RAA 2017 Vol. **17** No. **12**, 122 (10pp) doi: 10.1088/1674–4527/17/12/122

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Research in Astronomy and Astrophysics

Stellar structure model in hydrostatic equilibrium in the context of $f(\mathcal{R})$-gravity

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Received 2017 July 1; accepted 2017 August 24

Abstract In this work we present a stellar structure model from the $f(\mathcal{R})$-gravity point of view capable of describing some classes of stars (white dwarfs, brown dwarfs, neutron stars, red giants and the Sun). This model is based on $f(\mathcal{R})$-gravity field equations for $f(\mathcal{R}) = \mathcal{R} + f_2\mathcal{R}^2$, hydrostatic equilibrium equation and a polytropic equation of state. We compare the results obtained with those found by Newtonian theory. It has been observed that in these systems, where high curvature regimes emerge, stellar structure equations undergo modifications. Despite the simplicity of this model, the results are satisfactory. The estimated values of pressure, density and temperature of the stars are within those determined by observations. This $f(\mathcal{R})$-gravity model has proved to be necessary to describe stars with strong fields such as white dwarfs, neutron stars and brown dwarfs, while stars with weaker fields, such as red giants and the Sun, are best described by Newtonian theory.

Key words: cosmology: theory — stars: general — massive — brown dwarfs — white dwarfs — neutron

1 INTRODUCTION

$f(\mathcal{R})$-gravity is a class of theories that represent an approach to gravitational interaction. In this context, general relativity (GR) has to be extended in order to solve several issues. This approach considers modifications of the Einstein-Hilbert action in order to include higher-order curvature invariants with respect to the Ricci scalar (Schmidt 2007; Sotiriou & Faraoni 2010). From an astrophysical and cosmological point of view, the objective is to explain phenomena such as dark energy and dark matter under a geometric pattern (Capozziello et al. 2004, 2005, 2007; Martins & Salucci 2007; Böhmer et al. 2008; Nojiri & Odintsov 2007a; Battye et al. 2016; Hu & Sawicki 2007; Tsujikawa 2010) with the possibility that gravitational interaction depends on scales. In this sense, in principle, these theories do not require the introduction of new particles and preserve all successful results from Einstein's theory, based on the same fundamental physical principles. It is also known that some $f(\mathcal{R})$ gravity models can pass tests performed in the weak-field of the solar system (Hu & Sawicki 2007). In addition, a considerable number of viable $f(\mathcal{R})$ models are known, among which we highlight Nojiri & Odintsov (2003, 2007b), Faulkner et al. (2007), Faraoni (2006) and Cognola et al. (2008). Furthermore, no extended gravity model, until this moment, can handle all phenomenology ranging from the quantum scale to the cosmological scale (Nojiri & Odintsov 2011). Another problem is that the description of the $f(\mathcal{R})$ theory is mostly equivalent to the description associated with the hypothesis of dark components. This interpretation arises from the fact that the degrees of freedom present in the $f(\mathcal{R})$ theory can be expressed by an effective energy-momentum tensor able to give rise to dark matter effects (Capozziello et al. 2012; Sotiriou & Faraoni 2010). From this picture emerges the necessity of observations capable of preserving or excluding one of these theories.

In this framework, several works related to relativistic stars considering extended gravity theories have been developed. In Cooney et al. (2010), using the method of perturbative constraints and corrections of the form $\mathcal{R}^{n+1}$, they showed that the predicted mass-radius relation for neutron stars differs from that calculated in GR. In Capozziello et al. (2016), the mass-radius diagram for a static neutron star was obtained in $f(\mathcal{R})$ gravity for two functions: $f(\mathcal{R}) = \mathcal{R} + \alpha\mathcal{R}^2(1+\gamma\mathcal{R})$ and $f(\mathcal{R}) = \mathcal{R}^{1+\epsilon}$. New terms related to curvature corrections emerged and modified the evolution of the mass-radius relation. In Astashenok et al. (2017), realistic models of relativistic stars in $f(\mathcal{R}) = \mathcal{R} + \alpha R^2$ gravity have been explored. In this context, the authors presented a study on the existence of neutron and quark stars for various α with no intermediate approximation in the system of equations. On the other hand, in Astashenok et al. (2015b), quark star models with realistic equation of state in nonperturbative $f(\mathcal{R})$ gravity have been considered. The authors showed that it is possible to discriminate modified theories of gravity from GR due to the gravitational redshift of the thermal spectrum emerging from the surface of the star. In Arapoğlu et al. (2016), a stellar structure model in $f(\mathcal{R}) = \mathcal{R} + \alpha\mathcal{R}^2$ theory was considered, using the method of matched asymptotic expansion to handle the higher order derivatives in field equations. Solutions were found for uniform density stars matching the Schwarzschild solution outside the star. The mass-radius relations were obtained, in which the dependence of maximum mass on α could be observed. In Astashenok et al. (2013), neutron star models in perturbative $f(\mathcal{R})$ have been considered with realistic equations of state. The mass-radius relations for $f(\mathcal{R}) = \mathcal{R} + \beta\mathcal{R}\Big[\exp(\frac{-\mathcal{R}}{\mathcal{R}_0}) - 1\Big]$ and $\mathcal{R}^2$ models with logarithmic and cubic corrections were obtained. In the case of cubic corrections, stable star configurations at high central density were obtained. Such an effect could give rise to more compact stars than in GR. In Alavirad & Weller (2013), considering a logarithmic $f(\mathcal{R})$ theory, the authors showed that the model exhibits a chameleon effect which completely eliminates the effect of the modification on scale exceeding a few radii, but close to the surface of the neutron star, the deviation from GR can significantly affect the surface redshift. In Astashenok et al. (2014), the authors showed that for a simple hyperon equation of state it is possible to obtain the maximal neutron star mass (which satisfies the recent observational data) in higher-derivative models with power-law terms as $f(\mathcal{R}) = \mathcal{R} + \gamma\mathcal{R}^2 + \beta\mathcal{R}^3$. In Astashenok et al. (2015a), the authors studied neutron stars with strong magnetic fields. They took into account models derived from $f(\mathcal{R})$ and $f(G)$ theories where functions of the Ricci curvature invariant $\mathcal{R}$ and the Gauss-Bonnet invariant G were respectively considered. In this model, the maximal mass of a neutron star had a considerable increase in $f(\mathcal{R})$ with cubic corrections.

In the model discussed in this work we propose second-order corrections in the Ricci scalar adopting $f(\mathcal{R}) = \mathcal{R} + f_2\mathcal{R}^2$. Such orders in the correction are an extension of GR and are particularly interesting in cosmology, since they allow the construction of a self-consistent inflationary model (Starobinsky 1980). We present the results obtained for a stellar model in hydrostatic equilibrium according to the $f(\mathcal{R})$-gravity theory. The goal is to control precisely how the results deviate from those obtained through GR in order to see how strong gravity regimes affect the pressure, temperature and density. We also compare these results with those related to the Newtonian theory, already known in the literature (Chandrasekhar 1957).

In our model we consider a polytropic equation of state, since this equation plays an important role in stellar structure models. This equation correctly represents the stellar gas behavior and, consequently, solves the fundamental problem of these structures together with the hydrostatic equilibrium equation. The motivation to employ the polytropic equation in the study of stellar structure is the simple nature of the polytropic structure and its correspondence with known classes of stars. Such simplicity provides a basis for the incorporation of additional effects (such as rotation), and thus an insight into the nature of the effects on true stars (Horedt 2004).

The interest of this model, in the present context, is manifested in the fact that, due to the expressive gravitational field, the interiors of stars can be seen as appropriate places to test alternative theories of gravity. In these regions, high curvature regimes can emerge and modify the stellar structure. In this way, we aim to show that the pressure, temperature and density can be consistently reached by extended theories of gravity, such as $f(\mathcal{R})$-gravity for $f(\mathcal{R}) = \mathcal{R} + f_2\mathcal{R}^2$ and how the expected changes occur in the values of these quantities.

This paper is organized as follows: In Section 2 we derive the modified Poisson and Lane-Emden equations through the Newtonian limit of $f(\mathcal{R})$-gravity. In Section 3 we obtain the modified stellar structure equations. In Section 4 we present the numerical solutions for pressure, temperature and density obtained for neutron stars, brown dwarfs, white dwarfs, red giants and the Sun, and we compare those with results obtained by Newtonian theory. Finally, we draw our conclusions in Section 5. In this work we will denote R as the radius of the star and $\mathcal{R}$ as the Ricci scalar.

2 THE LANE-EMDEN EQUATION FOR A MODEL DESCRIBED BY $f(\mathcal{R})$ THEORY

Considering a spherically symmetric self-gravitating system in equilibrium, we adopt the hydrostatic equilibrium equation presented below

$$\frac{d\phi}{dr} = \frac{1}{\rho}\frac{dp}{dr}, \tag{1}$$

where $\rho(r)$ is the matter density, p is the pressure and ϕ is the gravitational potential. Equation (1) is a Newtonian limit of the equation resulting from conservation of the stress-energy tensor for a perfect fluid in hydrostatic equilibrium. It can be also achieved through the Newtonian limit of the Tolman-Oppeheimmer-Volkoff equation (the equation employed to describe a spherically symmetric astrophysical system in equilibrium in GR (Rezzolla & Zanotti 2013; Landau & Lifshitz 1987)).

We assume a polytropic equation of state

$$p = k\rho^{\gamma}, \tag{2}$$

where k is the polytropic constant and γ is the polytropic exponent. Then we insert Equation (2) into Equation (1), obtaining

$$\frac{d\phi}{dr} = \gamma k \rho^{\gamma-2}\frac{d\rho}{dr}. \tag{3}$$

For $\gamma \neq 1$, integration of the above equation results in

$$\rho = \Big(\frac{\gamma-1}{\gamma k}\Big)^{\frac{1}{\gamma-1}} \phi^{\frac{1}{\gamma-1}} = \left[\frac{\phi}{(n+1)k}\right]^n, \tag{4}$$

where the chosen integration constant is $\phi = 0$ on the surface ($\rho = 0$). The constant n is known as the polytropic index and is defined as $n = \frac{1}{\gamma-1}$. Through Equations (2) and (4), we obtain the following expression for the pressure

$$p = \frac{\rho\phi}{(n+1)}. \tag{5}$$

To describe a stellar structure model by $f(\mathcal{R})$-gravity (Capozziello & Faraoni 2011; Capozziello & de Laurentis 2011) we adopt the action represented below

$$S = \int d^4x\sqrt{-g}\Big[f(\mathcal{R}) + \chi\mathcal{L}_m\Big], \tag{6}$$

where the Ricci scalar is only a function of metric tensor $\mathcal{R} \equiv \mathcal{R}(g)$, $f(\mathcal{R}) = \mathcal{R} + f_2\mathcal{R}^2$ and $\chi = 8\pi G$, with G denoting the gravitational constant. It is worth mentioning that we adopt the metric $(-,+,+,+)$ and natural units.

By varying the action according to metric formalism, we obtain the modified Einstein field equation in $f(\mathcal{R})$ theory

$$f'(\mathcal{R})\mathcal{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(\mathcal{R}) + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)f'(\mathcal{R}) = -\chi T_{\mu\nu}, \tag{7}$$

where $f'(\mathcal{R}) = \frac{df(\mathcal{R})}{d\mathcal{R}}$ and $T_{\mu\nu}$ is the energy-momentum tensor.
The trace of the above field equation reads

$$f'(\mathcal{R})\mathcal{R} - 2f(\mathcal{R}) + 3\Box f'(\mathcal{R}) = -\chi T^{\sigma}_{\sigma}. \tag{8}$$

In this model we will address the situation where particles in the system move at a very low speed (compared to the speed of light) and the gravitational field which they are subjected to is considered weak and static. Such requirements refer to the Newtonian limit (Weinberg 1972). In this way, writing Equations (7) and (8) in this limit, we have

$$\mathcal{R}^{(2)}_{00} + \frac{\mathcal{R}^{(2)}}{2} + \frac{1}{3m^2}\nabla^2\mathcal{R}^{(2)} = -\chi\rho, \tag{9}$$

$$\Big(1 + \frac{1}{m^2}\nabla^2\Big)\mathcal{R}^{(2)} = -\chi\rho, \tag{10}$$

where $m^2 = -\frac{1}{6f_2}$. Considering $(1 + \frac{1}{m^2}\nabla^2)$ as an operator, we can pass this term to the right-hand side of the equation and expand it in a Taylor series to the order of m^{-2}. Hence we rewrite Equation (10) as

$$\mathcal{R}^{(2)} \approx -\chi\Big(1 - \frac{1}{m^2}\nabla^2\Big)\rho. \tag{11}$$

It is important to emphasize here that the validity of this equation is assured for all the cases studied in this paper, since the term $\frac{1}{m^2}\nabla^2$ should always be small. This fact can be understood by observing that m is directly related to the coefficient f_2 and that it must be small since it is a correction of GR. There are some works where bounds for this coefficient are estimated and in all cases small values are presented for this correction coefficient (Berry & Gair 2011; Aviles et al. 2013).

Substituting the Newtonian limit for the temporal component of the Ricci tensor given by $\mathcal{R}^{(2)}_{00} = \nabla^2\phi$ (Weinberg 1972) and inserting Equations (4) and (11) in (9), we obtain in spherical coordinates the following equation

$$\frac{\partial^2\phi}{\partial r^2} + \frac{2}{r}\frac{\partial\phi}{\partial r} + \frac{4\pi G}{3m^2[(n+1)k]^n}\frac{\partial^2\phi^n}{\partial r^2} + \frac{8\pi G}{3m^2[(n+1)k]^n}\frac{1}{r}\frac{\partial\phi^n}{\partial r} = -\frac{4\pi G}{[(n+1)k]^n}\phi^n. \tag{12}$$

By defining dimensionless variables

$$z=\frac{r}{\xi},\ \ \omega(z)=\frac{\phi}{\phi_c}=\Big(\frac{\rho}{\rho_c}\Big)^{\frac{1}{n}}, \tag{13}$$

where the index c refers to the center of the star and

$$\xi=\sqrt{\frac{[(n+1)k]^n}{4\pi G}\phi_c^{(1-n)}}, \tag{14}$$

we obtain the Lane-Emden equation for this $f(\mathcal{R})$-gravity model

$$\begin{aligned}\frac{d^2\omega}{dz^2}+\frac{2}{z}\frac{d\omega}{dz}+\omega^n+\frac{1}{3m^2\xi^2}\frac{d^2\omega^n}{dz^2}\\ +\frac{2}{3m^2\xi^2}\frac{1}{z}\frac{d\omega^n}{dz}=0.\end{aligned} \tag{15}$$

Making $m\to\infty$, i.e. $f_2\to 0$, we recover the Lane-Emden equation for the model described by Newtonian gravity (see Capozziello et al. 2011). Through this equation it is possible to determine the physical quantities of the system such as pressure, density and temperature. Therefore, such models allow a simple description of stars and planets.

It is worth mentioning that the modified Lane-Emden equation was initially obtained in Capozziello et al. (2011) and subsequently in Farinelli et al. (2014). In the analysis presented in these cited articles and also in this work, the field equations for $f(\mathcal{R})$ gravity (in metric formalism), the polytropic equation and the hydrostatic condition, in the Newtonian limit, were considered as a system of equations. The difference lies in the fact that in Capozziello et al. (2011) and Farinelli et al. (2014), the modified Lane-Emden equation results in an integro-differential equation while in this work, it results only in a second-order differential equation. This happens due to the approach applied in Equation (10). Due to this approximation, we were able to write the modified Lane-Emden equation as shown in (15). However, we also verified that both cases present approximately the same solutions.

3 THE STELLAR STRUCTURE EQUATIONS ACCORDING TO $f(\mathcal{R})$-GRAVITY

Through the solution of the Lane-Emden equation we can write expressions for the physical quantities of stellar structure such as radius, mass, temperature, matter density and pressure. Certain values of n provide a description for a class of stars, for example, for $n=1$ the solution represents a neutron star, for $n=1.5$ we have the closest solution to completely convective stars such as red giants and brown dwarfs, and for $n=3$ the solution corresponds to a fully radiative star such as the Sun and stars with degenerate nuclei like white dwarfs (Chandrasekhar 1957; Hansen et al. 2004). Searching for a star description in this work, we analyze solutions only for the following values of n: 1, 1.5 and 3.

Through Equations (13) and (14), we obtain the radius $R=\xi z_{(n)}$ of the star given by

$$R=\sqrt{\frac{[(n+1)k]^n}{4\pi G}\phi_c^{(1-n)}}z_{(n)}, \tag{16}$$

where k is the polytropic constant and $z_{(n)}$ is the first zero of the solution. Since the boundary of the star is indicated by $\omega=0$, i.e. where $z=z_{(n)}$, in Newtonian theory we have radius of the star R and mass M of the star defined through the gravitational potential as (Eddington 1926)

$$R=(r)_{\omega=0}\ ,\ \ GM=\Big(-r^2\frac{d\phi}{dr}\Big)_{\omega=0}. \tag{17}$$

Through the dimensionless variables, we write

$$R'=(z)_{\omega=0}=z_{(n)}\ ,\ \ M'=\Big(-z^2\frac{d\omega}{dz}\Big)_{\omega=0}. \tag{18}$$

Therefore, the known data will be the radius R and mass M (see Eddington 1926).

In the same way, in the $f(\mathcal{R})$ theory, the radius and mass of the star will be the same (they are data), so we again define

$$R=(r)_{\omega=0},\ \ \ GM=\Big(-r^2\frac{d\phi}{dr}\Big)_{\omega=0}. \tag{19}$$

In dimensionless variables we write Equation (18) with corrections, obtaining

$$R'_{f(\mathcal{R})}=(z)_{\omega=0}=z_{(n)}, \tag{20}$$

$$M'_{f(\mathcal{R})}=\Bigg[\Big(-z^2\frac{d\omega}{dz}\Big)+\frac{1}{3m^2\xi^2}\Big(-z^2\frac{d\omega^n}{dz}\Big)\Bigg]_{\omega=0} \tag{21}$$

Thus $R'_{f(\mathcal{R})}$ and $M'_{f(\mathcal{R})}$ are corrections to R' and M' due to modified gravity, and R and M are the known data.

Therefore, in the $f(\mathcal{R})$-gravity model we can write

$$\frac{R}{R'_{f(\mathcal{R})}}=\sqrt{\frac{[(n+1)k]^n}{4\pi G}\phi_c^{(1-n)}}. \tag{22}$$

Note that in this case we have additional correction terms due to the $f(\mathcal{R})$-gravity. It is also important to note that the values of $z_{(n)}$ for the Newtonian model and $f(\mathcal{R})$-gravity model described by (18) and (20) differ from each other since they were derived from different solutions of the Lane-Emden equation (standard equation and modified equation for the $f(\mathcal{R})$ theory).

The mass $M(z)$ interior to z is given by (Chandrasekhar 1957)

$$M(z) = \int_0^{\xi z} 4\pi\rho r^2 dr = 4\pi\xi^3\rho_c \int_0^z z^2\omega^n dz. \quad (23)$$

Using the Lane-Emden equation (15), we integrate over the entire star, obtaining

$$M = 4\pi\xi^3\rho_c\left[\left(-z^2\frac{d\omega}{dz}\right) + \frac{1}{3m^2\xi^2}\left(-z^2\frac{d\omega^n}{dz}\right)\right]_{z_{(n)}} \quad (24)$$

Substituting expression (14) that defines ξ and expression (4), we have

$$M = 4\pi\left\{\frac{[(n+1)k]^n}{4\pi G}\phi_c^{(1-n)}\right\}^{\frac{3}{2}} \times \frac{\phi_c^n}{[(n+1)k]^n} \times\left[\left(-z^2\frac{d\omega}{dz}\right) + \frac{1}{3m^2\xi^2}\left(-z^2\frac{d\omega^n}{dz}\right)\right]_{z_{(n)}}. \quad (25)$$

Here we define a parameter α as

$$\alpha = m\xi = m\sqrt{\frac{[(n+1)k]^n}{4\pi G}\phi_c^{(1-n)}}. \quad (26)$$

Actually α is the free parameter of this model related to correction f_2 from modified gravity theory. From this definition, we rewrite Equation (25) as

$$GM = \left\{\frac{[(n+1)k]^n}{4\pi G}\right\}^{\frac{1}{2}} \times \phi_c^{\frac{(3-n)}{2}} \times\left[\left(-z^2\frac{d\omega}{dz}\right) + \frac{1}{3\alpha^2}\left(-z^2\frac{d\omega^n}{dz}\right)\right]_{z_{(n)}}. \quad (27)$$

According to (21), we can rewrite (27) as

$$\frac{GM}{M'_{f(\mathcal{R})}} = \left\{\frac{[(n+1)k]^n}{4\pi G}\right\}^{\frac{1}{2}}\phi_c^{\frac{(3-n)}{2}}. \quad (28)$$

Here M is the mass data of the star and $M'_{f(\mathcal{R})}$ is the correction calculated from the Lane-Emden solution.

Central condensation is defined as the ratio between the central density of the configuration and its mean density. Thus through the mean density of a star with radius $R = \xi z_{(n)}$ and expression (24) for mass, we obtain

$$\frac{\rho_c}{\bar{\rho}} = \left(-\frac{3}{z}\frac{d\omega}{dz} - \frac{1}{\alpha^2}\frac{1}{z}\frac{d\omega^n}{dz}\right)^{-1}_{z_{(n)}}. \quad (29)$$

In order to write ϕ_c as a function of mass and radius, we multiply (22) and (28), obtaining

$$\frac{GM}{M'_{f(\mathcal{R})}}\frac{R'_{f(\mathcal{R})}}{R} = \phi_c. \quad (30)$$

Thus, by replacing (29) and (30) in (5) in the central region (where $z = 0$) and through the definition of ω by (13) and the polytropic equation of state (2), we find an expression for the pressure

$$p = \frac{\bar{\rho}}{(n+1)}\frac{GM}{M'_{f(\mathcal{R})}}\frac{R'_{f(\mathcal{R})}}{R} \times\left(-\frac{3}{z}\frac{d\omega}{dz} - \frac{1}{\alpha^2}\frac{1}{z}\frac{d\omega^n}{dz}\right)^{-1}_{z_{(n)}}\omega^{(n+1)}. \quad (31)$$

The central pressure p_c is defined for $\omega = 1$.

To determine the central temperature of the configurations, we consider the ideal gas law and Equation (13), obtaining

$$T = \frac{p_c\mu m_\mu}{\rho_c k_B}\omega, \quad (32)$$

where k_B is the Boltzmann constant, μ is the atomic mass and m_μ is the atomic mass unit. Using (5) and (30), we write (32) as

$$T = \frac{G\mu m_\mu}{(n+1)k_B}\frac{M}{M'_{f(\mathcal{R})}}\frac{R'_{f(\mathcal{R})}}{R}\,\omega, \quad (33)$$

from which we define the central temperature T_c when $\omega = 1$. It is observed here that for $\alpha \to \infty$, i.e. $m \to \infty$ and $f_2 \to 0$ (keeping ξ constant), we recover stellar structure equations for the Newtonian model (see Chandrasekhar 1957; Eddington 1926).

4 THE POLYTROPIC SOLUTIONS

In this section, we obtain polytropic solutions for the stellar structure model according to Newtonian theory and $f(\mathcal{R})$-gravity theory for the following classes of stars: neutron stars, brown dwarfs, white dwarfs, red giants and the Sun. We present the results for pressure, temperature and density inside the star according to the value of n and the value of α which is related to f_2.

The curves that describe the behavior of the pressure inside the stars were obtained through expression (31) for the $f(\mathcal{R})$ theory and the same expressions with $f_2 = 0$ for the Newtonian model (see Chandrasekhar 1957; Eddington 1926). We numerically solve the Lane-Emden equation for the respective cases in order to obtain ω as a function of z, consequently p as a function of

z, and finally, as a function of the distance r from center to surface (where $r = R$) of the star. It is worth noting that in order to compare the curves, for all cases, we have normalized the radius.

To represent a neutron star in the solution $n = 1$, we choose a star, with mass $M = 2.01\,M_\odot$ and radius $R = 1.87 \times 10^{-5}\,R_\odot$, as *PSR J0348+0432* (Antoniadis et al. 2013). In the same way, we obtain the pressure values for a star described by $n = 1.5$ with mass $M = 0.053\,M_\odot$ and radius $R = 0.1\,R_\odot$, exemplified by brown dwarf *Teide 1* (Rebolo et al. 1996). Similarly, for $n = 1.5$ we calculate the pressure for a red giant star with mass $M = 1.5\,M_\odot$ (Ohnaka et al. 2013; Tsuji 2008) and radius $R = 44.2\,R_\odot$ (Richichi & Roccatagliata 2005), represented by *Aldebaran*. For $n = 3$, we obtain the pressure in a white dwarf with mass $M = 1.5\,M_\odot$ (Teerikorpi et al. 2009) and radius $R = 0.008\,R_\odot$ (Holberg et al. 1998), exemplified by *Sirius B*, and also use the Sun. Below we have curves with pressure values at the center and the surface of the stars for some values of α. The chosen values for α in this work were made after many computational tests. We verified that for values greater than 5, the Newtonian theory was always recovered for all cases studied. Thus we set an upper limit at $\alpha = 5$ and we investigated what would be the values of α which provide results (for pressure, temperature and density) closest to the observational data for the chosen stars. According to these tests for the chosen stars, in general, the values which provide results closest to the observational data were $\alpha = 0.5,\ 1,\ 5$. It is also important to explain that it is not possible to calculate the surface pressure of the star exactly where $\omega = 0$. In this case, according to Equation (31), when $\omega \to 0$, we have $p \to 0$. So in order to solve this problem, we make an approximation. We choose a point near the surface (point where ω is close to zero, but not zero) to calculate the surface pressure. This does not cause a problem because, as was verified in computational tests, the values obtained for the pressure in this region near the surface do not vary much (small variations are only in decimal digits). Therefore, we consider this approximation to obtain pressure on the surface. The same applies to surface temperature and density.

According to Zhao (2015), the estimated core pressure in a neutron star is 5.01×10^{34} Pa, therefore as can be seen through Figure 1(a) the central pressure results that are closest to this estimated value are those corresponding to the Newtonian model and $f(\mathcal{R})$-gravity model with $\alpha = 1$ and $\alpha = 5$.

In the case of a brown dwarf, we verify through Figure 1(d) that central pressure values for $\alpha = 1$, $\alpha = 5$ and the Newtonian model have the same order of magnitude as the expected value for this kind of star, about 10^{16} Pa according to Auddy et al. (2016). As can be seen in Figure 1(g), all values obtained for the core pressure in the reported red giant have the same order of magnitude as expected for a star in that category, approximately 10^8 Pa. For a white dwarf, analyzing Figure 1(j) we conclude that among the values of core pressure found, those that best fit the estimated value (approximately 4.95×10^{24} Pa in accordance with Teerikorpi et al. (2009)) are obtained for the Newtonian model and $f(\mathcal{R})$-gravity with $\alpha = 5$. Through Figure 1(m) we observe that, for the Newtonian model and $\alpha = 5$, we find the best results for pressure in the Sun's core, since according to Williams (2016) the estimated value is 2.477×10^{16} Pa.

In order to analyze the behavior of the temperature in the same stars, we use Equation (33) for the $f(\mathcal{R})$-gravity model and the version of these equations for the Newtonian theory (Chandrasekhar 1957). It should be remembered that each type of star has an estimated value for atomic mass μ according to its composition (see table 1). Also, in order to obtain the temperature as a function of R (normalized), we have solved the Lane-Emden equation with the temperature equations mentioned above. Likewise, the surface temperature was calculated considering an ω value close to it. According to Hong et al. (2016), the expected value for the central temperature of a neutron star is 1.50×10^{11} K, therefore observing Figure 1(b) we conclude that the results closest to this value are obtained for $\alpha = 0.5$ and $\alpha = 1$. Unfortunately, satisfactory data for surface temperature of this star were not found in the literature. As can be seen in Figure 1(e), the results that best fit the expected central and surface temperature values for this brown dwarf correspond to $\alpha = 1$, since the surface temperature is 2700 K and the core temperature is 2.7×10^6 K, according to Auddy et al. (2016).

As can be seen in Figure 1(h), the central temperatures in this red giant have the same order of magnitude for all α values considered. However, an exact value for temperature in the center of a star like that is not known, since data found in the literature diverge. Therefore it was not possible to confront the values obtained for the models portrayed in this work in a secure way. The surface temperature in this red giant, according to its spectral classification, is about 3.8×10^3 K (Gray et al. 2006). Thus we find that the best values are those

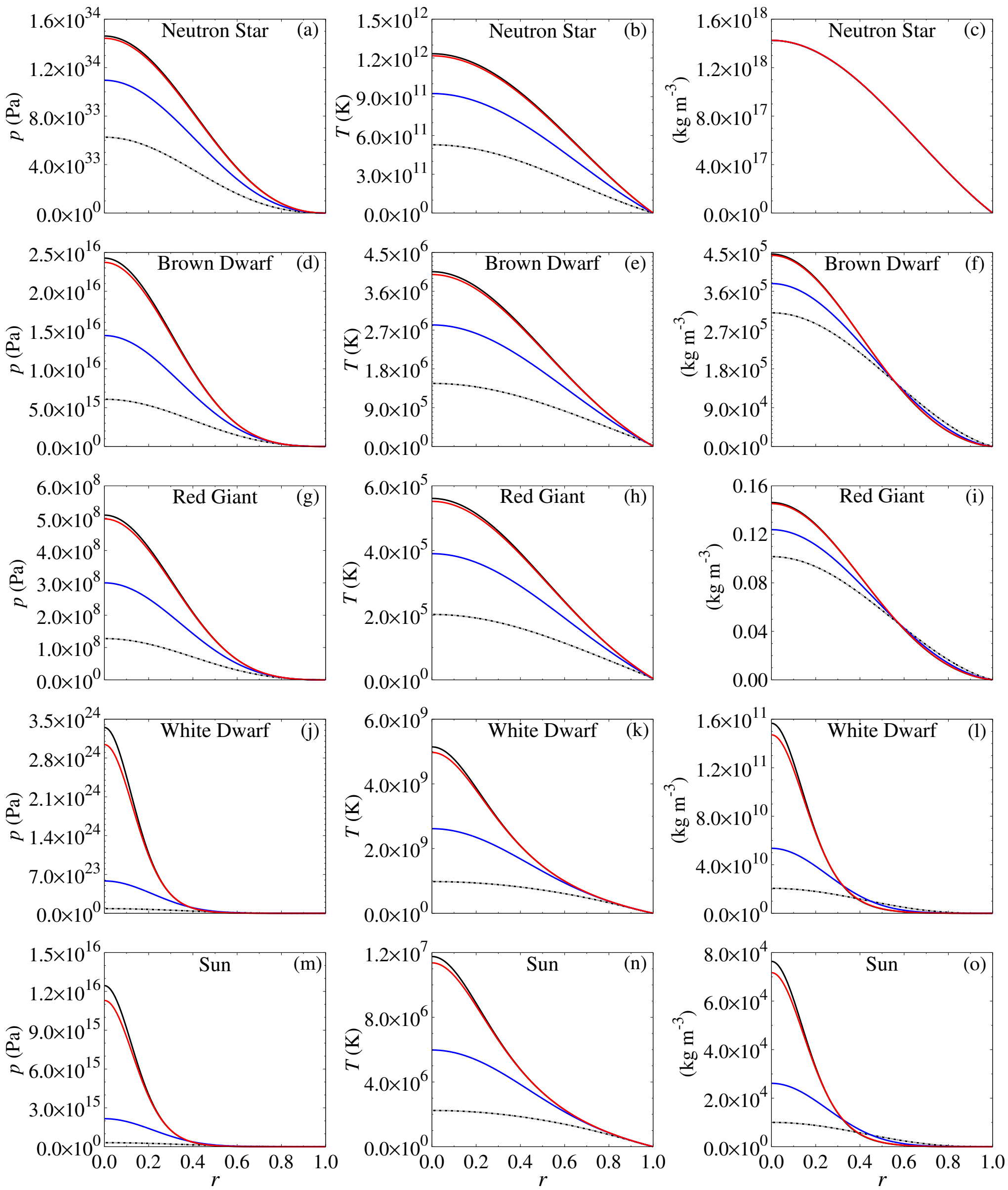

Fig. 1 Behavior of pressure, temperature and density within some classes of stars (neutron star, brown dwarf, red giant, white dwarf and the Sun) described by Newtonian and $f(\mathcal{R})$-gravity stellar structure models. The $f(\mathcal{R})$-gravity model adopted here considers $f(\mathcal{R}) = \mathcal{R} + f_2\mathcal{R}^2$. The curves in *black* represent the Newtonian model results, the *dot-dashed* curves represent the $f(\mathcal{R})$-gravity model results for $\alpha = 0.5$, the *blue curves* are for $\alpha = 1$ and the red curves are for $\alpha = 5$.

related to $\alpha = 5$ and the Newtonian model. In the case of a white dwarf, we verify from Figure 1(k) that all values for the central temperature are one or two orders of magnitude higher than what is expected for this star. In compliance with Teerikorpi et al. (2009), the estimated surface temperature is 2.52×10^4 K and estimated central temperature is 2.20×10^7 K. For the surface temperature, the value relative to $\alpha = 1$ generates the best result. According to Williams (2016), the estimated surface temperature of the Sun is 5780 K and its central temperature is 1.571×10^7 K. Thus, as can be seen in Figure 1(n), among the values obtained for temperature in the Sun's core, those that best fit the data are related to $\alpha = 5$ and the Newtonian model. For the surface temperature, all values are compatible with the data.

To determine the behavior of matter density in the stellar interior, we solve Equation (29) considering the mean density calculated through the mass M and the radius R of the star. Undoubtedly it is expected in all situations that the density will decrease as it approaches

Table 1 Estimates of atomic mass and composition (mass fraction) for each type of star. The mass fraction of hydrogen is represented by X, the mass fraction of helium by Y and the heavy metals by Z (Wilking et al. 1999; Cercignani & Kremer 2002; Asplund et al. 2009).

Star	Composition (Mass Fraction)	μ
Neutron Star	Neutrons	1.00
Brown Dwarf	$X = 0.70, Y = 0.28, Z = 0.02$	0.62
White Dwarf	$X = 0, Y = 0, Z = 1$	2.00
Red Giant	$X = 0, Y = 0.98, Z = 0.02$	1.34
Sun	$X = 0.73, Y = 0.25, Z = 0.02$	0.60

the surface as observed with temperature and pressure. The density at the surface can be obtained considering a point near the interface of the star that is connected with the outside where ω vanishes, according to the boundary conditions adopted. For a neutron star (see Fig. 1(c)), we have the case where all the curves coincide, so the values obtained for the central and superficial density do not present considerable differences. Therefore, in this situation, all values of central density are close to the reported value for a neutron star with the aspects described above. According to Zhao (2015) the expected value of the core density is $1.50 \times 10^{18}\,\mathrm{kg\,m^{-3}}$. In the case of a brown dwarf, the central density is estimated between $10^{3}\,\mathrm{kg\,m^{-3}}$ and $10^{6}\,\mathrm{kg\,m^{-3}}$. Therefore, through Figure 1(f), we conclude that the core density resulting from the models discussed in this work presents values within the order of magnitude expected for a brown dwarf with the characteristics mentioned previously (Burrows & Liebert 1993; Rebolo et al. 1996). For red giants, the core densities, shown in Figure 1(i), have the same order of magnitude. Again, data found in the literature for this star are not congruent, which makes it difficult to compare with the results obtained. For the central density of a white dwarf, according to Figure 1(l), the best results are those found for $\alpha = 0.5$ and $\alpha = 1$, since the estimated value is $3.30 \times 10^{10}\,\mathrm{kg\,m^{-3}}$, in compliance with Teerikorpi et al. (2009). Observing Figure 1(o), we verify that the central densities obtained for the Sun present values with an order of magnitude lower than that assured by the data: $1.622 \times 10^{5}\,\mathrm{kg\,m^{-3}}$ according to Williams (2016), including the Newtonian polytropic model, which is considered as a reasonable model for describing the Sun.

Through Figure 1, it has been found that in all cases, curves show a more rapid decrease as α decreases. Therefore, the more the model $f(\mathcal{R})$ departs from Newtonian theory, the lower the predicted pressure, temperature and density in the center of the star.

5 CONCLUSIONS

In this work we analyzed stellar structure from the point of view of $f(\mathcal{R})$-gravity. The reason that led us to adopt such an approach is the fact that higher-order curvature corrections can emerge in intense gravitational field regimes, as occurs within stars. In this scheme, it is reasonable to assume that the emergence of these corrections may generate effects on pressure, temperature and density, for example. Thus, in the stellar structure model equations, new terms related to the curvature corrections lead to different behaviors of these magnitudes. It is worth mentioning that since these quadratic terms arise in strong field regimes, in the solar system scale we have the weak field scheme where only linear terms of the Ricci scalar $\mathcal{R}$ are relevant.

In order to perform the analysis of stellar structure under the focus of this extended theory of gravity, we started with an action that represents the $f(\mathcal{R})$-gravity models, adopting as function $f(\mathcal{R}) = \mathcal{R} + f_2\mathcal{R}^2$, the hydrostatic equilibrium equation and the polytropic equation of state. These equations are employed in the non-relativistic approximation, making the model a simple description of the complex behavior of these stellar object classes. The goal is to illustrate the effects of the $f(\mathcal{R})$-gravity theory adopted in this work on an easily understandable model. According to this assumption, we numerically solved the equations responsible for stellar structure, evidencing the role that the corrections in the curvature play in these equations. In this way, the expressions for pressure, density and temperature depend strictly on the values of these corrections. Interpreting these additional terms as corrections of GR, we control deviations from the model with respect to Einstein's theory. It is worth mentioning that in solutions from the $f(\mathcal{R})$-gravity model, we noticed the presence of an extra degree of freedom α related to the correction f_2.

Through Figure 1(a) – 1(o), temperature, pressure and density of different classes of stars, described by Newtonian theory and the $f(\mathcal{R})$-gravity model, were compared numerically. These quantities were plotted against the normalized radius of the star. Observing these graphics, we found that by increasing the correction term f_2, and consequently decreasing α (according to Eq. (26), keeping ξ constant, α and f_2 are inversely proportional), these physical quantities within the star exhibit a faster decrease and lower core values. In this way, the more the extended theory moves away from Newtonian theory (more α decreasing), the lower the pressure and the temperature become inside the star. With respect to density, for $n = 1$ the curves coincide for all values of α chosen, including for the Newtonian case. For the other values of n, we observed the same behavior that was described for the pressure and temperature: the lower α, the lower the central density of the star and the faster the curve decreases. All this leads us to believe that the magnitude of gravitational corrections changes the stellar structure. Therefore, by increasing the value of parameter f_2, the original pressure, density and temperature of GR are affected by a term that modifies the mass and radius of the star in some way. It is important to emphasize that we did not use exotic matter, only the usual (baryonic) matter. We adopted the *Jordan frame*, so that the results of the gravitational sector were corrected while the matter sector was not affected. Thus, the geodesic structure was not changed and the standard polytropic state equation could be assumed.

By analyzing the graphics it can be seen that, in general, the best description of neutron stars, brown dwarfs and white dwarfs was achieved by the parameter $\alpha = 1$. This fact corroborates the expected behavior of the physical quantities in stars with intense gravitational fields. It is worth mentioning that this value of α represents a case in which curvature is more accentuated and, consequently, the gravitational field is more intense. Therefore, in this work we show that, in these cases, for a better description, a model of stellar structure according to the $f(\mathcal{R})$-gravity theory is needed, since the contribution of the corrections in the quantities is more expressive. Another issue to be discussed is based on the analysis of results obtained for neutron stars. For these stars, some results found did not show significant differences between the values obtained by the $f(\mathcal{R})$ and Newtonian models, although the model with $\alpha = 1$ was closer than the observed values for this star. Differences as apparent as those observed in the white dwarfs were expected, since neutron stars also have intense gravitational fields. We justify this fact by noting that the equation of state used here (polytropic equation) is not the most adequate, since it does not consider the quantum effects present in this type of star. The red giants and the Sun (stars with weak fields) were better represented by the Newtonian model and $\alpha = 5$ (in all cases reported in this work, the curves corresponding to this parameter value are almost coincident with the Newtonian's curve). This reinforces a fact observed in the graphics: for values $\alpha = 5$, we have already been able to recover the Newtonian description for the stars. In this way, we can conclude through this work that in the case where the associated gravitational field is weak, the best description is obtained through a Newtonian model, not necessarily the approach through an $f(\mathcal{R})$ theory.

As demonstrated, this work was intended to indicate the possibility of describing some classes of stars in polytropic models under a different assumption about gravity. The study of these systems in this approach may be important for testing $f(\mathcal{R})$-gravity theories, since strong gravitational field regimes are located in stars. Despite the simplicity of the model, the results are satisfactory. The estimated values for pressure, density and temperature are within those determined by observations. The results of this work can be extended to stars with magnetic and rotating fields, for example, and for different equations of state.

We also emphasize that the results obtained in this work could not be compared with other results from other articles, including those cited throughout this manuscript, except in the case of Newtonian theory, whose comparison was performed and presented here. Until now, all papers found in the literature have different motivations and objectives from those explored by us.

To sum up, we have determined the density, pressure and temperature fields for stars by using an $f(\mathcal{R})$-gravity model and compared with the results that came out from the Newtonian theory. The stars analyzed were of two types: with strong fields such as white dwarfs, neutron stars and brown dwarfs and with weak fields such as red giants and the Sun. The $f(\mathcal{R})$-gravity model has proved to be necessary for the description of stars with strong fields, and as was expected the Newtonian theory provides a good description for stars with weak fields.

Acknowledgements The authors acknowledge the financial support of CNPq (Brazil).

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