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Application of a new semi-analytical method to periodic motion due to the J_{22} tesseral harmonic *

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Abstract We revisit the issue of constructing the first-order periodic solution that incorporates the J_{22} tesseral harmonic and developing a new semi-analytical solution that may apply to any orbital eccentricity in [0,1). In our work, the solution is expressed in a finite compact form composed of several definite integrals with varying integration intervals constrained in [0,π], in which the traditional Hansen coefficients are no longer involved. Numerical experiments are also given and compared with the traditional series expansion method, and the results show that the derived solution is capable of dealing with highly eccentric orbits. Therefore, the solution given can provide a new technique to analyze the perturbation characteristics arising from the J_{22} harmonic.

Key words: celestial mechanics

1 INTRODUCTION

The motion of a satellite orbiting around a central body is influenced by many perturbations. Among these, the inhomogeneous central gravitational perturbation plays an important role in its motion. Although numerical methods may provide greater accuracy, analytical theories (Brouwer 1959; Kozai 1959; Garfinkel 1958; Aksnes 1970) have an advantage of showing a clearly dynamical picture of the motion. Analytical investigations provide an extensive understanding of the gravitational effects on satellite orbits (Hori 1966; Deprit 1969, 1981; Alfriend & Coffey 1984; Wnuk 1999).

One of the interesting issues associated with the analysis of longitude-dependent tesseral harmonic perturbation has been extensively investigated by many researchers since the work of Kaula (1966). Generally, the first-order solutions of tesseral harmonics present short-period effects, and occasionally long-period effects occur due to resonances (Kaula 1966; Wnuk 1988; Wnuk & Breiter 1990; Rossi 2008; Sampaio et al. 2012). The coupling effects between them have been investigated by several authors and their characteristics were also shown (Wnuk & Breiter 1991; Metris et al. 1993; Palacián 2007; Zhou et al. 2012). Besides the classical methods of elliptic motion expansion (Kaula 1966), the relegation methods based on canonical simplifications of a Hamiltonian system with the elimination of the parallax (Deprit 1981; Deprit et al. 2001) were also used to deal with the tesseral perturbation (Segerman & Coffey 2000; Palacián 2007). Recently, Lara et al. (2013) compared the two methods in detail, both analytically and numerically.

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Although relegation methods make a closed form of the solution possible, the traditional series expansion methods are still widely used in aerospace engineering. A theory developed by Proulx et al. (1981) deals with the solution that is represented in the form of double Fourier series of the mean-longitude and the Greenwich sidereal time, where the Hansen coefficients were evaluated in an efficient way (Proulx & McClain 1988). However, the series expansion methods show a slow convergence because Hansen coefficients converge very slowly when the eccentricity increases up to 1. To solve problems that arise in the case of highly eccentric orbits, a series of work was presented by Brumberg (1992), Brumberg & Fukushima (1994), and Brumberg et al. (1995), in which the authors replaced the traditional mean, true or eccentric anomaly by an elliptic anomaly and represented the solution in a more compact and more quickly convergent form than the conventional methods. This method is a good approach in celestial mechanics, where the concept of anomalies is extended and a Hansen-like coefficient is also introduced. However, this method is complicated and not widely used in mathematics, which has limited its application.

In the present work, we revisit the tesseral problem with only the J_{22} harmonic considered, and we present here a new first-order semi-analytical solution that is applicable to any eccentricity in [0,1). The solution is represented in a finite compact form, composed of the definite integrals with varying integration intervals constrained in [0,π]. In addition, the semi-analytical solution is also suitable for numerical computation. In Section 2, we give the perturbation function arising from the J_{22} harmonic. Our theory and method are presented in detail in Section 3. In Section 4, we show numerical results from the solution, and then compare it with the traditional series method. Finally, we present a brief discussion in Section 5.

2 PERTURBATION MODEL

The perturbing function of a satellite orbiting around a central body due to the J_{22} harmonic may be expressed in Keplerian elements as follows

$$U_{22} = \frac{3J_{22}\mu a_e^2}{r^3}\left[\cos^4\frac{I}{2}\cos(2u+2\Omega_r)+\sin^4\frac{I}{2}\cos(2u-2\Omega_r)+\frac{\sin^2 I}{2}\cos(2\Omega_r)\right], \quad (1)$$

where $J_{22}=\sqrt{C_{22}^2+S_{22}^2}$. C_{22} and S_{22} are the corresponding spherical harmonic coefficients. a_e is the mean equatorial radius of the central body. a, e, I, Ω and ω are the semimajor axis, eccentricity, inclination, longitude of the ascending node and argument of periastron of the satellite orbit, respectively. $r = a(1-e^2)$ is the radial distance and μ is the product of the gravitational constant and mass of the central body. $u = f + \omega$ and $\Omega_r = \Omega - S_0 - n_r(t-t_0)$. f is the true anomaly. n_r is the angular velocity of the central body. S_0 is the local sidereal time (or the hour angle of the vernal equinox) at time t_0 and the longitude λ_{22}. λ_{22} is computed by

$$\cos(2\lambda_{22}) = \cos\left(\frac{C_{22}}{\sqrt{C_{22}^2+S_{22}^2}}\right), \quad \sin(2\lambda_{22}) = \sin\left(\frac{S_{22}}{\sqrt{C_{22}^2+S_{22}^2}}\right).$$

To make the problem dimensionless, a new length scale $[L]$ and time scale $[T]$ are introduced, where $[L]=a_e$, $[T]=\sqrt{\mu/a_e^3}$. Thus Equation (1) can be rewritten as

$$U_{22} = \frac{3J_{22}}{r^3}\left[\cos^4\frac{I}{2}\cos(2u+2\Omega_r)+\sin^4\frac{I}{2}\cos(2u-2\Omega_r)+\frac{\sin^2 I}{2}\cos(2\Omega_r)\right]. \quad (2)$$

In the expression for U_{22}, there are two fast variables: f and Ω_r. However, Ω_r can be considered as a slow variable for a central body with slow rotation, such as the Moon, Venus, etc. Here we emphasize that only cental bodies with fast rotation are considered in this work, such as the Earth, Mars, Jupiter, asteroid (433) Eros, etc., even though the solution is also valid for in general for bodies with slow rotation. Moreover, the non-resonance condition is also assumed, which ensures the applicability of our solution.

3 THEORY AND SOLUTION

3.1 Lagrange's Planetary Equations

The perturbations to six Keplerian elements arising from perturbing function U_{22} are given in the following Lagrange's planetary equations

$$\begin{aligned}
\frac{\mathrm{d}a}{\mathrm{d}t} &= \frac{2}{n_\mathrm{s} a}\frac{\partial U_{22}}{\partial M},\\
\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{1-e^2}{n_\mathrm{s} a^2 e}\frac{\partial U_{22}}{\partial M} - \frac{\sqrt{1-e^2}}{n_\mathrm{s} a^2 e}\frac{\partial U_{22}}{\partial \omega},\\
\frac{\mathrm{d}i}{\mathrm{d}t} &= \frac{1}{n_\mathrm{s} a^2\sqrt{1-e^2}\sin i}(\cos i\frac{\partial U_{22}}{\partial \omega} - \frac{\partial U_{22}}{\partial \Omega}),\\
\frac{\mathrm{d}\Omega}{\mathrm{d}t} &= \frac{1}{n_\mathrm{s} a^2\sqrt{1-e^2}\sin i}\frac{\partial U_{22}}{\partial i},\\
\frac{\mathrm{d}\omega}{\mathrm{d}t} &= \frac{\sqrt{1-e^2}}{n_\mathrm{s} a^2 e}\frac{\partial U_{22}}{\partial e} - \cos i\frac{\mathrm{d}\Omega}{\mathrm{d}t},\\
\frac{\mathrm{d}M}{\mathrm{d}t} &= n_\mathrm{s} - \frac{1-e^2}{n_\mathrm{s} a^2 e}\frac{\partial U_{22}}{\partial e} - \frac{2}{n_\mathrm{s} a}\frac{\partial U_{22}}{\partial a},
\end{aligned} \tag{3}$$

where M is the mean anomaly and $n_\mathrm{s} = \sqrt{1/a^3}$ is the mean motion of a satellite. The first-order perturbation σ_{22}^i due to U_{22} is represented in terms of mean elements as follows

$$\sigma_{22}^i(t) = \int\left\{\left(\frac{\mathrm{d}\sigma^i}{\mathrm{d}t}\right) + \left[\kappa^i\frac{\partial n_\mathrm{s}}{\partial a}a_{22}(t)\right]\right\}\mathrm{d}t \quad (i=1,2,...,6), \tag{4}$$

where σ^i $(i=1,2,...,6)$ represents six Keplerian elements, which are a, e, I, Ω, ω and M. $\frac{\mathrm{d}\sigma^i}{\mathrm{d}t}$ are given by Equation (3), such that $\kappa^i = 0$ when $i = 1,2,...,5$ and $\kappa^6 = 1$.

We cannot obtain the explicit analytical solution of Equation (4) directly because of rapid variation in the two terms f and t. We will follow the conventional Fourier expansion method in multiples of the mean anomaly and some additional mathematical techniques are introduced.

3.2 Two Trigonometric Series

In order to derive the formula, two trigonometric series are first introduced as follows

$$\sum_{k=1}^{\infty}\left[\frac{\cos kx}{k+\beta} - \frac{\cos kx}{k-\beta}\right] = \pi\frac{\cos(\beta x_m - \beta\pi)}{\sin\beta\pi} - \frac{1}{\beta}, \tag{5a}$$

$$\sum_{k=1}^{\infty}\left[\frac{\sin kx}{k+\beta} + \frac{\sin kx}{k-\beta}\right] = -\pi\frac{\sin(\beta x_m - \beta\pi)}{\sin\beta\pi}, \tag{5b}$$

where $\beta\in\mathbb{R}$ and $\beta\notin\mathbb{Z}$, $x\in\mathbb{R}$ and $x\neq 2k\pi$, $k\in\mathbb{Z}$. $x_m = \mathrm{mod}(x, 2\pi)$. Herein the function mod (X, Y) means the modulus of division of X by Y. Therefore, we have $x_m\in[0, 2\pi)$.

Equation (5) is adopted when we derive the semi-analytical expressions of $a_{22}(t)$, $e_{22}(t)$, $I_{22}(t)$, $\Omega_{22}(t)$, $\omega_{22}(t)$ and $M_{22}(t)$.

3.3 Mathematical Treatments

To deal with the integrals, we introduce the following three types of differential equations

$$\begin{aligned}\dot{\mathbf{I}}_{p,q}^{n,m} &= \left(\frac{a}{r}\right)^n \sin\left(mf + q\Omega_r + p\omega\right) n_\mathrm{s} \\ &= \left(\frac{a}{r}\right)^n \sin\left(mf - qn_r t + s\right) n_\mathrm{s}, \end{aligned} \tag{6a}$$

$$\begin{aligned}\dot{\mathbf{J}}_{p,q}^{n,m} &= \left(\frac{a}{r}\right)^n \cos\left(mf + q\Omega_r + p\omega\right) n_\mathrm{s} \\ &= \left(\frac{a}{r}\right)^n \cos\left(mf - qn_r t + s\right) n_\mathrm{s}, \end{aligned} \tag{6b}$$

$$\dot{\mathbf{K}}_{p,q}^{n,m} = n_\mathrm{s} \mathbf{I}_{p,q}^{n,m}, \tag{6c}$$

where

$$\dot{\mathbf{I}}_{p,q}^{n,m} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{I}_{p,q}^{n,m}, \dot{\mathbf{J}}_{p,q}^{n,m} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{J}_{p,q}^{n,m}, \dot{\mathbf{K}}_{p,q}^{n,m} = \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{K}_{p,q}^{n,m}$$

and n, m, p, q are integers, $s = p\omega + q\left(\Omega - S_0 + n_r t_0\right)$, and f and r are functions of time t. Equation (6c) is simply used for $M_{22}(t)$. r, f and t are involved in the integration. Note that only the primitive functions $\mathbf{I}_{p,q}^{n,m}$, $\mathbf{J}_{p,q}^{n,m}$ and $\mathbf{K}_{p,q}^{n,m}$ that do not contain constant terms are considered. For Equations (6a)–(6c), according to the conventional approach of Fourier expansion in multiples of M, we have

$$\mathbf{I}_{p,q}^{n,m} = -\sum_{k=-\infty}^{k=\infty} X_k^{-n,m} \frac{\cos\left(kM - qn_r t + s\right)}{k - q\alpha}, \tag{7a}$$

$$\mathbf{J}_{p,q}^{n,m} = \sum_{k=-\infty}^{k=\infty} X_k^{-n,m} \frac{\sin\left(kM - qn_r t + s\right)}{k - q\alpha}, \tag{7b}$$

$$\mathbf{K}_{p,q}^{n,m} = -\sum_{k=-\infty}^{k=\infty} X_k^{-n,m} \frac{\sin\left(kM - qn_r t + s\right)}{\left(k - q\alpha\right)^2}, \tag{7c}$$

where $\alpha = n_r/n_\mathrm{s}$ and the Fourier coefficients $X_k^{-n,m}$ are Hansen coefficients, defined by

$$X_k^{-n,m} = \frac{1}{\pi}\int_0^\pi \left(\frac{a}{r}\right)^n \cos\left(mf - kM\right) \mathrm{d}M. \tag{8}$$

Note that all the expressions of Equation (7) apply to $0 \leq e < 1$.

To continue the derivation of Equation (7), we will show the detailed derivation process for Equation (7a). Substituting Equation (8) into Equation (7a) yields

$$\mathbf{I}_{p,q}^{n,m} = -\sum_{k=-\infty}^{k=\infty} \left(\frac{1}{\pi}\int_0^\pi \left(\frac{a}{r'}\right)^n \cos\left(mf' - kM'\right) \mathrm{d}M'\right) \frac{\cos\left(kM - qn_r t + s\right)}{k - q\alpha},$$

where the mark $'$ represents that this variable is involved in the integration.

With the assumption of the non-resonance condition that $q\alpha \notin \mathbb{Z}$, the sequence of summation and integration may be exchanged as follows

$$\begin{aligned}\mathbf{I}_{p,q}^{n,m} = -\frac{1}{\pi}\int_0^\pi \left(\frac{a}{r'}\right)^n &\left\{\cos mf' \sum_{k=-\infty}^{k=\infty}\left[\frac{\cos kM' \cos\left(kM - qn_r t + s\right)}{k - q\alpha}\right]\right. \\ &\left. + \sin mf' \sum_{k=-\infty}^{k=\infty}\left[\frac{\sin kM' \cos\left(kM - qn_r t + s\right)}{k - q\alpha}\right]\right\} \mathrm{d}M'.\end{aligned}$$

With the product-to-sum formulae, we have

$$
\begin{aligned}
\mathbf{I}_{p,q}^{n,m} = & -\frac{1}{2\pi}\int_0^{\pi}\left(\frac{a}{r'}\right)^n \times \\
& \left\{\cos mf' \sum_{k=-\infty}^{k=\infty}\left[\frac{\cos\left(k{z_1}'-qn_rt+s\right)+\cos\left(k{z_2}'+qn_rt-s\right)}{k-q\alpha}\right]\right. \\
& \left.+\sin mf' \sum_{k=-\infty}^{k=\infty}\left[\frac{\sin\left(k{z_1}'-qn_rt+s\right)+\sin\left(k{z_2}'+qn_rt-s\right)}{k-q\alpha}\right]\right\}\mathrm{d}M',
\end{aligned}
$$

where ${z_1}' = M' + M$ and ${z_2}' = M' - M$. Separating the summation, we obtain

$$
\begin{aligned}
\mathbf{I}_{p,q}^{n,m} = & -\frac{1}{2\pi}\int_0^{\pi}\left(\frac{a}{r'}\right)^n\left\{-\cos mf'\frac{2\cos\left(-qn_rt+s\right)}{q\alpha}\right. \\
& +\cos mf'\sum_{k=1}^{k=\infty}\left[-\sin\left(-qn_rt+s\right)\left(\frac{1}{k+q\alpha}+\frac{1}{k-q\alpha}\right)\sin k{z_1}'\right. \\
& -\cos\left(-qn_rt+s\right)\left(\frac{1}{k+q\alpha}-\frac{1}{k-q\alpha}\right)\cos k{z_1}' \\
& +\sin\left(-qn_rt+s\right)\left(\frac{1}{k+q\alpha}+\frac{1}{k-q\alpha}\right)\sin k{z_2}' \\
& \left.-\cos\left(-qn_rt+s\right)\left(\frac{1}{k+q\alpha}-\frac{1}{k-q\alpha}\right)\cos k{z_2}'\right] \\
& +\sin mf'\sum_{k=1}^{k=\infty}\left[\cos\left(-qn_rt+s\right)\left(\frac{1}{k+q\alpha}+\frac{1}{k-q\alpha}\right)\sin k{z_1}'\right. \\
& -\sin\left(-qn_rt+s\right)\left(\frac{1}{k+q\alpha}-\frac{1}{k-q\alpha}\right)\cos k{z_1}' \\
& +\cos\left(-qn_rt+s\right)\left(\frac{1}{k+q\alpha}+\frac{1}{k-q\alpha}\right)\sin k{z_2}' \\
& \left.\left.+\sin\left(-qn_rt+s\right)\left(\frac{1}{k+q\alpha}-\frac{1}{k-q\alpha}\right)\cos k{z_2}'\right]\right\}\mathrm{d}M'.
\end{aligned}
$$

Considering that the measure of the points in the set $\{M'|M'+M = 2k\pi, k \in \mathbb{N}\} \cup \{M'|M'-M = 2k\pi, k \in \mathbb{N}\}$ which do not satisfy Equation (5) is zero, thus we obtain

$$
\begin{aligned}
\mathbf{I}_{p,q}^{n,m} = & \frac{\cos\left(-qn_rt+s\right)}{\sin q\alpha\pi}\int_0^{\pi}\left(\frac{a}{r'}\right)^n \\
& \times\left[\cos\left(mf'-q\alpha\frac{{z_{1m}}'+{z_{2m}}'}{2}+q\alpha\pi\right)\cos\left(q\alpha\frac{{z_{1m}}'-{z_{2m}}'}{2}\right)\right]\mathrm{d}M' \\
& -\frac{\sin\left(-qn_rt+s\right)}{\sin q\alpha\pi}\int_0^{\pi}\left(\frac{a}{r'}\right)^n \\
& \times\left[\cos\left(mf'-q\alpha\frac{{z_{1m}}'+{z_{2m}}'}{2}+q\alpha\pi\right)\sin\left(q\alpha\frac{{z_{1m}}'-{z_{2m}}'}{2}\right)\right]\mathrm{d}M',
\end{aligned}
\tag{9}
$$

where $z_{1m}{}' = \mathrm{mod}(z_1{}', 2\pi)$, $z_{2m}{}' = \mathrm{mod}(z_2{}', 2\pi)$. The result can be further formulated as

$$\begin{aligned}\mathbf{I}_{p,q}^{n,m} =& \frac{\cos\left(\delta_{p,q} - q\alpha\pi\right)}{\sin q\alpha\pi} \int_0^{\pi} \left(\frac{a}{r'}\right)^n \cos\left(mf' - q\alpha M'\right) \mathrm{d}M' \\ &+ \int_{\pi}^{M} \left(\frac{a}{r'}\right)^n \sin\left(mf' - q\alpha M' + \delta_{p,q}\right) \mathrm{d}M',\end{aligned} \tag{10}$$

where $\delta_{p,q} = q\left(\Omega_r + \alpha M\right) + p\omega$ and $M = \mathrm{mod}\left(M_0 + n_{\mathrm{s}}\left(t - t_0\right), 2\pi\right)$.

For $\mathbf{J}_{p,q}^{n,m}$, applying (10) and performing a partial derivative with respect to s, we can easily obtain

$$\begin{aligned}\mathbf{J}_{p,q}^{n,m} =& -\frac{\sin\left(\delta_{p,q} - q\alpha\pi\right)}{\sin q\alpha\pi} \int_0^{\pi} \left(\frac{a}{r'}\right)^n \cos\left(mf' - q\alpha M'\right) \mathrm{d}M' \\ &+ \int_{\pi}^{M} \left(\frac{a}{r'}\right)^n \cos\left(mf' - q\alpha M' + \delta_{p,q}\right) \mathrm{d}M'.\end{aligned} \tag{11}$$

For $\mathbf{K}_{p,q}^{n,m}$, considering the following equation

$$\mathbf{K}_{p,q}^{n,m} = n_{\mathrm{s}}\left(t - t_0\right) \mathbf{I}_{p,q}^{n,m} - \frac{1}{q}\frac{\partial}{\partial\alpha}\mathbf{J}_{p,q}^{n,m}$$

then we have

$$\begin{aligned}\mathbf{K}_{p,q}^{n,m} =& -\frac{1}{2}\int_0^{\pi} \left(\frac{a}{r'}\right)^n \times \left[\frac{2\pi}{\sin^2 q\alpha\pi} \cos\left(mf' - q\alpha\frac{z_{1m}{}' + z_{2m}{}'}{2}\right)\right. \\ &\times \sin\left(q\alpha\frac{z_{1m}{}' - z_{2m}{}'}{2} - qn_r t + s\right) \\ &- \frac{z_{1m}{}'}{\sin q\alpha\pi} \cos\left(mf' - q\alpha z_{1m}{}' + q\alpha\pi + qn_r t - s\right) \\ &\left.+ \frac{z_{2m}{}'}{\sin q\alpha\pi} \cos\left(mf' - q\alpha z_{2m}{}' + q\alpha\pi - qn_r t + s\right)\right] \mathrm{d}M',\end{aligned} \tag{12}$$

or

$$\begin{aligned}\mathbf{K}_{p,q}^{n,m} =& M \cdot \mathbf{I}_{p,q}^{n,m} - \frac{\pi \sin\delta_{p,q}}{\sin^2 q\alpha\pi} \int_0^{\pi} \left(\frac{a}{r'}\right)^n \cos\left(mf' - q\alpha M'\right) \mathrm{d}M' \\ &+ \frac{\sin\left(\delta_{p,q} - q\alpha\pi\right)}{\sin q\alpha\pi} \int_0^{\pi} \left(\frac{a}{r'}\right)^n M' \sin\left(mf' - q\alpha M'\right) \mathrm{d}M' \\ &- \int_{\pi}^{M} \left(\frac{a}{r'}\right)^n M' \sin\left(mf' - q\alpha M' + \delta_{p,q}\right) \mathrm{d}M'.\end{aligned} \tag{13}$$

Now, we have already converted the infinite series defined by Equation (7) to definite integrals shown in Equations (10), (11) and (13) with an integration interval $[0, \pi]$ and a varying interval $[\pi, M]$. These results have compact forms compared with the infinite series given in Equation (7) and are also valid for $0 \leq e < 1$. The definite integral parts of Equations (10), (11) and (13) may be evaluated with numerical quadrature methods, thus they may be regarded as semi-analytical expressions of Equation (7) in this sense.

However, we note that $\delta_{p,q}$ is a discontinuous function in time because of the mod() function that applies to M, which indicates that each term of the definite integrals in Equations (10), (11), and (13) is discontinuous whereas the summation results are continuous.

3.4 The Semi-Analytical Solution in Terms of Keplerian Elements

Using the formulae above, we derive the first-order perturbation expressions due to J_{22} in terms of Keplerian elements.

For J_{22}, $q = 2$ is substituted in Equations (10), (11) and (13). Noting that

$$\mathbf{I}^{n,m}_{p,q} = -\mathbf{I}^{n,-m}_{-p,-q}\,, \quad \mathbf{J}^{n,m}_{p,q} = \mathbf{J}^{n,-m}_{-p,-q}\,, \quad \mathbf{K}^{n,m}_{p,q} = -\mathbf{K}^{n,-m}_{-p,-q}\,,$$

and applying the following shorter notation

$$\mathbf{I}^{n,m}_{p} = \mathbf{I}^{n,m}_{p,2}\,, \quad \mathbf{J}^{n,m}_{p} = \mathbf{J}^{n,m}_{p,2}\,, \quad \mathbf{K}^{n,m}_{p} = \mathbf{K}^{n,m}_{p,2}\,, \quad \delta_p = \delta_{p,2}\,,$$

we obtain

$$a_{22}(t) = 2a^2U_{22} - \frac{12J_{22}\alpha}{a}\left(\cos^4\frac{I}{2}\mathbf{I}^{3,2}_2 + \sin^4\frac{I}{2}\mathbf{I}^{3,-2}_{-2} + \frac{\sin^2 I}{2}\mathbf{I}^{3,0}_0\right), \tag{14a}$$

$$e_{22}(t) = \frac{\eta^2}{2ae}a_{22}(t) + \frac{6J_{22}\eta}{a^2e}\left(\cos^4\frac{I}{2}\cdot\mathbf{I}^{3,2}_2 - \sin^4\frac{I}{2}\cdot\mathbf{I}^{3,-2}_{-2}\right), \tag{14b}$$

$$I_{22}(t) = \frac{3J_{22}\sin I}{a^2\eta}\left(\cos^2\frac{I}{2}\mathbf{I}^{3,2}_2 + \sin^2\frac{I}{2}\mathbf{I}^{3,-2}_{-2} + \mathbf{I}^{3,0}_0\right), \tag{14c}$$

$$\Omega_{22}(t) = \frac{3J_{22}}{a^2\eta}\left(-\cos^2\frac{I}{2}\mathbf{J}^{3,2}_2 + \sin^2\frac{I}{2}\mathbf{J}^{3,-2}_{-2} + \cos I\cdot\mathbf{J}^{3,0}_0\right), \tag{14d}$$

$$\omega_{22}(t) = \omega_1(t) - \cos I\cdot\Omega_{22}(t), \tag{14e}$$

$$\begin{aligned} M_{22}(t) = &-\eta\cdot\omega_1(t) + \frac{9J_{22}}{a^2}\left[\cos^4\frac{I}{2}\cdot\left(\mathbf{J}^{3,2}_2 + 2\alpha\mathbf{K}^{3,2}_2\right)\right.\\ &\left.-\sin^4\frac{I}{2}\cdot\left(\mathbf{J}^{3,-2}_{-2} + 2\alpha\mathbf{K}^{3,-2}_{-2}\right) + \frac{\sin^2 I}{2}\cdot\left(\mathbf{J}^{3,0}_0 + 2\alpha\mathbf{K}^{3,0}_0\right)\right], \end{aligned} \tag{14f}$$

where

$$\begin{aligned} \omega_1(t) = \frac{3J_{22}\eta}{2a^2e}&\left[\cos^4\frac{I}{2}\left(\mathbf{J}^{4,1}_2 + 5\mathbf{J}^{4,3}_2 - \frac{2}{1-e^2}\mathbf{J}^{3,1}_2 + \frac{2}{1-e^2}\mathbf{J}^{3,3}_2\right)\right.\\ &+\sin^4\frac{I}{2}\left(\mathbf{J}^{4,-1}_{-2} + 5\mathbf{J}^{4,-3}_{-2} - \frac{2}{1-e^2}\mathbf{J}^{3,-1}_{-2} + \frac{2}{1-e^2}\mathbf{J}^{3,-3}_{-2}\right)\\ &\left.+\frac{3}{2}\sin^2 I\left(\mathbf{J}^{4,1}_0 + \mathbf{J}^{4,-1}_0\right)\right] \end{aligned} \tag{15}$$

and $\eta = \sqrt{1-e^2}$. The indirect term induced by $a_{22}(t)$ is included in Equation (14f). The following relation

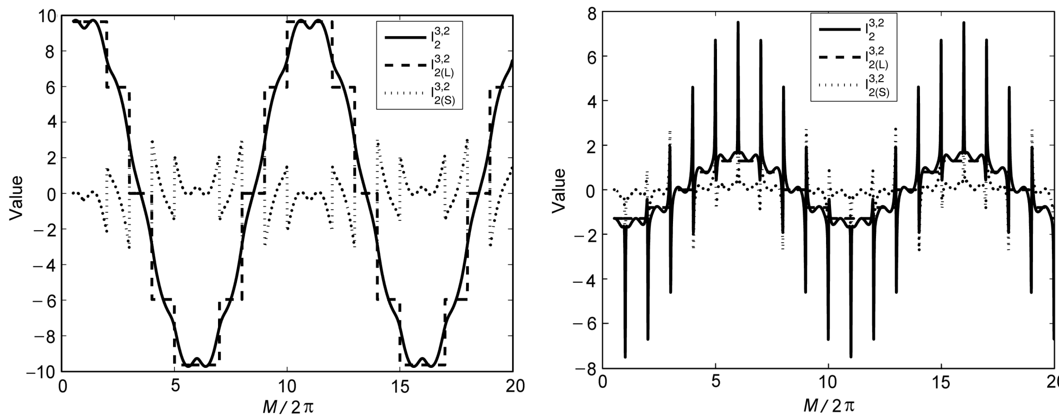
$$n_\mathrm{s}\int\frac{\partial U_{22}}{\partial M}\mathrm{d}t = U_{22} - \int\frac{\partial U_{22}}{\partial t}\mathrm{d}t \tag{16}$$

is utilized during the derivation of Equation (14a).

3.5 Notes on $\mathbf{I}^{n,m}_p$ and $\mathbf{J}^{n,m}_p$

In particular, for $\mathbf{I}^{n,m}_p$ and $\mathbf{J}^{n,m}_p$, each expression is divided into two parts, where one may be denoted as the L term $\mathbf{I}^{n,m}_{p(L)}$ and $\mathbf{J}^{n,m}_{p(L)}$ and the other as the S term $\mathbf{I}^{n,m}_{p(S)}$ and $\mathbf{J}^{n,m}_{p(S)}$, that is

$$\mathbf{I}^{n,m}_p = \mathbf{I}^{n,m}_{p(L)} + \mathbf{I}^{n,m}_{p(S)}, \quad \mathbf{J}^{n,m}_p = \mathbf{J}^{n,m}_{p(L)} + \mathbf{J}^{n,m}_{p(S)}, \tag{17}$$



(a) (b)

Fig. 1 Variations of $\mathbf{I}_p^{3,2}$, $\mathbf{I}_{2(L)}^{3,2}$ and $\mathbf{I}_{2(S)}^{3,2}$ with M for different e and α, where M is the mean anomaly without applying the mod function. α is set to approximate the resonance condition. The time span covers 20 orbital periods. (a) $e = 0.1, \alpha = 1.05$, (b) $e = 0.75, \alpha = 1.05$.

where

$$\begin{cases} \mathbf{I}_{p(L)}^{n,m} = A \cdot \cos\left(\delta_p - 2\alpha\pi\right), \\ \mathbf{I}_{p(S)}^{n,m} = \displaystyle\int_\pi^M \left(\frac{a}{r'}\right)^n \sin\left(mf' - q\alpha M' + \delta_{p,q}\right) \mathrm{d}M', \\ \mathbf{J}_{p(L)}^{n,m} = -A \cdot \sin\left(\delta_p - 2\alpha\pi\right), \\ \mathbf{J}_{p(S)}^{n,m} = \displaystyle\int_\pi^M \left(\frac{a}{r'}\right)^n \cos\left(mf' - q\alpha M' + \delta_{p,q}\right) \mathrm{d}M', \\ A = \dfrac{1}{\sin q\alpha\pi} \displaystyle\int_0^\pi \left(\frac{a}{r'}\right)^n \cos\left(mf' - q\alpha M'\right) \mathrm{d}M'. \end{cases}$$

Note that

$$\begin{aligned} \delta_p &= 2\left(\Omega_r + \alpha M\right) + p\omega \\ &= -4\alpha N\pi + p\omega + 2\left(\Omega - S_0 + \alpha M_0\right), \quad N \in \mathbb{Z}, \\ &M_0 + n_{\mathrm{s}}\left(t - t_0\right) = M + 2N\pi. \end{aligned}$$

Because of the discontinuity of δ_p at the periastron, the L terms and S terms are discontinuous functions of time.

The division above is a somewhat artificial treatment. Nevertheless, they have different properties in general. As 2α approaches a certain integer, the amplitude A will increase to as large as infinity, indicating that the L terms contain potential-resonance terms but the S terms do not.

Figure 1 shows the variation of $\mathbf{I}_2^{3,2}$, $\mathbf{I}_{2(L)}^{3,2}$, and $\mathbf{I}_{2(S)}^{3,2}$ with time near the resonance condition for different eccentricities, where we can clearly see that the L term $\mathbf{I}_{2(L)}^{3,2}$ depicts the averaged change in near-resonance cases.

Interestingly, the amplitude of the L term A can also be expressed in a kind of generalized Hansen coefficient as follows

$$A = \frac{\pi X_{2\alpha}^{-n,m}}{\sin 2\alpha\pi}. \tag{18}$$

Since $\alpha \in \mathbb{R}$, the definition given by Equation (8) has been generalized from $k \in \mathbb{Z}$ to $k \in \mathbb{R}$. Such treatment differs from those presented by Breiter et al. (2004) and Laskar (2005). For near-resonance cases, the value of $X_{2\alpha}^{-n,m}$ is essential because it directly reflects the amplitude of the resonance effect. Thus in the framework of the first-order solution, the evaluation of A shows us a simple way to evaluate the resonance strength for near-resonance cases.

4 COMPUTATION OF $\mathbf{I}_P^{N,M}$, $\mathbf{J}_P^{N,M}$ AND $\mathbf{K}_P^{N,M}$

First, for an estimation, we have

$$\begin{cases} |\mathbf{I}_p^{n,m}| < \dfrac{\pi(1-e)^{2-n}\eta^{-1}}{\sin 2\alpha\pi}\,, & \text{(19a)} \\[2ex] |\mathbf{J}_p^{n,m}| < \dfrac{\pi(1-e)^{2-n}\eta^{-1}}{\sin 2\alpha\pi}\,, & \text{(19b)} \\[2ex] |\mathbf{K}_p^{n,m}| < \dfrac{\pi^2(1-e)^{2-n}\eta^{-1}}{\sin^2 2\alpha\pi} + \dfrac{2\pi(1-e)^{2-n}\eta^{-1}}{\sin 2\alpha\pi}\,. & \text{(19c)} \end{cases}$$

The proofs are not complicated. For Equation (19a), considering Equation (9), we have

$$\begin{aligned} |\mathbf{I}_p^{n,m}| &< \left| \frac{1}{\sin 2\alpha\pi} \int_0^{\pi} \left(\frac{a}{r'}\right)^n \mathrm{d}M' \right| \\ &= \frac{1}{\sin 2\alpha\pi} \int_0^{\pi} \frac{1}{\eta} \left(\frac{a}{r'}\right)^{n-2} \mathrm{d}f' \\ &< \frac{1}{\sin 2\alpha\pi} \int_0^{\pi} \frac{1}{(1-e)^{n-2}\eta} \mathrm{d}f' \\ &< \frac{\pi(1-e)^{2-n}\eta^{-1}}{\sin 2\alpha\pi}\,. \end{aligned}$$

The proof of Equation (19b) is similar and we can also easily prove Equation (19c) using Equation (12).

Equations (19a)–(19c) are very rough estimations because of several overestimations. Nevertheless, they give the clear upper boundary of $\mathbf{I}_p^{n,m}$, $\mathbf{J}_p^{n,m}$ and $\mathbf{K}_p^{n,m}$ when $2\alpha \notin \mathbb{Z}^+$. However, it is hard for us to obtain this property if we use the infinite series solution directly, especially for high eccentricity cases. If the estimated upper boundary of $\mathbf{I}_p^{n,m}$, $\mathbf{J}_p^{n,m}$ or $\mathbf{K}_p^{n,m}$ is less than the error for some actual problems, the effect of J_{22} may be ignored.

The conventional analytical method to compute $\mathbf{I}_p^{n,m}$, $\mathbf{J}_p^{n,m}$ and $\mathbf{K}_p^{n,m}$ may involve two kinds of series. One is used to compute the Hansen coefficients and the other is the series Equation (7) used to compute the final results. Both of them converge slowly when large eccentricity is present, which makes the estimation of truncation error difficult.

However, for our solution, methods that rely on numerical quadrature, such as the adaptive Simpson's method (Lyness 1969), can be used to directly compute the definite integrals appearing in Equation (14) for a given precision.

For the purpose of improving numerical efficiency, the following representations are used for the case $0 \leq M \leq \pi$

$$\begin{cases} \mathbf{I}_p^{n,m} = \dfrac{\cos(\delta_p - 2\alpha\pi)}{\sin 2\alpha\pi} \displaystyle\int_0^f F_1 \mathrm{d}f' + \dfrac{\cos\delta_p}{\sin 2\alpha\pi} \displaystyle\int_f^\pi F_2 \mathrm{d}f', \\ \mathbf{J}_p^{n,m} = -\dfrac{\sin(\delta_p - 2\alpha\pi)}{\sin 2\alpha\pi} \displaystyle\int_0^f F_1 \mathrm{d}f' - \dfrac{\sin\delta_p}{\sin 2\alpha\pi} \displaystyle\int_f^\pi F_2 \mathrm{d}f', \\ \mathbf{K}_p^{n,m} = \dfrac{1}{\sin 2\alpha\pi} \left(\displaystyle\int_0^f G_1 \mathrm{d}f' + \displaystyle\int_f^\pi G_2 \mathrm{d}f' \right), \end{cases}$$

and the following equations are used for the case $\pi \leq M < 2\pi$

$$\begin{cases} \mathbf{I}_p^{n,m} = \dfrac{\cos(\delta_p - 2\alpha\pi)}{\sin 2\alpha\pi} \displaystyle\int_0^{2\pi - f} F_1 \mathrm{d}f' + \dfrac{\cos(\delta_p - 4\alpha\pi)}{\sin 2\alpha\pi} \displaystyle\int_{2\pi - f}^\pi F_2 \mathrm{d}f', \\ \mathbf{J}_p^{n,m} = -\dfrac{\sin(\delta_p - 2\alpha\pi)}{\sin 2\alpha\pi} \displaystyle\int_0^{2\pi - f} F_1 \mathrm{d}f' - \dfrac{\sin(\delta_p - 4\alpha\pi)}{\sin 2\alpha\pi} \displaystyle\int_{2\pi - f}^\pi F_2 \mathrm{d}f', \\ \mathbf{K}_p^{n,m} = \dfrac{1}{\sin 2\alpha\pi} \left(\displaystyle\int_0^{2\pi - f} G_1 \mathrm{d}f' + \displaystyle\int_{2\pi - f}^\pi G_3 \mathrm{d}f' \right), \end{cases}$$

where

$$\begin{cases} F_1 = \dfrac{1}{\sqrt{1-e^2}} \left(\dfrac{a}{r'}\right)^{n-2} \cos(mf' - 2\alpha M'), \\ F_2 = \dfrac{1}{\sqrt{1-e^2}} \left(\dfrac{a}{r'}\right)^{n-2} \cos(mf' - 2\alpha M' + 2\alpha\pi), \\ F_3 = \dfrac{1}{\sqrt{1-e^2}} \left(\dfrac{a}{r'}\right)^{n-2} M' \sin(mf' - 2\alpha M'), \\ F_4 = \dfrac{1}{\sqrt{1-e^2}} \left(\dfrac{a}{r'}\right)^{n-2} M' \sin(mf' - 2\alpha M' + 2\alpha\pi), \end{cases}$$

and

$$\begin{cases} G_1 = -\dfrac{\pi \sin\delta_p}{\sin 2\alpha\pi} F_1 + M \cos(\delta_p - 2\alpha\pi) F_1 + \sin(\delta_p - 2\alpha\pi) F_3, \\ G_2 = -\dfrac{\pi \sin\delta_p}{\sin 2\alpha\pi} F_1 + M \cos\delta_p F_2 + \sin\delta_p F_4, \\ G_3 = -\dfrac{\pi \sin(\delta_p - 4\alpha\pi)}{\sin 2\alpha\pi} F_1 + (M - 2\pi) \cos(\delta_p - 4\alpha\pi) F_2 \\ \qquad + \sin(\delta_p - 4\alpha\pi) F_4. \end{cases}$$

As mentioned above, the mark $'$ represents that the variable is involved in the integration. These equations restrict the integration intervals to $[0, \pi]$. Moreover, the original integration variable (mean anomaly) has been converted to a true anomaly, which will generally be more numerically efficient for cases of moderate and high eccentricity.

Numerical experiments are carried out for $\mathbf{I}_2^{3,2}$ with our method (denoted as method A). Averaged evaluation numbers of integrand functions in one orbit for a provided absolute precision of 10^{-3} are given with different eccentricities and five different celestial bodies, where the software, Matlab's built-in function 'quad' using adaptive Simpson quadrature, is used. A fixed radial distance at periastron $r_p = 1.05$ is assumed and α may be calculated for a given eccentricity. Table 1 lists the results.

A comparison with the traditional series expansion method (denoted as method B) based on Equation (7) is interesting. With a truncated order of D_1 and D_2, for $q = 2$, Equation (7) can be rewritten as

$$\mathbf{I}_p^{n,m} = -\sum_{k=-D_1}^{k=D_2} X_k^{-n,m} \frac{\cos\left(kM + 2\Omega_r + p\omega\right)}{k - 2\alpha},$$

$$\mathbf{J}_p^{n,m} = \sum_{k=-D_1}^{k=D_2} X_k^{-n,m} \frac{\sin\left(kM + 2\Omega_r + p\omega\right)}{k - 2\alpha},$$

$$\mathbf{K}_p^{n,m} = -\sum_{k=-D_1}^{k=D_2} X_k^{-n,m} \frac{\sin\left(kM + 2\Omega_r + p\omega\right)}{\left(k - 2\alpha\right)^2}.$$

However, comparing the methods used for computing in the two methods is naturally different. It is hard for us to compare their numerical efficiency fairly. Nevertheless, for a comparison, Table 2 lists the numbers of required computed Hansen coefficients ($D_1 + D_2$) for the same precision. The results clearly show the advantage of our method over the series expansion method becomes more and more obvious with the growth in eccentricity. The issue of fussy computation with Hansen coefficients is another disadvantage that is apparent in results from the latter method.

Table 1 Evaluation Numbers of Integrand for $\mathbf{I}_2^{3,2}$ — Method A

	$e = 0.2$	$e = 0.4$	$e = 0.6$	$e = 0.7$	$e = 0.8$	$e = 0.85$	$e = 0.9$	$e = 0.95$
Earth	28	28	28	28	32	31	38	98
Mars	28	28	28	28	39	33	36	47
Jupiter	26	28	28	29	35	64	58	84
(433) Eros	28	28	26	33	63	51	84	153
(4) Vesta	27	36	26	34	34	75	59	88

Table 2 Numbers of Computed Hansen Coefficients for $\mathbf{I}_2^{3,2}$ — Method B

	$e = 0.2$	$e = 0.4$	$e = 0.6$	$e = 0.7$	$e = 0.8$	$e = 0.85$	$e = 0.9$	$e = 0.95$
Earth	7	9	16	34	55	155	241	442
Mars	7	9	16	34	54	155	241	442
Jupiter	7	9	15	34	54	154	242	441
(433) Eros	7	9	16	33	54	154	242	439
(4) Vesta	7	9	15	33	54	154	242	439

5 CONCLUSIONS

In this work we develop a new semi-analytical theory to deal with the perturbation due to the J_{22} tesseral harmonic and derive the first order semi-analytical solution in terms of six Keplerian elements. Unlike the traditional series solution based on elliptic motion expansion, this solution, which is applicable to any eccentricity in [0,1), has a finite compact form with several definite integrals, and the Hansen coefficients that usually appear in other series expansion solutions are no longer involved.

The solution is based on three kinds of integrals $\mathbf{I}_p^{n,m}$, $\mathbf{J}_p^{n,m}$ and $\mathbf{K}_p^{n,m}$ which may be rewritten as the summation of several definite integrals. We find that the expressions of $\mathbf{I}_p^{n,m}$ and $\mathbf{J}_p^{n,m}$ naturally separate the potential-resonance terms from the others. In addition, the amplitudes of the potential-resonance terms are related to a kind of generalized Hansen coefficients. Therefore, we may evaluate the amplitude of the solution for near-resonance cases to give a rough judgment of the resonance strength due to the J_{22} perturbation. In addition, the boundaries of $\mathbf{I}_p^{n,m}$, $\mathbf{J}_p^{n,m}$ and $\mathbf{K}_p^{n,m}$ are also easily given, which reflects another merit of our method.

The numerical computation scheme of our solution is thoroughly investigated. The final integration intervals may be restricted in $[0, \pi]$. An adaptive Simpson quadrature method is used to perform the numerical experiments and the results show that our solution is suitable for computing in cases with highly eccentric orbits.

The advantage of the theory shown herein is its compactness and that it provides a new way to analyze the perturbation characteristics of J_{22}. By only considering J_{22}, this theory is useful, especially for highly eccentric orbits. Actually, the theory may also be applicable to other tesseral harmonics with higher orders and degrees and the corresponding first-order solutions also have finite terms. However, with the increase in order and degree, the corresponding definite integrals assume faster oscillations, which will thereby increase the amount of computation. In future work, we will take this problem into account to extend this theory to higher order and degree.

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