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Hawking radiation and thermodynamics of a Vaidya-Bonner black hole*

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Abstract Using Parikh's tunneling method, the Hawking radiation on the apparent horizon of a Vaidya-Bonner black hole is calculated. When the back-reaction of particles is neglected, the thermal spectrum can be precisely obtained. Then, the black hole thermodynamics can be calculated successfully on the apparent horizon. When a relativistic perturbation is applied to the apparent horizon, a similar calculation can also lead to a purely thermal spectrum. The first law of thermodynamics can also be derived successfully at the new supersurface near the apparent horizon. When the event horizon is thought of as a deviation from the apparent horizon, the expressions of the characteristic position and temperature are consistent with the previous viewpoint which asserts that the thermodynamics should be based on the event horizon. It is concluded that the thermodynamics should be constructed exactly on the apparent horizon while the event horizon thermodynamics is just one of the perturbations near the apparent horizon.

Key words: black hole physics — relativity — kinematics and dynamics

1 INTRODUCTION

In the 1970's, Hawking's discovery of black body thermal radiation (Hawking 1975, 1974) made the analogy between black hole dynamics and general thermodynamics more meaningful (Bekenstein 1973; Bardeen et al. 1973). When the surface gravity κ is regarded as temperature and the area A of the event horizon as entropy, the first law of thermodynamics is satisfied at the event horizon of static or stationary black holes because the apparent horizon and event horizon coincide with each other. However, this will be more complex in a dynamical black hole because the event horizon will separate from the apparent horizon. Where does Hawking radiation come from? Where can the thermodynamics be constructed? There are two different viewpoints in previous work. One is from Roberto Balbinot and other researchers (Balbinot 1986; Ren et al. 2006; Parikh 2004). They investigated black hole entropy and Hawking radiation on the event horizon of a dynamical black hole and showed that the radiation from the event horizon will have a corrected temperature expression which is not the same as that of a static or stationary black hole. Some of them even think that the radiation is not a perfect black body spectrum, because thermal equilibrium does not exist on the event horizon. The other is from Hajicek (1987), which suggests that Hawking radiation and thermodynamics

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should be on the apparent horizon instead of the event horizon, because the apparent horizon acts as the boundary of negative energy states. Considering the collapse of a spherical shell, Hiscock (1989) proposed to identify one-quarter of the area of the apparent horizon as the Bekenstein-Hawking area-entropy of a Vaidya black hole. Subsequently, using a null tetrad formalism, Collins (1992) derived a formula for the area change of the apparent horizon, which can be interpreted as a generalized first law of thermodynamics.

Recently, a series of trapped horizon definitions which can be applied to non-equilibrium black holes has been reviewed in Gourgoulhon et al. (2008). Hayward (1994, 2004a,b), Ashtekar et al. (2003), and Nielsen (2008) have investigated the relation between surface gravity, black hole entropy and total black hole mass. They concluded that the thermodynamics can be built on these trapped horizons. The apparent horizon, called the marginally outer trapped surface, can be regarded as one of the trapped horizons, so these conclusions are also applicable to the apparent horizon of a dynamical black hole.

In Section 2, following Parikh and Wilczek's tunneling method, we calculate the Hawking radiation from the apparent horizon of a Vaidya-Bonner black hole. For simplicity, we do not consider particles' back-reaction to space-time. After the calculation, we will find that the first law of thermodynamics can be derived successfully on the apparent horizon. In Section 3, considering a relativistic perturbation near the apparent horizon, we can also successfully calculate the first law of thermodynamics on the new supersurface near the apparent horizon. In Section 4, comparing the thermodynamical property of the event horizon to that of the new supersurface near the apparent horizon, we will find that the event horizon is just one of the new supersurfaces after applying a relativistic perturbation around the apparent horizon. Finally, we come to the conclusion that the thermodynamics should be constructed exactly on the apparent horizon.

2 TUNNELING THROUGH THE APPARENT HORIZON

The line element of a Vaidya-Bonner black hole can be written as

$$ds^2 = -\left[1 - \frac{2m(v)}{r} + \frac{Q(v)^2}{r^2}\right] dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{1}$$

where $m(v)$ and $Q(v)$ are, respectively, the mass and charge of the Vaidya-Bonner black hole, and v is the advanced Eddington time coordinate.

The apparent horizon $r_{\rm AH}$ of the Vaidya-Bonner black hole satisfies (Zhao 1999)

$$g_{vv} = -\left[1 - \frac{2m\,(v)}{r} + \frac{Q(v)^2}{r^2}\right] = 0. \tag{2}$$

Apparently, for a Vaidya-Bonner black hole, the apparent horizon and time-like limit surface coincide with each other

$$r_{\rm AH} = r_{\rm TLS} = m + \sqrt{m^2 - Q^2}. \tag{3}$$

We can easily find that the metric is regular on the apparent horizon and the Eddington coordinates satisfy Landau's coordinate clock synchronization. So, Parikh's framework can be used here.

The radial null geodesic is given by

$$\dot{r} \equiv \frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right). \tag{4}$$

The imaginary part of the action is

$${\rm Im}S = {\rm Im}\int p_r dr = {\rm Im}\iint dp_r dr = {\rm Im}\iint \frac{dH}{\dot{r}} dr, \tag{5}$$

where we have used the Hamilton equation $\frac{dH}{dp_r} = \dot{r}$. By substituting Equation (4) into Equation (5), we can obtain

$$\mathrm{Im}S = \int_0^{\omega}\int \frac{2dr}{1 - \frac{2m}{r} + \frac{Q^2}{r^2}}(-d\omega'), \tag{6}$$

where we have considered that the radiative field does not radiate energy during the tunneling process due to the instantaneous property of tunneling. However, the condition of energy conservation is used to conclude $dH = -d\omega^{'}$, where ω is the energy of the emitting particle. We can calculate Equation (6) as follows:

$$\mathrm{Im}S = \int_0^{\omega} \pi\frac{(m + \sqrt{m^2 - Q^2})^2}{\sqrt{m^2 - Q^2}}d\omega' = \pi\frac{(m + \sqrt{m^2 - Q^2})^2}{\sqrt{m^2 - Q^2}}\omega. \tag{7}$$

Using the WKB approximation, the emission rate can be given as

$$\Gamma \sim e^{-2\mathrm{Im}S} = e^{-2\pi\frac{(m+\sqrt{m^2-Q^2})^2}{\sqrt{m^2-Q^2}}\omega}. \tag{8}$$

The emission rate satisfies the Boltzmann distribution $e^{-\beta\omega}$ with inverse temperature

$$\beta = 2\pi\frac{(m + \sqrt{m^2 - Q^2})^2}{\sqrt{m^2 - Q^2}}, \tag{9}$$

so we have

$$T_{\mathrm{AH}} = \frac{\kappa}{2\pi} = \frac{\sqrt{m^2 - Q^2}}{2\pi(m + \sqrt{m^2 - Q^2})^2}. \tag{10}$$

According to the thermodynamical analogy in black hole physics, the entropy of the black hole is defined as $S = \frac{A}{4}$, where A is the area of the black hole's characteristic supersurface. Thinking of the apparent horizon, the entropy can be written as

$$S_{\mathrm{BH}} = \pi r_{\mathrm{AH}}^2 = \pi(m + \sqrt{m^2 - Q^2})^2, \tag{11}$$

so we get

$$\begin{aligned} T_{\mathrm{AH}}dS_{\mathrm{BH}} &= \frac{\sqrt{m^2 - Q^2}}{2\pi(m + \sqrt{m^2 - Q^2})^2}2\pi(m + \sqrt{m^2 - Q^2})\Big(dm + \frac{mdm - QdQ}{\sqrt{m^2 - Q^2}}\Big) \\ &= dm - V_0 dQ, \end{aligned} \tag{12}$$

where $V_0 = \frac{Q}{r_{\mathrm{AH}}} = \frac{Q}{m+\sqrt{m^2-Q^2}}$ is the electromagnetic potential on the apparent horizon.

This satisfies the first law of thermodynamics

$$dm\,(v) = \frac{\kappa}{2\pi}dS_{\mathrm{BH}} + V_0 dQ = T_{\mathrm{AH}}dS_{\mathrm{BH}} + V_0 dQ, \tag{13}$$

where κ is just the surface gravity of the apparent horizon.

3 TUNNELING THROUGH THE SUPERSURFACES NEAR THE APPARENT HORIZON

To investigate the Hawking radiation from the supersurfaces near the apparent horizon $r' = r_{\mathrm{AH}}(1+\delta)$, where δ is an infinitesimal parameter, we define a new radial coordinate as follows:

$$R = r - \frac{1}{2}\delta \cdot v, \tag{14}$$

so the Vaidya-Bonner metric in the new coordinates can be written as

$$ds^2 = -\left[1 - \frac{2m(v)}{r} + \frac{Q(v)^2}{r^2} - \delta\right] dv^2 + 2dvdR + r^2(d\theta^2 + \sin^2\theta d\varphi^2). \tag{15}$$

From $g_{vv} = 0$, the new "apparent horizon" and the time-like limit surface are easily obtained. Considering δ as an infinitesimal parameter, we have

$$r'_{\mathrm{AH}} = r'_{\mathrm{TLS}} = \frac{m + \sqrt{m^2 - (1-\delta)Q^2}}{1-\delta} = (m + \sqrt{m^2 - Q^2})\,(1+\delta) = r_{\mathrm{AH}}\,(1+\delta)\,. \tag{16}$$

We can easily find that the metric is regular on the new "apparent horizon" and the new coordinates satisfy Landau's coordinate clock synchronization. So, Parikh's framework can also be used here.

The radial null geodesic is

$$\dot{R} \equiv \frac{dR}{dv} = \frac{1}{2}\Big(1 - \frac{2m}{r} + \frac{Q^2}{r^2} - \delta\Big), \tag{17}$$

so we have

$$\mathrm{Im}S = \mathrm{Im}\int p_R dR = \mathrm{Im}\iint dp_R dR = \mathrm{Im}\iint \frac{dH}{\dot{R}} dR. \tag{18}$$

Putting Equation (17) into Equation (18), we have

$$\mathrm{Im}S = \mathrm{Im}\int_0^{\omega}\!\!\int \frac{2dR}{1 - \frac{2m(v)}{r} + \frac{Q^2}{r^2} - \delta}(-d\omega') = \frac{\pi[m + \sqrt{m^2 - (1-\delta)Q^2}]^2}{(1-\delta)^2\sqrt{m^2 - (1-\delta)Q^2}}\omega. \tag{19}$$

The emission rate is given by

$$\Gamma \sim e^{-2\mathrm{Im}S} = e^{-\frac{2\pi[m + \sqrt{m^2 - (1-\delta)Q^2}]^2}{(1-\delta)^2\sqrt{m^2 - (1-\delta)Q^2}}\omega}, \tag{20}$$

which corresponds to the purely thermal spectrum. The Boltzmann factor with inverse temperature is

$$\beta' = \frac{1}{T'} = \frac{2\pi[m + \sqrt{m^2 - (1-\delta)Q^2}]^2}{(1-\delta)^2\sqrt{m^2 - (1-\delta)Q^2}}\omega. \tag{21}$$

The thermodynamical temperature on the new supersurface can be written as

$$T' = \frac{(1-\delta)^2\sqrt{m^2 - (1-\delta)Q^2}}{2\pi[m + \sqrt{m^2 - (1-\delta)Q^2}]^2}. \tag{22}$$

The area of the supersurface r'_{AH} that can be regarded as the entropy of the black hole's thermal system will naturally be

$$S' = \frac{1}{4}A' = \pi r'^2_{\mathrm{AH}} = \pi\Big[\frac{m + \sqrt{m^2 - (1-\delta)Q^2}}{1-\delta}\Big]^2, \tag{23}$$

so we can obtain

$$\begin{aligned} T'dS' &= \frac{\sqrt{m^2-(1-\delta)Q^2}}{2\pi[m+\sqrt{m^2-(1-\delta)Q^2}]^2} 2\pi[m+\sqrt{m^2-(1-\delta)Q^2}]\left[dm+\frac{mdm-(1-\delta)QdQ}{\sqrt{m^2-(1-\delta)Q^2}}\right] \\ &= dm - V_0'dQ, \end{aligned} \tag{24}$$

where $V_0' = \frac{Q}{r'_{\rm AH}} = \frac{Q}{m+\sqrt{m^2-(1-\delta)Q^2}}$ is the electromagnetic potential on the new "apparent horizon."

By simple calculation, we can find that they obey the first law of thermodynamics as follows:

$$dm\,(v) = T'dS' + V_0'dQ. \tag{25}$$

Apparently, we can also derive thermodynamics on the new supersurface which is positioned after a perturbation near the apparent horizon.

4 EVENT HORIZON AS A PERTURBATION OF THE APPARENT HORIZON

The equation of the event horizon of the metric Equation (1) can be written as

$$(1-2\dot{r})r^2 - 2mr + Q^2 = 0, \tag{26}$$

so the event horizon is given as

$$r_{\rm EH} = \frac{m+\sqrt{m^2-(1-2\dot{r}_{\rm EH})Q^2}}{1-2\dot{r}_{\rm EH}}, \tag{27}$$

in which $\dot{r}_{\rm EH}$ is the derivative of the event horizon $r_{\rm EH}$ with respect to v. If we let $\delta = 2\dot{r}_{\rm EH}$, the new supersurface after a relativistic perturbation $r'_{\rm AH} = \frac{m+\sqrt{m^2-(1-\delta)Q^2}}{1-\delta}$ is exactly the event horizon of a Vaidya-Bonner black hole. Apparently, when $\dot{r}_{\rm EH} > 0$ and the dynamical black hole is growing larger, the event horizon will be outside of the apparent horizon. However, when $\dot{r}_{\rm EH} < 0$ and it radiates and becomes smaller, the event horizon will be inside the apparent horizon.

According to Fodor et al. (1996), we can find κ of a Vaidya-Bonner black hole as follows:

$$\kappa \mid_{r_{\rm EH}} = \Gamma^0_{00} \mid_{r_{\rm EH}} = \frac{m-\frac{Q^2}{r_{\rm EH}}}{r_{\rm EH}^2}. \tag{28}$$

Substituting Equation (27) into Equation (28), we can obtain

$$T_{r_{\rm EH}} = \frac{(1-2\dot{r}_{\rm EH})^2\sqrt{m^2-(1-2\dot{r}_{\rm EH})Q^2}}{2\pi[m+\sqrt{m^2-(1-2\dot{r}_{\rm EH})Q^2}]^2}. \tag{29}$$

If we set $\delta = 2\dot{r}_{\rm EH}$, obviously Equation (22) is consistent with Equation (29), so the thermodynamical temperature on the new supersurface is exactly the expression on the event horizon. The event horizon can be regarded as one of the new supersurfaces after a relativistic perturbation is applied around the apparent horizon.

5 CONCLUSIONS AND DISCUSSIONS

Using Parikh's tunneling method and neglecting the back-reaction, we have calculated the temperature and entropy on the apparent horizon of a Vaidya-Bonner black hole. We found that the first law of thermodynamics can be derived on the apparent horizon, and this is consistent with the conclusions from previous work (Hayward 1994, 2004a,b; Ashtekar et al. 2003). When considering a relativistic perturbation near the apparent horizon, the black hole thermodynamics can also be successfully derived on the new supersurfaces.

Letting $\delta = 2\dot{r}_{\rm EH}$, the new supersurface after a relativistic perturbation is exactly the event horizon of the Vaidya-Bonner black hole, and the temperature on the new supersurface is exactly the expression on the event horizon. Apparently, the event horizon is just one of the new supersurfaces after a relativistic perturbation is applied around the apparent horizon.

The thermodynamics should be constructed exactly on the apparent horizon, while the event horizon is just one of the perturbations near the apparent horizon in a Vaidya-Bonner black hole.

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