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Stellar Structure Model in the Post-Newtonian Approximation

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Abstract

In this work the influence of the post-Newtonian corrections to the equations of stellar structure are analyzed. The post-Newtonian Lane–Emden equation follows from the corresponding momentum density balance equation. From a polytropic equation of state the solutions of the Lane–Emden equations in the Newtonian and post-Newtonian theories are determined and the physical quantities for the Sun, the white dwarf Sirius B and neutron stars with masses $M \simeq 1.4\ M_\odot$, $1.8\ M_\odot$ and $2.0\ M_\odot$ are calculated. It is shown that the post-Newtonian corrections to the fields of mass density, pressure and temperature are negligible for the Sun and Sirius B, but for stars with strong fields the differences become important. For the neutron stars analyzed here the central pressure and the central temperature which follow from the post-Newtonian Lane–Emden equation are about fifty to sixty percent greater than those of the Newtonian theory and the central mass density is about three to four percent smaller.

Key words: stars: neutron – stars: fundamental parameters – hydrodynamics

1. Introduction

The investigation of the internal structure of stars is an old subject in the literature and this topic was extensively described in the seminal books by Eddington (Eddington 1926) and Chandrasekhar (Chandrasekhar 1957).

In astrophysics the Newtonian theory assumes a prominent role in the characterization of the structure and dynamics of stars, but also general relativity assumes an important role in astrophysics.

In the analysis of self-gravitating systems it is important to have an approximation scheme that provides a Newtonian description in the lowest order and relativistic effects as higher order perturbations. To that end the post-Newtonian theory can supply the desired relativistic corrections to the Newtonian theory.

The post-Newtonian theory was proposed by Einstein, Infeld and Hoffmann (Einstein et al. 1938) and refers to the solution of Einstein's field equations from a method of successive approximations to the inverse power of the speed of light (for a description of the method see, e.g., the books Weinberg 1972; Capozziello & Faraoni 2011; Poisson & Will 2014; Kremer 2022a). The full Eulerian hydrodynamic equations in the first post-Newtonian approximation were derived by Chandrasekhar (Chandrasekhar 1965) and the corresponding ones in the second post-Newtonian approximation by Chandrasekhar and Nutku (Chandrasekhar & Nutku 1969).

The post-Newtonian approximation is important in analyzing several problems: the equations of motion of binary pulsars (Epstein 1977; Futamase & Itoh 2007), neutron stars (Shinkai 1999; Gupta et al. 2000), galaxy rotation curves (Agón et al. 2011; Kremer et al. 2016), Jeans instability (Nazari et al. 2017; Noh & Hwang 2021; Kremer 2021), spherical accretion (Kremer & Mehret 2021) and stationary spherical self-gravitating systems (Kremer 2022b), among others.

In the last years the equations of stellar structure were analyzed within the framework of the $f(R)$ theory where a modified Lane–Emden equation was derived (Farinelli et al. 2014; Capozziello & De Laurentis 2012; André & Kremer 2017).

The aim of this work is to investigate the influence of the post-Newtonian corrections in the equations of stellar structure which follow from the solution of the post-Newtonian Lane–Emden equation. This equation is obtained from the post-Newtonian momentum density balance equation for a stationary self-gravitating system where a polytropic equation of state is considered. The physical quantities related with the mass density, pressure and temperature of a star are explicitly expressed in terms of the variables of the post-Newtonian Lane–Emden equation. From the polytropic solutions of the Lane–Emden equations in the Newtonian and post-Newtonian theories, the physical quantities for the Sun, white dwarf Sirius B and for neutron stars with masses $M \simeq 1.4\ M_\odot$, $1.8\ M_\odot$ and $2.0\ M_\odot$ are calculated. From the comparison of the Newtonian and post-Newtonian results for the physical quantities, it is shown that the post-Newtonian corrections to the fields of mass density, pressure and temperature are negligible for the Sun and Sirius B. However for stars with strong fields the differences between the two theories become important, since for the neutron stars analyzed here the central pressure and central temperature which follow from the post-Newtonian Lane–Emden equation are about fifty to sixty percent greater than those of the Newtonian theory and the central mass density is about three to four percent smaller.

This paper is outlined as follows: In Section 2, we introduce the post-Newtonian momentum density balance equation and the corresponding Poisson equations. The post-Newtonian Lane–Emden equation is derived in Section 3. In Section 4, we introduce the stellar structure equations in the post-Newtonian approximation. In Section 5, the numerical solutions for the mass density, pressure and temperature for the Sun, Sirius B and for the neutron stars are determined and the Newtonian and post-Newtonian values for these fields are compared. Finally, in Section 6, we close the paper with the conclusions.

2. Post-Newtonian Momentum Density Balance Equation

For a perfect fluid the energy-momentum tensor is given by

$$T^{\mu\nu} = (\epsilon + p)\frac{U^\mu U^\nu}{c^2} + pg^{\mu\nu}. \tag{1}$$

In the above equation, p is the hydrostatic pressure, U^μ the four-velocity (such that $U^\mu U_\mu = c^2$), $g^{\mu\nu}$ the metric tensor and ϵ the energy density which has two contributions, one refers to the mass density ρc^2 and the other to its internal energy density ε, i.e., $\epsilon = \rho c^2(1 + \varepsilon/c^2)$. Here we shall investigate a perfect fluid characterized by the polytropic equation of state $p = \kappa\rho^\gamma$, where κ is a constant and γ is related to the polytropic index $n = 1/(\gamma - 1)$. For a polytropic fluid the internal energy density is given by $\varepsilon = p/[\rho(\gamma - 1)] = np/\rho$.

In the derivation of the post-Newtonian approximations from Einstein's field equations in powers of the ratio v/c—where v is a typical speed of the system and c the speed of light—the components of the metric tensor in the first post-Newtonian approximation read (Chandrasekhar 1965; Kremer 2022a)

$$g_{00} = 1 - \frac{2U}{c^2} + \frac{2}{c^4}(U^2 - 2\Phi),$$
$$g_{0i} = \frac{\Pi_i}{c^3}, \qquad g_{ij} = -\left(1 + \frac{2U}{c^2}\right)\delta_{ij}, \tag{2}$$

where the Newtonian U, scalar Φ and vector Π_i gravitational potentials satisfy the Poisson equations

$$\nabla^2 U = -4\pi G\rho,$$
$$\nabla^2\Phi = -4\pi G\rho\left(V^2 + U + \frac{\varepsilon}{2} + \frac{3p}{2\rho}\right), \tag{3}$$

$$\nabla^2\Pi_i = -16\pi G\rho V_i + \frac{\partial^2 U}{\partial t\partial x^i}. \tag{4}$$

Here $\boldsymbol{V}$ is the hydrodynamic three-velocity and G the universal gravitational constant.

The balance of the momentum density in the first post-Newtonian approximation obtained from conservation of the energy-momentum tensor reads (Chandrasekhar 1965; Kremer 2022a)

$$\frac{\partial\sigma V_i}{\partial t} + \frac{\partial\sigma V_i V_j}{\partial x^j} + \frac{\partial}{\partial x^i}\left[p\left(1 - \frac{2U}{c^2}\right)\right]$$
$$-\rho\frac{\partial U}{\partial x^i}\left[1 + \frac{1}{c^2}\left(2V^2 + \varepsilon - 2U - \frac{p}{\rho}\right)\right]$$
$$-\frac{\rho}{c^2}V_j\left(\frac{\partial\Pi_i}{\partial x^j} - \frac{\partial\Pi_j}{\partial x^i}\right) + 4\frac{\rho}{c^2}V_i\left(\frac{\partial U}{\partial t} + V_j\frac{\partial U}{\partial x^j}\right)$$
$$-\frac{\rho}{c^2}\left(2\frac{\partial\Phi}{\partial x^i} + \frac{\partial\Pi_i}{\partial t}\right) = 0, \tag{5}$$

where σ is the following abbreviation introduced by Chandrasekhar (Chandrasekhar 1965)

$$\sigma = \rho\left[1 + \frac{1}{c^2}\left(V^2 + 2U + \varepsilon + \frac{p}{\rho}\right)\right]. \tag{6}$$

3. Post-Newtonian Lane–Emden Equation

For the description of stellar structure models in the post-Newtonian approximation, we start with the balance equation of momentum density (5) by considering stationary self-gravitating systems where the hydrodynamic three-velocity vanishes, i.e., $\boldsymbol{V} = 0$. Since in spherical coordinates the only dependence of the fields ρ, p, U and Φ is on the radial variable r, Equation (5) becomes

$$\left(1 - \frac{2U}{c^2}\right)\frac{dp}{dr} - \rho\frac{dU}{dr}\left[1 + \frac{1}{c^2}\left(\varepsilon + \frac{p}{\rho} - 2U\right)\right]$$
$$-\frac{2\rho}{c^2}\frac{d\Phi}{dr} = 0. \tag{7}$$

By neglecting the $1/c^2$ terms, the above equation reduces to the Newtonian limiting case $dp/dr = \rho dU/dr$.

Equation (7) can be rewritten—by taking into account that $\varepsilon = np/\rho$ and by considering terms up to $1/c^2$—as

$$\frac{1}{\rho}\frac{dp}{dr}\left(1 - \frac{n+1}{c^2}\frac{p}{\rho}\right) - \frac{d}{dr}\left(U + 2\frac{\Phi}{c^2}\right) = 0. \tag{8}$$

If we assume the polytropic equation of state $p = \kappa\rho^{\frac{n+1}{n}}$, the differential Equation (8) can be solved for the mass density ρ as a function of the gravitational potentials U and Φ, so that from the integration of the resulting equation we get

$$U + 2\frac{\Phi}{c^2} = (n+1)\kappa\rho^{\frac{1}{n}}\left(1 - \frac{\kappa(1+n)}{2c^2}\rho^{\frac{1}{n}}\right). \tag{9}$$

In the above equation it was considered that the gravitational potentials U and Φ and the mass density ρ vanish at the boundary of the star. The argument that U vanishes at the boundary is due to Eddington (Eddington 1926), and here we extend it to the post-Newtonian gravitational potential Φ.

We can solve (9) for ρ up to order $1/c^2$, yielding

$$\rho = \left[\frac{U + \frac{2\Phi}{c^2}}{(n+1)\kappa\left(1 - \frac{\kappa(1+n)}{2c^2}\rho^{\frac{1}{n}}\right)}\right]^n \approx \left(\frac{U}{(n+1)\kappa}\right)^n \left[1 + \frac{n}{c^2}\left(\frac{U}{2} + \frac{2\Phi}{U}\right)\right]. \tag{10}$$

The Poisson Equations (3) for the gravitational potentials U and Φ in spherical coordinates, for stationary systems ruled by a polytropic equation of state $p = \kappa\rho^{\frac{n+1}{n}}$ and $\varepsilon = np/\rho$, become

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dU}{dr}\right) = -4\pi G\rho,$$
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi}{dr}\right) = -4\pi G\rho\left(U + \frac{3+n}{2}\kappa\rho^{\frac{1}{n}}\right). \tag{11}$$

The combination of the two Poisson Equations (11) yields

$$\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(U + 2\frac{\Phi}{c^2}\right)\right] = -4\pi G\rho\left[1 + \frac{2}{c^2}\left(U + \frac{3+n}{2}\kappa\rho^{\frac{1}{n}}\right)\right]. \tag{12}$$

The elimination of the potentials U and Φ from (12) by using (9) results in the following differential equation for the mass density

$$\kappa(n+1)\frac{1}{r^2}\frac{d}{dr}\left[r^2\frac{d}{dr}\left(\rho^{\frac{1}{n}} - \frac{1+n}{2c^2}\kappa\rho^{\frac{2}{n}}\right)\right] = -4\pi G\rho\left[1 + \frac{5+3n}{c^2}\kappa\rho^{\frac{1}{n}}\right]. \tag{13}$$

The dimensionless Lane–Emden equation is obtained from the introduction of the dimensionless variables (Eddington 1926; Chandrasekhar 1957)

$$z = \frac{r}{a},\ u(z) = \left(\frac{\rho}{\rho_c}\right)^{\frac{1}{n}},\ a = \sqrt{\frac{(n+1)\kappa}{4\pi G}\rho_c^{\frac{1-n}{n}}}, \tag{14}$$

where ρ_c denotes the mass density at the center of the star.

The introduction of the new variables (14) into (13) leads to the Lane–Emden equation in the first post-Newtonian approximation

$$\left(1 - \frac{(1+n)p_c}{c^2\rho_c}u(z)\right)\left[\frac{d^2u(z)}{dz^2} + \frac{2}{z}\frac{du(z)}{dz}\right] - \frac{(1+n)p_c}{c^2\rho_c}\left(\frac{du(z)}{dz}\right)^2 = -u(z)^n\left(1 + \frac{(5+3n)p_c}{c^2\rho_c}u(z)\right), \tag{15}$$

where $p_c = \kappa\rho_c^{\frac{n+1}{n}}$ is the hydrostatic pressure at the center of the star.

An equivalent version of the first post-Newtonian approximation of the Lane–Emden equation is obtained from the multiplication of (15) by $[1 + (5+3n)p_c u(z)/c^2\rho_c]$ and considering terms up to order $1/c^2$, yielding

$$\left(1 - \frac{(6+4n)p_c}{c^2\rho_c}u(z)\right)\left[\frac{d^2u(z)}{dz^2} + \frac{2}{z}\frac{du(z)}{dz}\right] - \frac{(1+n)p_c}{c^2\rho_c}\left(\frac{du(z)}{dz}\right)^2 + u(z)^n = 0. \tag{16}$$

If in the above equation we do not consider the $1/c^2$–terms, the Newtonian limit of the Lane–Emden equation is recovered, namely

$$\frac{1}{z^2}\frac{d}{dz}\left(z^2\frac{du(z)}{dz}\right) = -u(z)^n. \tag{17}$$

Furthermore, by considering the perfect fluid equation of state for the hydrostatic pressure at the center of the star $p_c = \rho_c kT_c/m = \rho_c kT_c/\mu m_\mu$—where T_c represents the temperature at the star center, μ the mean molecular weight and m_μ the unified atomic mass—we can write

$$\frac{p_c}{\rho_c c^2} = \frac{kT_c}{mc^2} = \frac{kT_c}{\mu m_\mu c^2}. \tag{18}$$

Note that $p_c/\rho_c c^2$ represents the ratio of the thermal energy of the fluid at the star center kT_c and the rest energy of its particles mc^2.

In astrophysics, the Lane–Emden equation is used to describe thermodynamic system structures characterized by polytropic fluids, considering the gravitational interaction. This equation allows us to determine some physical quantities for these systems, such as pressure, density and temperature.

4. Physical Quantities of Stars

In this section we follow Eddington (Eddington 1926) and Chandrasekhar (Chandrasekhar 1957) and give the expressions for the mass, radius, pressure, mass density and temperature of the stars which follow from the Lane–Emden equation.

The Lane–Emden Equation (16) will be solved by considering the boundary conditions

$$u(0) = 1, \qquad \left.\frac{du(z)}{dz}\right|_{z=0} = 0. \tag{19}$$

The numerical solution of (16) represents a monotonically decreasing behavior of $u(z)$ and its first zero—denoted by $z|_{u=0} = R_*$—corresponding to the surface of the star. From (14) the radius of the star becomes

$$R = aR_* = \sqrt{\frac{(n+1)\kappa}{4\pi G}\rho_c^{\frac{1-n}{n}}}R_*. \tag{20}$$

For a sphere with radius R its inner mass $M(R)$ is given by

$$M(R) = \int_0^R 4\pi\sqrt{\gamma_*}\,\rho r^2 dr. \tag{21}$$

Here γ_* denotes the determinant of the spatial metric tensor, which by considering terms up to $1/c^2$ order reads

$$\sqrt{\gamma_*} = \sqrt{\frac{-g}{g_{00}}} = \left(1 + \frac{3U}{c^2}\right) = \left(1 + \frac{3(n+1)\kappa\rho^{\frac{1}{n}}}{c^2}\right)$$
$$= \left(1 + \frac{3(n+1)p_c}{c^2\rho_c}u(z)\right), \tag{22}$$

by taking into account (9), (14) and the expression for the determinant of the metric tensor in the first post-Newtonian approximation $g = -(1 + 4U/c^2)$.

The mass of the star which follows from the Lane–Emden Equation (16) is given by

$$M(R) = 4\pi a^3\rho_c\int_0^{R_*}\left(1 + \frac{3(n+1)p_c}{c^2\rho_c}u(z)\right)z^2u^n dz$$
$$= -4\pi a^3\rho_c\int_0^{R_*}\left\{\left(1 - \frac{(3+n)p_c}{c^2\rho_c}u(z)\right)\left[\frac{d^2u(z)}{dz^2} + \frac{2}{z}\frac{du(z)}{dz}\right]\right.$$
$$\left. - \frac{(1+n)p_c}{c^2\rho_c}\left(\frac{du(z)}{dz}\right)^2\right\}z^2dz = 4\pi\rho_c a^3 M_*. \tag{23}$$

In the second equality above we have considered only terms up to the $1/c^2$ order.

From the elimination of a and ρ_c from (23) by using (14) and (20) we get that the mass of the star becomes

$$M(R) = 4\pi\left[\frac{(n+1)\kappa}{4\pi G}\right]^{\frac{n}{n-1}}\left(\frac{R}{R_*}\right)^{\frac{n-3}{n-1}}M_*. \tag{24}$$

Now we can build the mass–radius relationships by taking into account (20) and (24), yielding

$$\frac{GM(R)}{M_*}\frac{R_*}{R} = (n+1)\kappa\rho_c^{\frac{1}{n}},$$
$$\left(\frac{GM(R)}{M_*}\right)^{n-1}\left(\frac{R_*}{R}\right)^{n-3} = \frac{[(n+1)\kappa]^n}{4\pi G}. \tag{25}$$

The quantities R_* and M_* can be determined from the Lane–Emden Equation (16) once the mass $M(R)$ and radius R of a star are known. Furthermore, for fixed values of the polytropic index n, the values of κ and ρ_c follow from (25).

We may also express the central mass density of the star as a function of the mean mass density of the star $\bar{\rho}$, namely

$$\bar{\rho} = \frac{M(R)}{4\pi R^3/3}, \qquad \text{hence} \qquad \rho_c = \frac{R_*^3}{3M_*}\bar{\rho}, \tag{26}$$

thanks to (20) and (23).

From the polytropic equation of state $p_c = \kappa\rho_c^{\frac{1+n}{n}}$ together with (25) and (26) we can determine the central pressure of the star

$$p_c = \frac{GM(R)}{M_*}\frac{R_*}{R}\frac{\rho_c}{n+1} = \frac{GM(R)}{M_*}\frac{R_*}{R}\frac{\bar{\rho}}{n+1}\frac{R_*^3}{3M_*}, \tag{27}$$

furthermore, from the equation of state of a perfect fluid we get the temperature at the center of the star

$$T_c = \frac{\mu m_\mu}{k}\frac{p_c}{\rho_c} = \frac{\mu m_\mu}{k(n+1)}\frac{GM(R)}{M_*}\frac{R_*}{R}. \tag{28}$$

The mass density, pressure and temperature as functions of the dimensionless radial distance z follow from the polytropic equation of state and (14), yielding

$$\rho(z) = \rho_c\, u(z)^n,\ p(z) = p_c\, u(z)^{n+1},\ T(z) = T_c\, u(z). \tag{29}$$

5. Polytropic Solutions of the Lane–Emden Equation

A star is identified as a self-gravitating spherically symmetrical mass of a highly ionized gas at equilibrium which is held together by its own gravity. Normally a star is considered to be composed of three kinds of species: hydrogen, helium and heavy elements, which for the purpose of the calculations are not specified.

If X, Y and Z denote the mass fraction of hydrogen, helium and heavy elements, respectively, for a mixture with these three species we must have that $X + Y + Z = 1$ and the mean molecular weight becomes (Chandrasekhar 1957)

$$\mu = \frac{1}{2X + 3Y/4 + Z/2} = \frac{4}{2 + 6X + Y}. \tag{30}$$

In this work we are interested in determining the influence of the post-Newtonian approximation in the stellar structures: neutron stars, white dwarfs and the Sun. Neutron stars are formed from a gravitational collapse of massive stars at the end of their life and practically have only neutrons so that $\mu = 1$. The mass fractions for the Sun are $X = 0.73$, $Y = 0.25$ and $Z = 0.02$ (Basu & Antia 2008) and its mean molecular weight is $\mu = 0.6$. White dwarfs are compact objects with low luminosity and here we shall investigate the white dwarf Sirius B—which is the companion that orbits around the star *Sirius*—where there exists almost all heavy metals $Z \approx 1$, devoid of hydrogen and helium so that $X = Y \approx 0$ and the mean molecular weight is $\mu = 2$.

The Sun has a radius $R_\odot = 6.96 \times 10^8$ m, a mass $M_\odot = 1.989 \times 10^{30}$ kg and the polytropic index usually adopted for it is $n = 3$. For white dwarf stars with higher masses the polytropic index can also be considered as $n = 3$ and Sirius B has mass $M = 1.5\ M_\odot$ and radius $R = 8.4 \times 10^{-3}\ R_\odot$.

Neutron stars are represented by an equation of state with a polytropic index $n \simeq 1$ (Lattimer & Prakash 2001) and we will focus our attention on neutron stars with masses $M \simeq 1.4\ M_\odot$,

Table 1
First Zeros, Central and Mean Mass Densities, Central Pressures and Central Temperatures Calculated From the Newtonian Lane–Emden Equation (17)

	R_*	M_*	$\bar{\rho}$ (kg m^{-3})	ρ_c (kg m^{-3})	p_c (Pa)	T_c (K)
Sun	6.90	2.02	1.41×10^3	7.64×10^4	1.25×10^{16}	1.18×10^7
Sirius B	6.90	2.02	2.89×10^9	1.56×10^{11}	3.34×10^{24}	5.14×10^9
1.4 $M_\odot$	3.14	3.14	7.06×10^{17}	2.33×10^{18}	2.21×10^{34}	1.14×10^{12}
1.8 $M_\odot$	3.14	3.14	1.17×10^{18}	3.87×10^{18}	5.14×10^{34}	1.59×10^{12}
2.0 $M_\odot$	3.14	3.14	1.44×10^{18}	4.76×10^{18}	7.27×10^{34}	1.84×10^{12}

Table 2
Values of the Ratio $p_c/\rho_c c^2 = kT_c/mc^2$

	Sun	Sirius B	1.4 $M_\odot$	1.8 $M_\odot$	2.0 $M_\odot$
kT_c/mc^2	1.19×10^{-6}	2.37×10^{-4}	1.05×10^{-1}	1.48×10^{-1}	1.70×10^{-1}

Table 3
First Zero, Central and Mean Mass Densities, Central Pressure and Central Temperature From the Post-Newtonian Lane–Emden Equation (16) For the Neutron Stars

	R_*	M_*	$\bar{\rho}$ (kg m^{-3})	ρ_c (kg m^{-3})	p_c (Pa)	T_c (K)
1.4 $M_\odot$	2.56	1.75	7.06×10^{17}	2.26×10^{18}	3.14×10^{34}	1.67×10^{12}
1.8 $M_\odot$	2.43	1.52	1.17×10^{18}	3.70×10^{18}	7.86×10^{34}	2.56×10^{12}
2.0 $M_\odot$	2.38	1.43	1.44×10^{18}	4.55×10^{18}	1.16×10^{35}	3.06×10^{12}

1.8 $M_\odot$ and 2.0 $M_\odot$. According to Özel et al. (2010), Özel & Freire (2016), the radii of neutron stars are in the range 8.3 km $\leqslant R \leqslant$ 12 km for all neutron stars. Here we adopted the following radii for the neutron stars: $R \simeq 9.8$ km for $M \simeq$ z_nbsp;1.4 $M_\odot$, $R \simeq 9$ km for $M \simeq 1.8$ $M_\odot$ and $R \simeq 8.7$ km for $M \simeq 2.0$ $M_\odot$. The radius of the neutron star corresponding to the mass $M \simeq 1.8$ $M_\odot$ was taken as $R \simeq 9$ km and the radii of the neutron stars with masses 1.4 $M_\odot$ and 2.0 $M_\odot$ were obtained by using the relationship $R \propto M^{-\frac{1}{3}}$.

First we analyze the results that follow from the Newtonian Lane–Emden equation for the Sun, Sirius B and the neutron stars. In Table 1 the first zeros were found as numerical solutions of the Newtonian Lane–Emden Equation (17) and the mean and central mass densities, central pressure and central temperature were calculated from (26), (27) and (28) when the post-Newtonian correction $p_c/c^2\rho_c$ was not considered. The polytropic indexes adopted are: $n = 1$ for the neutron stars and $n = 3$ for the Sun and Sirius B. We infer from this table that the Sun and Sirius B have the same first zeros, since they have the same polytropic index. Furthermore, the values of the central quantities for the neutron stars are several orders of magnitude greater than those of the white dwarf Sirius B and the same occurs when we compare the values of the central quantities of the latter with those of the Sun. This behavior follows from the fact that smaller radius and a greater mass lead to an increase in the values of the central quantities.

From the comparison of the Lane–Emden equations in the post-Newtonian (16) and Newtonian (17) theories we note that the difference between them lies in the terms that are multiplied by $p_c/\rho_c c^2 = kT_c/mc^2$, which corresponds to the ratio of the thermal energy of the fluid at the star center kT_c and the rest energy of its particles $mc^2 = \mu m_\mu c^2$. This parameter was determined from the values of the central temperature T_c given in Table 1 and which are shown in Table 2.

We may conclude from Table 2 that the values of the ratio $p_c/\rho_c c^2 = kT_c/mc^2$ for the Sun and Sirius B are very small so that the post-Newtonian corrections to the Lane–Emden equation are negligible and the values given in Table 1 for these stars remain practically unchanged.

The post-Newtonian corrections are important for more massive stars like the neutron stars, since their central temperature is at least three orders of magnitude greater than those of the Sun and Sirius B and the ratio of the thermal energy at the star center and the rest energy of the particle is $kT_c/mc^2 \approx 10^{-1}$. In Table 3 the first zero and the values for the central quantities—calculated from the post-Newtonian Lane–Emden Equation (16)—are given for the neutron stars. We may

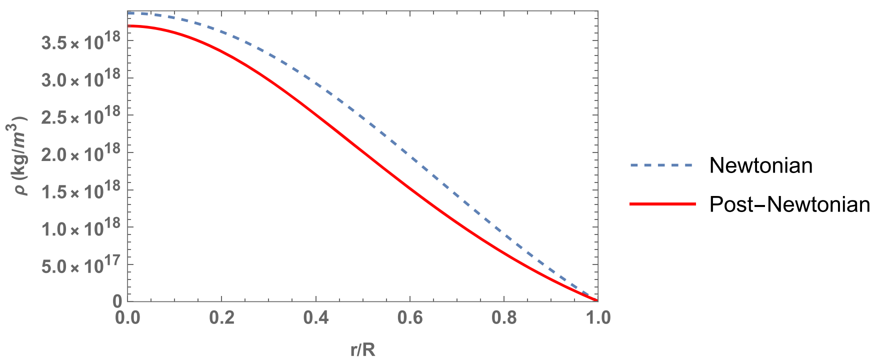

Figure 1. Mass densities ρ as functions of the normalized radius r/R for the neutron star of 1.8 $M_{\odot}$. Solid line—post-Newtonian solution, dashed line—Newtonian solution.

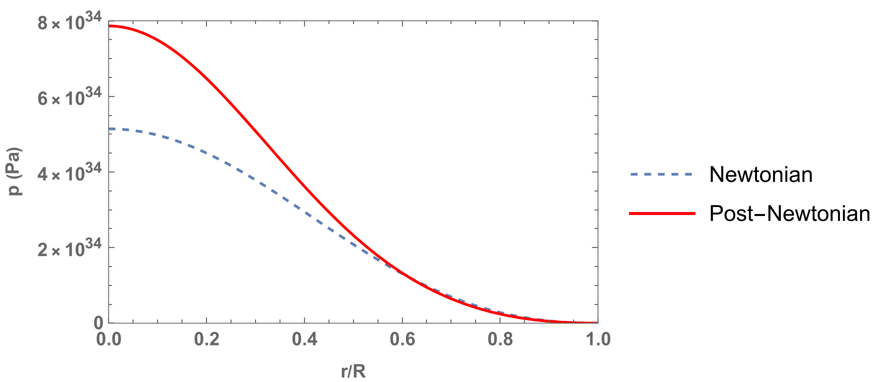

Figure 2. Pressures p as functions of the normalized radius r/R for the neutron star of 1.8 $M_{\odot}$. Solid line—post-Newtonian solution, dashed line—Newtonian solution.

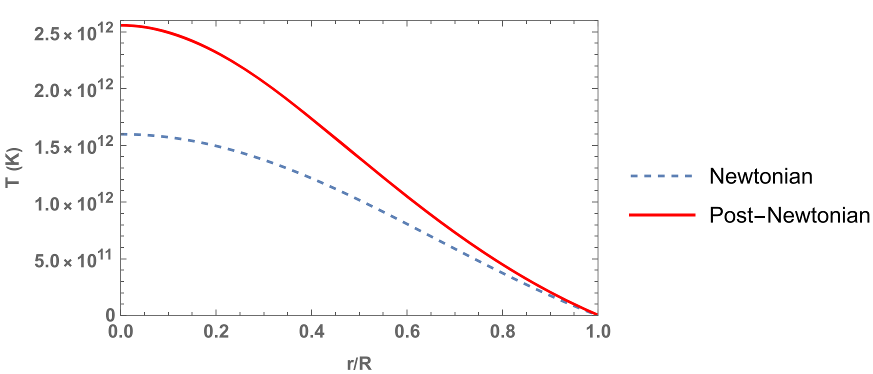

Figure 3. Temperatures T as functions of the normalized radius r/R for the neutron star of 1.8 $M_{\odot}$. Solid line—post-Newtonian solution, dashed line—Newtonian solution.

infer from the comparison of the values for the neutron stars given in Tables 1 and 2 that in the post-Newtonian theory the values for the central pressure and temperature are about fifty to sixty percent larger than those of the Newtonian theory, while the value for the central mass density is about three to four percent smaller.

From knowledge of the numerical solutions which follow from the Newtonian and post-Newtonian Lane–Emden equations for $u(z)$ and of the central quantities for ρ_c, p_c and T_c, one may obtain from (29) the behaviors of the mass density ρ, pressure p and temperature T as functions of the normalized radius r/R. In Figure 1 the mass density ρ for the neutron star with mass 1.8 $M_{\odot}$ is plotted as a function of normalized radius r/R, while Figures 2 and 3 represent the pressure p and temperature T, respectively. While the post-Newtonian solutions for the pressure and temperature are greater than those of the Newtonian ones, the Newtonian solution for the mass density is greater than the post-Newtonian solution. All three plots show that all fields have a monotonically decreasing behavior with respect to the normalized radius.

The value of the mass density at the crust can be obtained from the limiting value when $r/R \to 1$ and its value is of order 10^{15}, while from Figure 1 we infer that the mass density value at the center of the neutron star is of order 10^{18}. Both values are one order of magnitude greater than those reported in the literature. Note that here a polytropic equation of state was assumed and there are other equations of state that were proposed in the literature to describe properly the neutron stars (Haensel et al. 2007).

6. Conclusions

The aim of this work was to analyze the influence of the post-Newtonian corrections in the stellar structure equations. Starting from the post-Newtonian momentum density balance equation, the corresponding Lane–Emden equation was obtained. By assuming a polytropic equation of state, the solutions of the Lane–Emden equations in the Newtonian and post-Newtonian theories were determined. The physical quantities for the Sun, white dwarf Sirius B and neutron stars with masses $M \simeq 1.4\ M_\odot$, $1.8\ M_\odot$ and $2.0\ M_\odot$ were numerically calculated by considering the Newtonian and post-Newtonian solutions of the Lane–Emden equations. It was shown that the post-Newtonian corrections were negligible for the Sun and for Sirius B. For stars with strong fields the post-Newtonian corrections become important, so that for the neutron stars analyzed here the central pressure and the temperature which follow from the post-Newtonian Lane–Emden equation are about fifty to sixty percent greater than those of the Newtonian one and the central mass density is about three to four percent smaller.

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References

Agón, C. A., Pedraza, J. F., & Ramos-Caro, J. 2011, PhRvD, 83, 123007
André, R., & Kremer, G. M. 2017, RAA, 17, 122
Basu, S., & Antia, H. M. 2008, PhR, 457, 217
Capozziello, S., & De Laurentis, M. 2012, AnPhy, 524, 545
Capozziello, S., & Faraoni, V. 2011, Beyond Einstein Gravity (Dordrecht: Springer)
Chandrasekhar, S. 1957, An Introduction to the Study of Stellar Structure (Chicago, IL: Univ. Chicago Press)
Chandrasekhar, S. 1965, ApJ, 142, 1488
Chandrasekhar, S., & Nutku, Y. 1969, ApJ, 158, 55
Eddington, A. S. 1926, The Internal Constitution of the Stars (Cambridge: Cambridge Univ. Press)
Einstein, A., Infeld, L., & Hoffmann, B. 1938, AnMat, 39, 65
Epstein, R. 1977, ApJ, 216, 92
Farinelli, R., De Laurentis, M., Capozziello, S., & Odintsov, S. D. 2014, MNRAS, 440, 2894
Futamase, T., & Itoh, Y. 2007, LRR, 10, 2
Gupta, A., Gopakumar, A., Iyer, B. R., & Iyer, S. 2000, PhRvD, 62, 044038
Haensel, P., Potekhin, A. Y., & Yakovlev, D. G. 2007, Neutron Stars 1: Equation of State and Structure (New York: Springer)
Kremer, G. M. 2021, EPJC, 81, 927
Kremer, G. M. 2022a, Post-Newtonian Hydrodynamics: Theory and Applications (Newcastle upon Tyne: Cambridge Scholars Publishing)
Kremer, G. M. 2022b, Univ, 8, 179
Kremer, G. M., & Mehret, L. C. 2021, PhRvD, 104, 024056
Kremer, G. M., Richarte, M. G., & Weber, K. 2016, PhRvD, 93, 064073
Lattimer, J. M., & Prakash, M. 2001, ApJ, 550, 426
Nazari, E., Kazemi, A., Roshan, M., & Abbassi, S. 2017, ApJ, 839, 75
Noh, H., & Hwang, J.-C. 2021, ApJ, 906, 22
Özel, F., & Freire, P. 2016, ARA&A, 54, 401
Özel, F., et al. 2010, ApJL, 724, L199
Poisson, E., & Will, C. M. 2014, Gravity: Newtonian, Post-Newtonian, Relativistic (Cambridge: Cambridge Univ. Press)
Shinkai, H. 1999, PhRvD, 60, 067504
Weinberg, S. 1972, Gravitation and Cosmology: Principles and Applications of the General Theory of Relativity (New York: Wiley)