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Robertson-Walker cosmological models with perfect fluid in general relativity

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Abstract Einstein's field equations with variable gravitational and cosmological constants are considered in the presence of perfect fluid for a Robertson-Walker universe by assuming the cosmological term to be proportional to R^{-m} (R is a scale factor and m is a constant). A variety of solutions is presented. The physical significance of the cosmological models has also been discussed.

Key words: cosmological parameters — cosmology: variable cosmological term

1 INTRODUCTION

One of the greatest challenges in physics today is to explain the small positive value of the cosmological constant or equivalently, the energy density of the vacuum. The observed value of $7{\times}10^{-30}$ cm^{-3} is over 120 orders of magnitude smaller than the Planck density, 10^{93} g cm^{-3}, since the universe emerged from the big bang yet its value is thought to be set at that time.

The idea of a variable gravitational constant G in the framework of general relativity was first proposed by Dirac (1937). Lau (1985), working in the framework of general relativity, proposed a modification linking the variation of G with that of Λ. This modification allows us to use Einstein's field equations' form, which is unchanged since variation in Λ is accompanied by a variation of G. A number of authors investigated Friedman-Robertson-Walker (FRW) models and Bianchi models using this approach (Abdel-Rahman 1990; Berman & Som 1990; Sisteró 1991; Kalligas et al. 1992; Abdussattar & Vishwakarma 1997; Vishwakarma 2000, 2005; Pradhan & Otarod 2006; Singh et al. 2007; Singh & Tiwari 2008). Borges & Carneiro (2005) have considered that the cosmological term is proportional to the Hubble parameter in the FRW model and the Bianchi type-I model with variable G and Λ. Classification of the FRW universe with a cosmological constant and a perfect fluid in the equation of state have been studied by Ha et al. (2009). Recently, I have (Tiwari 2008, 2009, 2010) considered whether or not the cosmological term is proportional to the Hubble parameter in the Bianchi type-I model and FRW model with varying G and Λ. In this paper, we study homogeneous R-W space-time with variable G and Λ containing matter in the form of a perfect fluid. We obtain solutions of the field equations assuming that the cosmological term is proportional to R^{-m} (where R is a scale factor and m is constant). The paper is organized as follows. Basic equations of the models are given in Section 2, and their solution is in Section 3. We discuss the models and conclude our results in Section 4.

2 THE MODEL AND FIELD EQUATIONS

We consider the spatially homogeneous and isotropic Robertson-Walker (R–W) line element given by

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\right], \tag{1}$$

where $R(t)$ is the scale factor and $k = -1$, 0, or $+1$ is the curvature parameter for an open, flat or closed universe, respectively.

We assume that cosmic matter is represented by the energy-momentum tensor of a perfect fluid

$$T_{ij} = (\rho + p)\upsilon_i\upsilon_i - pg_{ij}, \tag{2}$$

where ρ is the energy density of cosmic matter, p is its pressure, and υ_i is the four velocity vector such that $\upsilon_i\upsilon^i = 1$. We take the equation of state

$$p = (\omega - 1)\rho, \quad 1 \leq \omega \leq 2. \tag{3}$$

The Einstein field equations with variable G and Λ are given by Weinberg (1972)

$$R_{ij} - \frac{1}{2}Rg_{ij} = 8\pi G(t)T_{ij} + \Lambda(t)g_{ij}. \tag{4}$$

For the metric Equation (1) and energy – momentum tensor Equation (2) in a comoving system of coordinates, the field Equation (4) yields

$$-\frac{2\ddot{R}}{R} - \frac{\dot{R}^2}{R^2} - \frac{k}{R^2} = 8\pi Gp - \Lambda, \tag{5}$$

$$\frac{3\dot{R}^2}{R^2} + \frac{3k}{R^2} = 8\pi G\rho + \Lambda. \tag{6}$$

In view of the vanishing of divergence in the Einstein tensor, we have

$$8\pi G\left\{\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R}\right\} + 8\pi\rho\dot{G} + \dot{\Lambda} = 0. \tag{7}$$

The usual energy conservation equation $T^j_{i,j} = 0$ yields

$$\dot{\rho} + 3(\rho + p)\frac{\dot{R}}{R} = 0. \tag{8}$$

Then Equation (7) reduces to

$$8\pi\rho\dot{G} + \dot{\Lambda} = 0. \tag{9}$$

This expression implies that Λ is a constant whenever G is constant. Using Equation (3) in Equation (8) and then integrating, we obtain

$$\rho = \frac{k_1}{R^{3\omega}}, \tag{10}$$

where k_1 is a constant of integration.

For zero curvature ($k = 0$), Equations (5) and (6) can be rewritten in terms of Hubble parameter H and deceleration parameter q as

$$H^2(2q - 1) = 8\pi Gp - \Lambda, \tag{11}$$

$$3H^2 = 8\pi G\rho + \Lambda, \tag{12}$$

where $q = -1 - \frac{\dot{H}}{H^2} = -\frac{R\ddot{R}}{\dot{R}^2}$ and expansion scalar $\theta = 3H = \frac{3\dot{R}}{R}$. Overduin & Cooperstock (1998) define the critical density ρ_c, vacuum density ρ_v and density parameter Ω as

$$\rho_c = \frac{3H^2}{8\pi G}, \tag{13}$$

$$\rho_v = \frac{\Lambda}{8\pi G}, \tag{14}$$

and

$$\Omega = \frac{\rho}{\rho_c} = \frac{8\pi G\rho}{3H^2}. \tag{15}$$

3 SOLUTION OF THE FIELD EQUATIONS

The system of Equations (3), (5), (6) and (8) supply only four equations in five unknowns (R, ρ, p, G and Λ). One extra equation is needed to solve the system completely. The phenomenological Λ decay scenarios have been considered by a number of authors. Chen & Wu (1990) considered $\Lambda \propto a^{-2}$ (a is the scale factor of the Robertson-Walker metric). Hoyle et al. (1997) considered $\Lambda \propto a^{-3}$ while $\Lambda \propto a^{-m}$ (a is a scale factor and m is a constant) was considered by Olson & Jordan (1987); Pavón (1991); Maia & Silva (1994); Silveira & Waga (1994, 1997) and Bloomfield Torres & Waga (1996).

Thus we take the decaying vacuum energy density

$$\Lambda = \frac{a}{R^m}, \tag{16}$$

where a is a constant. From Equations (9), (10) and (16) one finds that

$$G = \frac{aR^{3\omega - m}}{8k_1(3\omega - m)} \equiv G_0 R^{3\omega - m}. \tag{17}$$

There Equations (5) and (6) can be written in the form of perfect fluid $p_{\text{eff}} = (\omega_{\text{eff}} - 1)\,\rho_{\text{eff}}$, with

$$\rho_{\text{eff}} = \left(k_1 G_0 + \frac{a}{8\pi}\right)\frac{1}{G_n R^m}, \tag{18}$$

where G_n is the Newtonian Gravitational constant and ω_{eff} can be given explicitly in terms of other constants.

Now from Equations (3), (11), (12) and (16) we get a differential equation

$$2\dot{H} + 3\omega H^2 - \frac{a\omega}{R^m} = 0. \tag{19}$$

On integrating Equation (19), we have the scale factor

$$R = \left(\frac{m}{2}\sqrt{\frac{a\omega}{3\omega - m}}t + t_0\right)^{2/m}, \tag{20}$$

where t_0 is a constant of integration. The integration constant is related to the choice of origin of time.

Now we analyze scenarios for different values of ω.

3.1 Matter Dominated Solution (Cosmology for $\omega = 1$)

For $\omega = 1$ from Equation (20), we obtain the scale factor

$$R = \left(\frac{m}{2}\sqrt{\frac{a}{3-m}}t + t_0\right)^{2/m}. \tag{21}$$

In this case, the spatial volume V, matter density ρ, pressure p, gravitational constant G, and cosmological constant Λ are given by

$$V = \left[\frac{m}{2}\sqrt{\left(\frac{a}{3-m}\right)}t + t_0\right]^{6/m}, \tag{22}$$

$$\rho = \frac{k_1}{\left[\frac{m}{2}\sqrt{\left(\frac{a}{3-m}\right)}t + t_0\right]^{6/m}}, \tag{23}$$

$$p = 0, \tag{24}$$

$$G = \frac{G_0}{\left[\frac{m}{2}\sqrt{\left(\frac{a}{3-m}\right)}t + t_0\right]^{\frac{2(m-3)}{m}}}, \tag{25}$$

$$\Lambda = \frac{a}{\left[\frac{m}{2}\sqrt{\left(\frac{a}{3-m}\right)}t + t_0\right]^{2}}, \tag{26}$$

$$\theta = \frac{3\sqrt{a}}{\sqrt{3-m}}\left[\frac{m}{2}\sqrt{\left(\frac{a}{3-m}\right)}t + t_0\right]^{-1}, \tag{27}$$

$$\sigma = \frac{c}{\sqrt{3}}\left[\frac{m}{2}\sqrt{\left(\frac{a}{3-m}\right)}t + t_0\right]^{6/m}. \tag{28}$$

The density parameter is given by

$$\Omega = \frac{m}{3}. \tag{29}$$

The deceleration parameter q for the model is

$$q = \frac{m}{2} - 1. \tag{30}$$

The vacuum energy density ρ_{v} and critical density ρ_{c} are given by

$$\rho_{\mathrm{v}} = \frac{c(3-m)}{m}\left[\frac{m}{2}\sqrt{\left(\frac{a}{3-m}\right)}t + t_0\right]^{6/m}, \tag{31}$$

$$\rho_{\mathrm{c}} = \frac{3c}{m}\left[\frac{m}{2}\sqrt{\left(\frac{a}{3-m}\right)}t + t_0\right]^{-6/m}. \tag{32}$$

We observe that for $0 < m < 3$ the spatial volume V is zero at $t = t'$, where $t' = -2t_0/\left(m\sqrt{\frac{a}{3-m}}\right)$ and expansion scalar θ is infinite, which shows that the universe starts evolving with zero volume at $t = t'$ with an infinite rate of expansion. The scale factor also vanishes at $t = t'$ and hence the space-time exhibits point type singularity in the initial epoch. The energy density and shear scalar diverge at the initial singularity. As t increases, the scale factors and spatial volume increase but the expansion scalar decreases. Thus, the rate of expansion slows down with increasing time. Also, $\rho, \sigma, \rho_{\upsilon}, \rho_{\rm c}$, and Λ decrease as t increases. As $t \to \infty$, the scale factor and volume become infinite, $\rho, \sigma, \rho_{\upsilon}, \rho_{\rm c}$, and Λ tend to zero. Therefore, the model would essentially give an empty universe for large time t. Gravitational constant $G(t)$ is zero at $t = t'$ and as t increases, $G(t)$ also increases.

A partial list of cosmological models in which the gravitational constant G is increasing with time is contained in Abdel-Rahman (1990); Chow (1981); Levitt (1980) and Milne (1935). The ratio $\frac{\sigma}{\theta} \to 0$ as $t \to \infty$, so the model approaches isotropy for large values of t. Thus, the model represents a shearing, non-rotating and expanding model of the universe with a big bang start approaching isotropy at late times.

Further, it is observed that when $2 < m < 3$, $q > 0$; $q = 0$ for $m = 2$ and for $0 < m < 2$, $q < 0$. Therefore, the universe begins with decelerating expansion changes and the expansion changes from a decelerating phase to an accelerating one. This cosmological scenario is in agreement with SNe Ia astronomical observations (Knop et al. 2003; Riess et al. 1998, 2004; Spergel et al. 2007; Tegmark et al. 2004; Perlmutter et al. 1998) and it presents a unified description of the evolution of the universe.

3.2 Zel'dovich fluid distribution (Cosmology for $\omega = 2$)

Corresponding to the equation of state $\rho = p$, this equation of state has been widely used in general relativity to obtain stellar and cosmological models for utterly dense matter (Zel'dovich 1962). In this case, from Equation (18) the scale factor R becomes

$$R = \left[\frac{m}{2}\sqrt{\left(\frac{2a}{6-m}\right)}t + t_0\right]^{2/m}. \tag{33}$$

Here, spatial volume V, matter density ρ, pressure p, gravitational constant G, and cosmological constant Λ are given by

$$V = \left[\frac{m}{2}\sqrt{\left(\frac{2a}{6-m}\right)}t + t_0\right]^{6/m}, \tag{34}$$

$$\rho = p = \frac{k}{\left[\frac{m}{2}\sqrt{\left(\frac{a}{6-m}\right)}t + t_0\right]^{12/m}}, \tag{35}$$

$$G = \frac{G_0}{\left[\frac{m}{2}\sqrt{\left(\frac{a}{6-m}\right)}t + t_0\right]^{\frac{2(m-3)}{m}}}, \tag{36}$$

$$\Lambda = \frac{a}{\left[\frac{m}{2}\sqrt{\left(\frac{2a}{6-m}\right)}t + t_0\right]^{2}}. \tag{37}$$

Expansion scalar θ and shear σ are given by

$$\theta = 3\sqrt{\frac{2a}{6-m}}\left[\frac{m}{2}\sqrt{\left(\frac{2a}{6-m}\right)}t + t_0\right]^{-1}, \tag{38}$$

$$\sigma = \frac{k}{\sqrt{3}} \left[\frac{m}{2} \sqrt{\left(\frac{2a}{6-m} \right)} t + t_0 \right]^{6/m}. \quad (39)$$

The density parameter is given by

$$\Omega = \frac{\rho}{\rho_c} = \frac{m}{6}. \quad (40)$$

The deceleration parameter q for the model is

$$q = \frac{m}{2} - 1. \quad (41)$$

The vacuum energy density ρ_v and critical density ρ_c are given by

$$\rho_v = \frac{k(6-m)}{m} \left[\frac{m}{2} \sqrt{\left(\frac{2a}{6-m} \right)} t + t_0 \right]^{12/m}, \quad (42)$$

$$\rho_c = \frac{6k}{m} \left[\frac{m}{2} \sqrt{\left(\frac{2a}{6-m} \right)} t + t_0 \right]^{12/m}. \quad (43)$$

Here, we observe that for $m < 6$ the spatial volume V is zero at $t = -t_0/\left[m/2\sqrt{2a/(6-m)}\right] = t''$ and the expansion scalar θ is infinite at $t = t''$, which shows that the universe starts evolving with zero volume and infinite rate of expansion at $t = t''$. Initially at $t = t''$, the energy density ρ, pressure p, cosmological constant Λ and shear scalar σ are infinite. As t increases, the spatial volume increases but the expansion scalar decreases. Thus the expansion rate decreases as time increases. As t tends to ∞, the spatial volume V becomes infinitely large. As t increases, all the parameters p, ρ, Λ, θ, ρ_c, and ρ_v decrease and tend to zero asymptotically. Therefore, the model essentially gives an empty universe for large t. The ratio $\frac{\sigma}{\theta} \to 0$ as $t \to \infty$, which shows that the model approaches isotropy for large values of t. Furthermore, we observe that $\Lambda \propto \frac{1}{t^2}$ which follows from the model of Kalligas et al. (1992); Berman & Som (1990); Berman (1990); Berman et al. (1989) and Bertolami (1986a,b). This form of Λ is physically reasonable as observations suggest that Λ is very small in the present universe.

3.3 Radiation Dominated Solution ($\rho = 3p$) Cosmology for $\omega = 4/3$

In this case from Equation (20) we obtain

$$R = \left[\frac{m}{2} \sqrt{\left(\frac{4a}{3(4-m)} \right)} t + t_0 \right]^{2/m}. \quad (44)$$

Here, matter density ρ, pressure p, gravitational constant G, and cosmological constant Λ are given by

$$\rho = \frac{k}{\left(\frac{m}{2} \sqrt{\left\{ \frac{4a}{3(4-m)} \right\}} t + t_0 \right)^{8/m}}, \quad (45)$$

$$p = \frac{k}{3\left(\frac{m}{2} \sqrt{\left\{ \frac{4a}{3(4-m)} \right\}} t + t_0 \right)^{8/m}}, \quad (46)$$

$$G = G_0 \left[\frac{m}{2} \sqrt{\left\{ \frac{4a}{3(4-m)} \right\}} t + t_0 \right]^{\frac{2}{m}(4-m)}, \tag{47}$$

$$\Lambda = \frac{a}{\left[\frac{m}{2} \sqrt{\left\{ \frac{4a}{3(4-m)} \right\}} t + t_0 \right]^2}. \tag{48}$$

Expansion scalar θ and shear σ are given by

$$\theta = 3\sqrt{\frac{4a}{3(4-m)}} \left[\frac{m}{2} \sqrt{\left\{ \frac{4a}{3(4-m)} \right\}} t + t_0 \right]^{-1}, \tag{49}$$

$$\sigma = \frac{k}{\sqrt{3}} \left[\frac{m}{2} \sqrt{\left\{ \frac{4a}{3(4-m)} \right\}} t + t_0 \right]^{-6/m}. \tag{50}$$

The density parameter is given by

$$\Omega = \frac{\rho}{\rho_c} = \frac{m}{4}. \tag{51}$$

The deceleration parameter q for the model is

$$q = \frac{m}{2} - 1. \tag{52}$$

The vacuum energy density and critical density ρ_c are given by

$$\rho_v = \frac{k(4-m)}{m\left(\frac{m}{2} \sqrt{\left\{ \frac{4a}{3(4-m)} \right\}} t + t_0 \right)^{8/m}}, \tag{53}$$

$$\rho_c = \frac{4k}{m\left(\frac{m}{2} \sqrt{\left\{ \frac{4a}{3(4-m)} \right\}} t + t_0 \right)^{8/m}}. \tag{54}$$

Here, we observe that for $0 < m < 4$ the spatial volume V is zero at $t = t'''$, where $t''' = (-t_0)/[m/2\sqrt{(4a)/3(4-m)}]$ and expansion scalar θ is infinite, which shows that the universe starts evolving with zero volume at $t = t'''$ with an infinite rate of expansion. The scale factor also vanishes at $t = t'''$ and hence the space-time exhibits point type singularity in the initial epoch. The energy density, which is a shear scalar, diverges at the initial singularity. As t increases, the scale factors and spatial volume increase but the expansion scalar decreases. Thus, the rate of expansion slows down with increases in time. Also $\rho, \sigma, \rho_\upsilon, \rho_c$, and Λ decrease as t increases. As $t \to \infty$, the scale factor and volume become infinite whereas $\rho, \sigma, \rho_\upsilon, \rho_c$, and Λ tend to zero. Therefore, the model would essentially give an empty universe for large time t. The ratio $\frac{\sigma}{\theta} \to 0$ as $t \to \infty$, so the model approaches isotropy for large values of t. Hence, the model represents a shearing, non-rotating and expanding model of the universe with a big bang start which approaches isotropy at late times.

4 CONCLUSION

In this paper, we have studied a spatially homogeneous and isotropic R-W line element with variable gravitational constant $G(t)$ and cosmological constant $\Lambda(t)$. The field equations have been solved exactly by using a law of variation of scale factor with a variable cosmological term, i.e. a cosmological term that scales as $\Lambda \propto R^{-m}$ (where R is a scale factor). Three exact R-W models have been obtained in Sections 3.1, 3.2 and 3.3. Expressions for some important cosmological parameters were obtained for all the models and the physical behavior of the models were discussed in detail. In all the cases, the models represent a shearing, non-rotating and expanding model of the universe with a big-bang start which approaches isotropy at late times. It is interesting that the proposed variation law provides an alternative approach to obtain exact solutions of Einstein's field equations. It presents a unified description of the evolution of the universe which starts with a decelerating expansion and expands with acceleration at late times. Recent observational data (Knop et al. 2003; Riess et al. 1998, 2004; Spergel et al. 2007; Tegmark et al. 2004; Perlmutter et al. 1998) strongly suggest this acceleration occurs. Also, gravitational constant $G(t)$ is zero at initial singularity and it is increasing with increasing time. The cosmological constant $\Lambda(t) \propto 1/t^2$ which follows from the model of Kalligas et al. (1992); Berman & Som (1990); Berman (1990); Berman et al. (1989) and Bertolami (1986b). This form of Λ is physically reasonable as observations suggest that Λ is very small in the present universe. Finally, the solutions presented in the paper are new and useful for a better understanding of the evolution of the universe in R-W space-time with variable G and Λ.

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