of family of periodic orbits of the first kind and the period is equal to  $\pi$ . The other two families, (denoted as family  $I_{3:1e}$ ,  $II_{3:1e}$ ), bifurcate from a period orbit  $E_2$  of the CRTBP model, which lies on the stable family of resonant periodic orbits of the second kind, i.e., family  $II_{3:1}$  and the period is equal to  $2\pi$ . One of the families  $I_e$  and  $II_e$  is stable and the other is unstable. The detailed methods of computing the 3:1 resonant periodic families in the ERTBP model are performed by Hadjidemetriou (1992, 1993a). Besides, the averaged Hamiltonian has been widely used to explain the 3:1 resonant motion of the ERTBP model in the Sun-Jupiter system by Wisdom (1982, 1985, 1987); Hadjidemetriou (1991); Ferraz-Mello & Klafke (1991). Extensive studies combining the periodic orbits and the averaged model about the 3:1 resonance in the Sun-Jupiter system are made by Hadjidemetriou (1992, 1993a, 1993b), Grau (1995), Antoniadou & Libert (2018a). Antoniadou & Libert (2018a) extends the work of Hadjidemetriou to the high eccentricity of both the asteroid and the second primary. They indicated that family  $I_{\rho}$  possesses an unstable segment before it becomes stable again. Stable periodic orbits can be found in family  $I_c$  when the eccentricities of both the asteroid and the second primary are high enough. Furthermore, two new unstable families are generated in the ERTBP model from the new bifurcation point found by the authors in the CRTBP model.

Here we give an example of the bifurcations near the 5:3 resonance corresponding to the ERTBP model with the method developed in Hadjidemetriou (1992) and present the results in the left frame of Figure 10. Two pairs of 5:3 resonant periodic families of the planar ERTBP model for the Sun-Jupiter system are vividly shown in the  $x_0$ -e' plane. Similar to the case of 3:1 resonance studied in Hadjidemetriou (1992), the first orbit of the family  $I_{5:3c}$  and the first orbit of the family  $II_{5:3c}$  are the same orbit of the CRTBP model (e' = 0) at  $t = \pi/2$ , and t = 0, which lies in the unstable region AB of the first kind periodic family and its period is equal to  $3\pi$ . Therefore, the initial states of family  $I_{5:3c}$  and  $II_{5:3c}$  are on the negative and positive x-axis, respectively. Representative orbits of family  $I_{5:3c}$  (orbit-17) and  $II_{5:3c}$  (orbit-20) are shown in the second column of Figure 11. Family  $I_{5:3e}$  and  $II_{5:3e}$  bifurcate from a periodic orbit of the CRTBP model, which lies on the stable branch of the second kind 5:3 periodic family, and its period is equal to  $6\pi$ . Example orbits of family  $I_{5:3e}$  (orbit-18) and  $II_{5:3e}$  (orbit-21) are shown in the last column of Figure 11. Different from the case of the 3:1



Figure 10. The bifurcations of the ERTBP model in the Sun-Jupiter system ( $\mu = \mu_{SJ}$ ), shown in the  $x_0$ -e' plane. Left: Bifurcations near 5:3 resonance. Right: Bifurcations near 7:4 resonance.

resonance, family  $I_{5:3c}$  and family  $I_{5:3e}$  are actually the same family in the  $x_0-e'$  plane and in the  $x_0-\dot{y}-e'$  space.

Besides, exterior second order resonance 3:5, 5:7, and 7:9 of the ERTBP model in the Sun-Neptune system have been systematically studied in (Kotoulas & Voyatzis 2005; Voyatzis & Kotoulas 2005). Three bifurcation points exist in the case of 3:5 resonance, which belong to different families near the 3:5 resonance. There are five bifurcation points for both 5:7 and 7:9 resonances. The generated families of the two families have the same qualitative features.

#### 4.1.3. Other Higher Order Resonances

Stability of periodic resonance orbits (3:1, 4:1, 5:1, and 6:1) in the ERTBP model for different values of the eccentricity of the second primary for  $\mu = 0.1$  and  $\mu = 0.5$  are investigated in Kribbel & Dvorak (1988). Detailed bifurcations of the 4:1 resonance of the ERTBP model for the Sun-Jupiter system are analyzed in Hadjidemetriou (1993b). Hadejidemetriou indicates that there are three periodic orbits of the CRTBP model near the 4:1 resonance, from which families of periodic orbits of the ERTBP model bifurcate. Extensive studies about the stability of these generated families for greater eccentricity values of the second primary are conducted in Antoniadou & Libert (2018a). Moreover, the third order exterior resonances (4:7, 5:8, 7:10) of the ERTBP model in the Sun-Neptune system are investigated in (Kotoulas & Voyatzis 2005; Voyatzis & Kotoulas 2005). Bifurcation points and stability of the generated families are vividly shown. Besides, research on periodic orbits of the ERTBP model for the Sun-Jupiter-Saturn system (2:5 resonance) has been conducted by many researchers (Kwok & Nacozy 1982, 1985; Michtchenko & Ferraz-Mello 2001).

Taking the 7:4 resonance in the Sun-Jupiter system as an example, we present the bifurcations of the third order resonance of the ERTBP model in the right frame of Figure 10. Similar to the case of 4:1 resonance studied in Hadjidemetriou (1993b), there are three periodic orbits (denoted as orbit-BP1, BP2 ad BP3) of the circular problem near the 7:4 resonance, from each of which two periodic families of the elliptic problem bifurcate (one family with Jupiter at perihelion, the other with Jupiter at aphelion). The orbit-BP1 lies on the first kind periodic family and its period equals  $8\pi/3$ , from which family  $I_{7:4c}$  and  $II_{7:4c}$  bifurcate. The orbit-BP2 and orbit-BP3 are on the second kind periodic family with their period equal to  $8\pi$ , from which the other two pairs of periodic families, family  $I_{7:4e}$  and  $II_{7:4e}$  as well as family  $III_{7:4e}$ and IV7:4e bifurcate. We present these three pairs of 7:4 resonant periodic families of the planar ERTBP model for the Sun-Jupiter system in the right frame Figure 10. Six example orbits belonging to family  $I_{7:4c}$  (orbit-19),  $II_{7:4c}$  (orbit-22),  $I_{7:4e}$ (orbit-20),  $II_{7:4e}$  (orbit-23),  $III_{7:4e}$  (orbit-21) and  $IV_{7:4e}$  (orbit-24) are shown in Figure 12. It is not easy to find that family  $II_{7:4c}$  and  $II_{7:4e}$  belongs to one family in the  $x_0$ -e' plane, and so do the family  $I_{7:4e}$  and  $III_{7:4e}$ . Details of the second and third order bifurcated families in the ERTBP model and their stability will appear in our future work.

#### 4.2. The Asymmetric Resonant Periodic Families

Different from the symmetric cases, much less work has been carried out for asymmetric periodic orbits in the ERTBP model. (Voyatzis & Kotoulas 2005) find that many bifurcation points along the symmetric periodic orbits and conjectured the existence of asymmetric periodic orbits in the elliptic problem. (Antoniadou et al. 2011) computed and showed the existence

1 1.5

1.5



**Figure 11.** Example orbits in the ERTBP model (e' = 0.2): (1) Column 1: example orbit in the family  $I_{2:1e}$  (orbit-16) and in the family  $I_{2:1e}$  (orbit-19). (2) Column 2: example orbit in the family  $I_{5:3e}$  (orbit-17) and in the family  $I_{5:3e}$  (orbit-20). (3) Column 3: example orbit in the family  $I_{5:3e}$  (orbit-18) and in the family  $I_{5:3e}$  (orbit-21). The larger and smaller blue dots represent the positions of  $P_1$  and  $P_2$ , respectively.



**Figure 12.** Example orbits in the ERTBP model: (1) Column 1: example orbit in the family  $I_{7:4c}$  (orbit-22: e' = 0.2) and in the family  $I_{7:4c}$  (orbit-25: e' = 0.04). (2) Column 2: example orbit in the family  $I_{7:4e}$  (orbit-23: e' = 0.1) and in the family  $I_{7:4e}$  (orbit-26: e' = 0.04). (3) Column 3: example orbit in the family  $I_{7:4e}$  (orbit-24: e' = 0.1) and in the family  $I_{7:4e}$  (orbit-27: e' = 0.2). The larger and smaller blue dots represent the positions of  $P_1$  and  $P_2$ , respectively. The green dots indicate the initial phases of the example orbits.

of such families in the ERTBP model and examined their continuation to the general model.

# 4.3. Resonant Periodic Families in 3D-ERTBP Model

There are two ways to compute the resonant families of periodic orbits in the 3D-ERTBP model, one by bifurcating from the 3D-periodic families of the 3D-CRTBP model, and the other by vertically bifurcating from the families of the planar ERTBP model. In the Sun-Neptune system, (Kotoulas & Hadjidemetriou 2002) computed families of periodic orbits of the 3D-RTBP, at the resonances 1:2, 2:3, and 3:4 with both methods. An extensive study on the 1:2 resonance has also been done by Kotoulas (2005). Besides, a symplectic mapping model based on the averaged Hamiltonian is constructed by Hadjifotinou & Hadjidemetriou (2002), which is used to study the 2:3 and the 3:4 exterior resonances for the 3D-ERTBP model in the Sun-Neptune system by Hadjifotinou & Hadjidemetriou (2002), Kotoulas & Hadjidemetriou (2003), Kotoulas & Voyatzis (2004).

Spatial resonant periodic orbits corresponding to the planetary systems are also investigated by researchers. Resonant periodic orbits in three-dimensional exoplanetary systems and their stability are analyzed for 2:1 resonance by Antoniadou & Voyatzis (2013) and for 4:3, 3:2, 5:2, 3:1, and 4:1 resonances by Antoniadou & Voyatzis (2014). Besides, spatial families which emanate from the circular family for the 5:2 and 7:3 resonances are shown by Antoniadou & Voyatzis (2017). Extensive studies of Antoniadou & Libert (2018a, 2018b) related to 3:2, 2:1, 5:2, 3:1, 4:1, and 5:1 interior resonances from 2D-RTBP to 3D-RTBP (circular and elliptic cases) are also performed by Antoniadou & Libert (2019).

#### 5. Conclusion

Resonant periodic orbits in the restricted three-body problem are reviewed. Starting from the unperturbed two-body problem, resonant periodic orbits in two models are presented, including the CRTBP and ERTBP, each of which includes the planar as well as the three-dimensional case. Works related to the symmetric and asymmetric periodic orbits in these two models are introduced, respectively. Bifurcations of resonant periodic families and their stability are presented in detail for the first order, the second order, and other higher order resonances.

In the CRTBP model, the near circular family D continued from the unperturbed model bifurcates at the first order resonances, from which two families of symmetric periodic families are generated, differing in phase only. The continuation of circular family D from  $\mu = 0$  to  $\mu > 0$  is possible at the second order resonances. Due to the perturbation of the second primary, an unstable region exists near the second order resonance, from which symmetric families of the second order resonances are generated. In addition to the first and the second order resonances, families near other higher order resonances and their properties are also reviewed.

Investigations on the resonant periodic families in the ERTBP model were made by researchers to reveal the influence of the secondary's orbital eccentricity. It is found that families of the elliptic problem bifurcate from periodic orbits of the circular problem which reside on the resonant elliptic branches of the first order resonances and the circular branches of the second order resonances. The period of these bifurcation orbits is always equal to  $2\pi$ , or a multiple of it. Furthermore, families of asymmetric periodic orbits bifurcating from symmetric periodic orbits in the CRTBP and ERTBP are determined. Vertical critical orbits exist in the resonant periodic families of the planar CRTBP and ERTBP, from which the three-dimensional families are generated and investigated by researchers.

MMRs are very important in shaping the current structure of the solar system, and resonant periodic family is one approach to study them. Therefore, studies on the resonant periodic families in the restricted three-body model of the Sun-planet system, especially for the Sun-Jupiter and Sun-Neptune systems, are performed by many researchers. These studies give us a better understanding of these resonance phenomena in the solar system (e.g., Kirkwood gaps, trans-Neptunian objects). Meanwhile, there is a growing interest in extrasolar planetary systems. MMRs are also detected and analyzed with the resonant periodic orbits in the extrasolar planetary system by researchers.

One final remark is that: the current review only focuses on the direct (prograde) resonance orbits, leaving the 1:1 resonance, i.e., co-orbital motion and the retrograde orbits untouched. These resonances also have their correspondence in the solar system (for example, the Trojans, and the retrograde asteroids) and are the research interest of many historical and current studies.

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