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Effects of the cosmological constant on chaos in an FRW scalar field universe *

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Abstract The dependence of chaos on two parameters of the cosmological constant and the self-interacting coefficient in the imaginary phase space for a closed Friedman-Robertson-Walker (FRW) universe with a conformally coupled scalar field, as the full understanding of the dependence in real phase space, is investigated numerically. It is found that Poincaré plots for the two parameters less than 1 are almost the same as those in the absence of the cosmological constant and self-interacting terms. For energies below the energy threshold of 0.5 for the imaginary problem in which there are no cosmological constant and self-interacting terms, an abrupt transition to chaos occurs when at least one of the two parameters is 1. However, the strength of the chaos does not increase for energies larger than the threshold. For other situations of the two parameters larger than 1, chaos is weaker, and even disappears as the two parameters increase.

Key words: cosmology: cosmological parameters — methods: numerical

1 INTRODUCTION

Within the last decade or so, two independent groups have been engaged in the study of chaotic behaviors in relativistic astrophysics. One deals with the geodesic or nongeodesic motion of particles in a given gravitational field (Levin 2000; Wu & Huang 2003; Wu et al. 2006a,b; Wu & Zhang 2006; Xie & Huang 2006). For instance, the problem of whether spinning compact binaries exhibit chaos is so important that the onset of chaos in the binary system would have a bad effect on the detection of gravitational waves (Hartl & Buonanno 2005; Gopakumar & Königsdörffer 2005; Wu & Xie 2007, 2008). The other pays attention to the time evolution of a gravitational field itself. The study of the nonlinear behavior of the mixmaster cosmology is a typical example (Imponente & Montani 2001). On the other hand, Page (1984) noted that there is a fractal with positive Hausdorff-Besicovitch dimension in an FRW universe minimally coupled to a massive scalar field. Since then, the dynamical evolution of such cosmological models, with a cosmological constant term and scalar fields conformally or minimally coupled to the curvature, has been particularly appreciated (Calzetta & Hasi 1993; Cornish & Levin 1996; Kamenshchik et al. 1997; de Oliveira et al. 1997; Monerat et al. 1998; Beck 2004; Toporensky 2006; Gerakopoulos et al. 2008). As an emphasis, the cosmological constant can give a reasonable explanation for the current accelerating expansion of the universe. In addition, the universe is created bursting with matter modeled by conformally or minimally coupled fields.

Because chaotic dynamics can lead to observable implications for the realistic universe, there are a number of studies in this field. Minimally coupled fields are not integrable in general (Maciejewski

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et al. 2008). In fact, Cornish & Shellard (1998) found chaotic scattering in minimally coupled fields. Similarly, chaos occurs in preinflationary FRW universes (Monerat et al. 1998). On the other hand, non-minimally coupled fields can give more physical information than minimally coupled ones. For example, they are interesting in the context of the description of dark energy in some sense. It is worth noting that conformally coupled fields are only integrable in four known cases (Maciejewski et al. 2008). For other cases, they are nonintegrable, even chaotic. There was initially some doubt on the results of chaos in conformally coupled fields without the cosmological constant Λ and the self-interacting coefficient λ between the article of Calzetta & Hasi (1993) and that of Castagnino et al. (2001). A previous paper presents evidence of chaotic behavior of the model (Calzetta & Hasi 1993). However, a main conclusion of the article (Castagnino et al. 2001) is that there is no chaos in the same system. Motter & Letelier (2002) pointed out that, although the system is nonchaotic, it should be nonintegrable in the physical region. For a detailed exploration, Jorás & Stuchi (2003) studied the existence of mechanisms of the transition to global chaos in this problem by use of Poincaré sections. In practice, an earlier work (Helmi & Vucetich 1997) applies the Painlevé theory of differential equations to provide a brief analysis of the dynamics of an FRW universe with a conformally coupled, real, self-interacting, massive scalar field for the case of $\Lambda \neq 0$ and $\lambda \neq 0$.

Two points should be emphasized in the study of the dynamical evolution of the cosmological models. One is that it is vital to use coordinate-independent measures of chaos in general relativity. Lyapunov exponents, as a usual coordinate-dependent indicator of chaos, would become ambiguous in a curved space (Wu & Huang 2003; Wu et al. 2006a). By contrast, the method of fractal basin boundaries is regarded as a topologically invariant procedure (Cornish & Levin 1996). The analysis of Poincaré sections is also a good tool to search for cantori or stochastic layers when the system can complete many cycles for a useful picture to emerge. The other relates to the artificial complexification of the phase space. Although an imaginary solution for the complexification of the model is unphysical, the approach still retains the same dynamical characteristic of bifurcations and chaos in the real system before the universe collapses (Jorás & Stuchi 2003; Maciejewski et al. 2008). That is to say, complexification may be helpful to the understanding of the chaotic dynamics of many real systems.

Although some references (Cornish & Levin 1996; de Oliveira et al. 1997; Monerat et al. 1998) have reported that the cosmological constant term plays an important role in the dynamical evolution of other cosmological models, none has given an answer to the problem of the strength of chaos as the cosmological constant increases. In view of this, unlike the paper of Castagnino et al. (2001), our main purpose is to numerically investigate how the cosmological constant Λ affects the dynamical transition of an imaginary solution related to a closed FRW cosmological scalar field in imaginary phase space instead of the real system in real phase space. The rest of this paper presents the details. Section 2 gives the cosmological model studied. At once its corresponding Hamiltonian is written in the complexified space. Then, our numerical results of the imaginary problem are shown in Section 3. Finally, Section 4 summarizes our conclusions.

2 COSMOLOGICAL MODELS

A homogeneous and isotropic cosmological model can be described by an FRW universe with the metric of the form

$$ds^2 = a(\eta)^2\Big[-d\eta^2 + \frac{dr^2}{1-Kr^2} + r^2 d\Omega^2\Big], \tag{1}$$

where a, K and η denote the scale factor, curvature index and the conformal time in sequence. In addition, $d\Omega^2$ refers to the line element on a two-dimensional sphere. Supposing the universe is filled with matter, the general action with the cosmological constant Λ and field potential V reads

$$I = \frac{c^4}{16\pi G}\int[\Re - 2\Lambda - \frac{1}{2}(\nabla_\alpha\overline{\psi}\nabla^\alpha\psi + V(\psi) + \xi\Re|\psi|^2) - \varrho]\sqrt{-g}d^4x, \tag{2}$$

where $\Re$, ξ and ϱ are the Ricci scalar, the coupling constant and the density of the matter fluid, respectively. g is the determinant of the metric (1). For the case of a conformally coupled scalar field, the coupling constant is often chosen as $\xi = \frac{1}{6}$. Note that the potential $V(\psi)$ for the complex scalar field, as a function of the modulus of ψ, usually is a quadratic mass term plus a quartic self interaction term with parameter λ. In this sense, the action becomes

$$I = \frac{c^4}{16\pi G}\int[\Re - 2\Lambda - \frac{1}{2}(\nabla_\alpha\overline{\psi}\nabla^\alpha\psi + \frac{1}{6}\Re|\psi|^2 + \frac{m^2}{\hbar^2}|\psi|^2) - \frac{\lambda}{4!}|\psi|^4]\sqrt{-g}d^4x. \quad (3)$$

In light of Equations (1) and (3), it is easy to get the Lagrangian

$$L = 6(\ddot{a}a + Ka^2) - \frac{1}{2}\ddot{a}a|\psi|^2 + \frac{1}{2}|\dot{\psi}|^2a^2 - \frac{m^2}{2\hbar^2}a^4|\psi|^2 - \frac{\lambda}{4!}a^4|\psi|^4 - 2\Lambda a^4 - \frac{1}{2}Ka^2|\psi|^2. \quad (4)$$

Given $\psi = \sqrt{12}\phi exp(i\theta)/a$, then one can obtain

$$\mathcal{L} = 6[\dot{\phi}^2 + \phi^2\dot{\theta}^2 - \dot{a}^2 + K(a^2 - \phi^2) - \frac{m^2}{\hbar^2}a^2\phi^2 - \frac{\Lambda}{3}a^4 - \lambda\phi^4]. \quad (5)$$

This corresponds to the following Hamiltonian

$$H = \frac{1}{24}(p_\phi^2 + \frac{p_\theta^2}{\phi^2} - p_a^2) + 6[K(\phi^2 - a^2) + \frac{m^2}{\hbar^2}a^2\phi^2 + \frac{\Lambda}{3}a^4 + \lambda\phi^4], \quad (6)$$

where $p_a = -12\dot{a}$, $p_\phi = 12\dot{\phi}$ and $p_\theta = 12\phi^2\dot{\theta}$. As θ is a cyclic variable, the momentum p_θ is a constant. Here, only the case of $p_\theta = 0$, in which a scalar field is equivalent to a real field after a unitary rotation in the complex ψ plane, is considered. By rescaling the constants and momenta, the associated Hamiltonian in dimensionless variables is expressed as

$$H = \frac{1}{2}(p_\phi^2 - p_a^2) + \frac{1}{2}[K(\phi^2 - a^2) + m^2a^2\phi^2] + \frac{1}{4}(\Lambda a^4 + \lambda\phi^4). \quad (7)$$

See the paper of Maciejewski et al. (2008) as well as that of Helmi & Vucetich (1997) for more details. It should be particularly emphasized that general relativity requires that the Hamiltonian (7) should be subject to the constraint on the vanishing energy condition, namely, $H \equiv 0$. On the other hand, when a radiation component is added, that is, the density of matter ϱ in the action (2) is chosen suitably such that a constant energy E connected with the density can be absorbed by the Hamiltonian (7), one gets $\mathcal{H} = H - E$ (de Oliveira et al. 1997; Jorás & Stuchi 2003; Maciejewski et al. 2008). Although $\mathcal{H}$ still fits for the vanishing energy constraint $\mathcal{H} \equiv 0$, a nonzero value of the energy in Equation (7) becomes physically admissible. Hereafter, an energy E is identical to a Hamiltonian quantity H.

Adopting the complexified phase space, namely, a canonical transformation

$$a \to ia, \quad p_a \to -ip_a \quad (i = \sqrt{-1}), \quad (8)$$

one has a new Hamiltonian of the type

$$H = \frac{1}{2}(p_a^2 + p_\phi^2) + \frac{K}{2}(a^2 + \phi^2) - \frac{1}{2}m^2a^2\phi^2 + \frac{1}{4}(\Lambda a^4 + \lambda\phi^4). \quad (9)$$

For a closed cosmology with $K = 1$, Equation (9) seems to be two nonlinear coupled harmonic oscillators, but it is complicated because of the existence of quartic terms. Equation (9) is an imaginary problem of the real problem (7). As claimed in the Introduction, the imaginary system is unphysical, but the chaoticity and bifurcations in the imaginary phase space are basically coincided with those in the real phase space before the universe collapses. Therefore, it is interesting to discuss the dynamics of the imaginary problem. Next, we shall focus on the relation between the dynamics of the universe and the self-interacting term as well as the cosmological constant term for the imaginary problem.

3 NUMERICAL RESULTS

Jorás & Stuchi (2003) studied the transition to global chaos with energy E in imaginary phase space for the imaginary problem (9) without the cosmological constant and the self-interacting terms. In other words, the system is of the form

$$H = \frac{1}{2}(p_a^2 + p_\phi^2) + \frac{K}{2}(a^2 + \phi^2) - \frac{1}{2}m^2 a^2 \phi^2 \tag{10}$$

with $K = 1$ and $m^2 = 1$. They found four fixed points corresponding to four nontrivial solutions. Then the value of energy at the fixed points, $E_B = 0.5$, was given. The energy was called a threshold because the motion can be split into two components of the center and the saddle when any energy spans this critical value.

Now, in our simulations, we always take $m^2 = 1$. Initial conditions of a, ϕ and p_ϕ are given in various admissible regions, while the starting (positive) value of p_a can be solved from Equation (9) for a given energy E and parameters Λ and λ. A fifth-order Runge-Kutta-Fehlberg algorithm combined with one of the velocity correction methods of energy (Wu et al. 2007; Ma et al. 2008) is employed so that the energy of the system (9) at every integration step can almost be accurate to the double precision of the machine, 10^{-16}. Let us consider two cases of the energy E.

Case 1: $E = 0.46$. — When we consider $\Lambda = \lambda = 0$ in Equation (9), or equivalently for the system (10), the energy $E = 0.46$ is below the energy threshold, $E_B = 0.5$. As shown on the Poincaré section $\phi = 0$ ($p_\phi > 0$) in Figure 1(a), the trajectories are all quasi-periodic KAM tori. It is worth noting that there are two hyperbolic points, which may easily give rise to the occurrence of dynamical instability or transition when some of the dynamical parameters E, Λ and λ are varied. To observe this, at first we fix $\Lambda = 0$ and consider various possible values of λ. Chaos does not occur until λ is close to 1. In fact, the variation of λ seems to not have any effect on the dynamical transition in this case. See a comparison between Figure 1(a) and (b) for more information. However, there is an abrupt transition from nonchaotic to chaotic behavior when $\lambda = 1$. That is, larger chaotic areas are yielded near the two hyperbolic points in Figure 1(c). To our surprise, the chaotic areas become thinner and thinner when λ (> 1) grows gradually. Especially for $\lambda = 3$ in Figure 1(e), the dynamics are nearly nonchaotic. Thus, as a result, we can conclude that there is an abrupt change to chaos when the self-interacting coefficient λ is identical to 1, but the chaos for the system (9) with $\Lambda = 0$ and energies below the energy threshold is weaker or nonexistent as the coefficient spans 1 and gets larger. Without doubt, the effect of the parameter Λ on chaos in the system (9) with $\lambda = 0$ should be the same because there is a complete symmetry with respect to interchanging Λ and λ.

Now let us evaluate the effect on chaos of varying both the cosmological constant and the self-interacting coefficient. Similar to the above case, chaos does not appear when the two parameters are smaller than 1, as shown in Figure 1(f). Still, there is a dramatic change in the dynamics from regular to chaotic when at least one of the two parameters is equal to 1. Figure 1(g) describes the situation where two smaller chaotic areas are clustered at the separatrix for $\Lambda = \lambda = 1$. In addition, the volume of phase space shrinks. The strength of the chaos decreases, and the chaos even dies out, as both the cosmological constant and the self-interacting coefficient increase (see Fig. 1(h)–(j) for more details).

Case 2: $E = 0.59$. — The energy $E = 0.59$, 0.09 larger than the energy threshold, corresponds to the case considered in the paper of Jorás & Stuchi (2003), where the cosmological constant and the self-interacting terms in the system (9) are turned off. There are larger chaotic regions as well as a large number of islands due to resonances (Fig. 2(a)). Such a Poincaré plot can almost remain the same when either Λ or λ is smaller than 1 (Fig. 2(b) and (f)). Unlike in case 1, in Figure 2(c) and (g), the chaotic layers for at least one of Λ and λ equaling 1 become slightly thinner than those for any one of Λ and λ smaller than 1. Still, we can find that chaos is gradually weaker, and even disappears, when Λ and λ increase and at least one of them is larger than 1, as shown in Figure 2(d), (e), and (h)–(j).

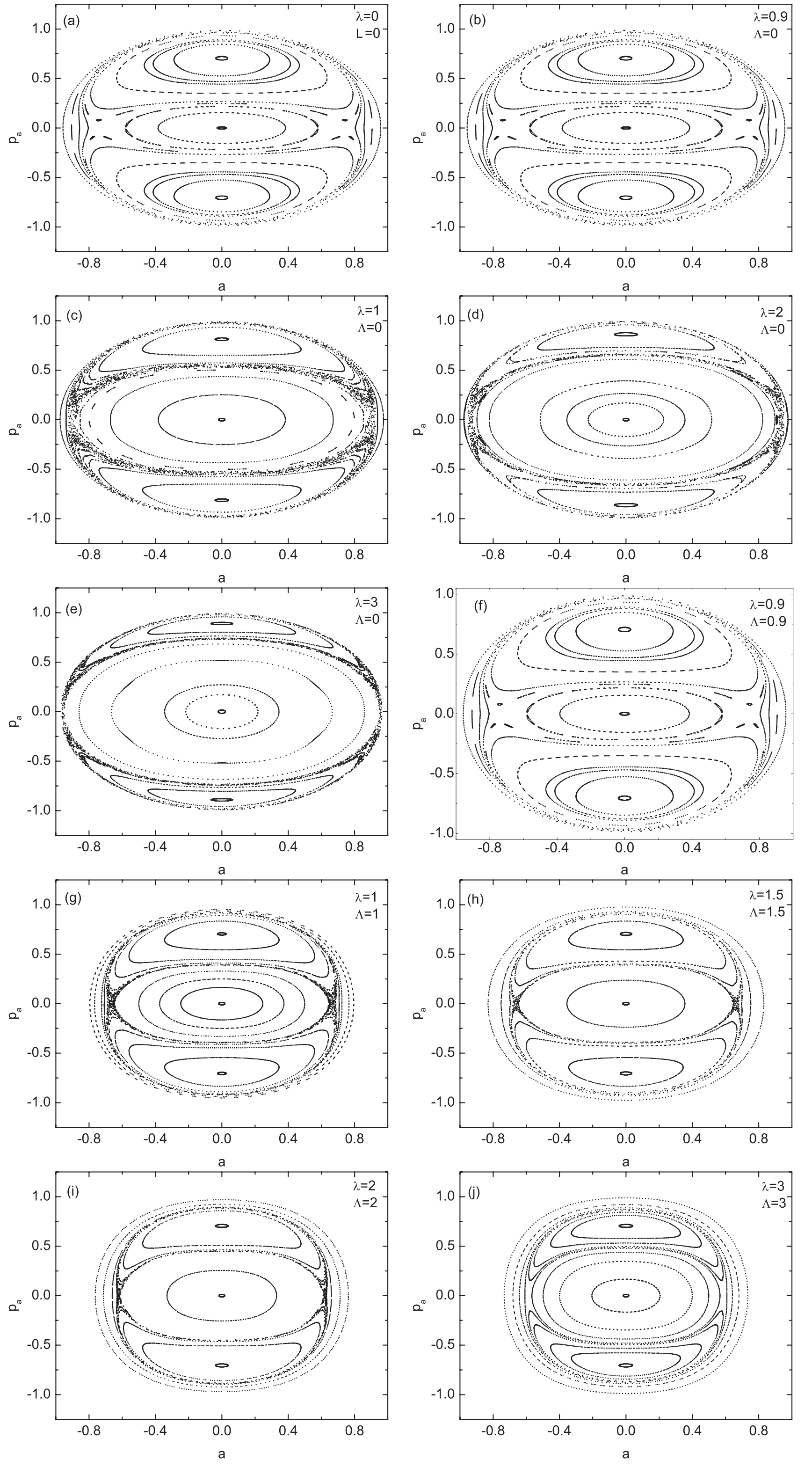

Fig. 1 Poincaré sections on the plane $\phi = 0$ ($p_\phi > 0$) with $E = 0.46$ for the imaginary problem (9).

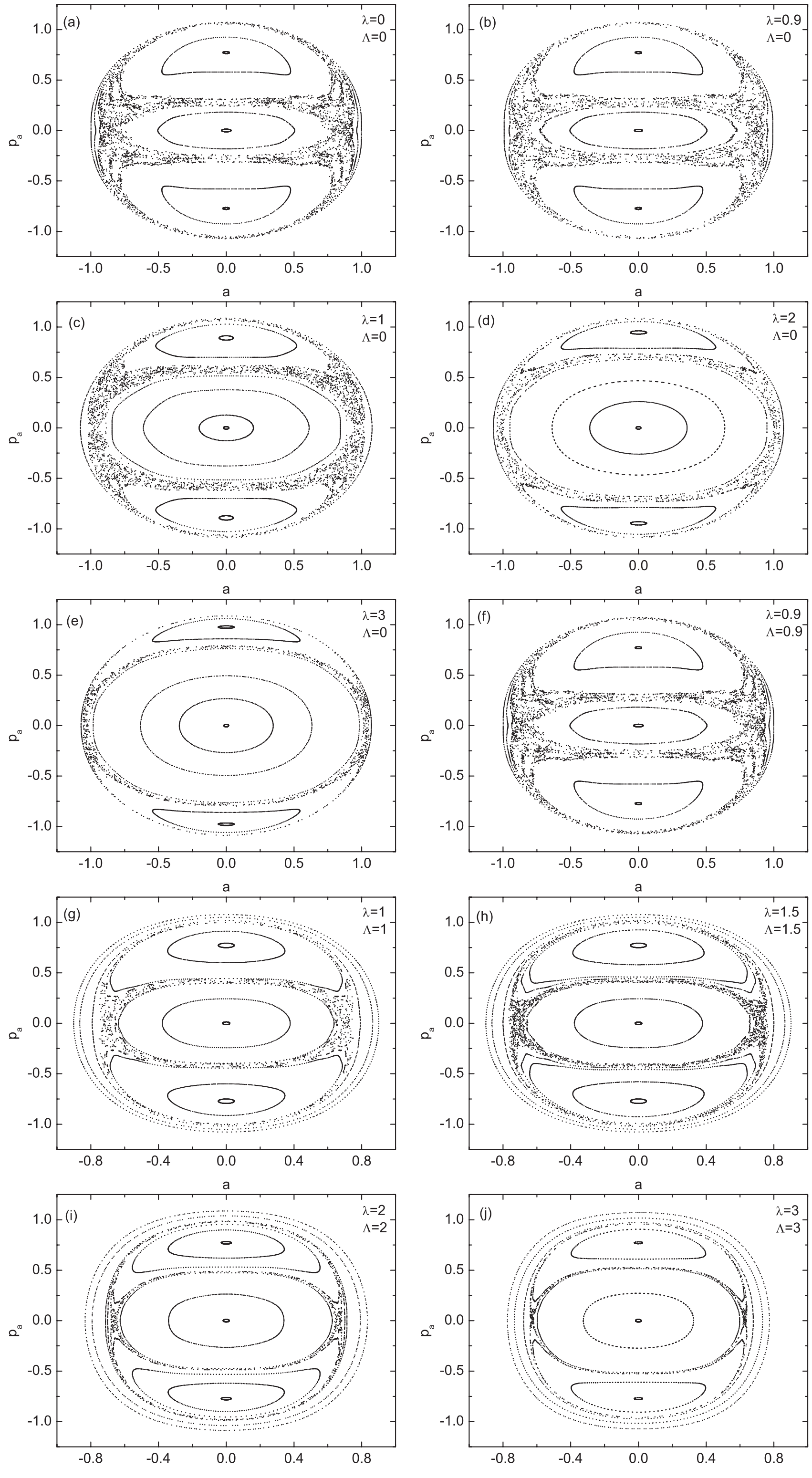

Fig. 2 Same as Fig. 1, but for $E = 0.59$.

4 CONCLUSIONS

Using Poincaré sections, we numerically examine the effect of varying the cosmological constant Λ and self-interacting coefficient λ on the transition to chaos in complexified space for a closed FRW cosmological model with a conformally coupled scalar field. The conclusions follow. Poincaré plots for the two parameters less than 1 are almost the same as those in the case of $\Lambda = \lambda = 0$. In particular, there is an abrupt transition to chaos for any energies below the energy threshold when at least one of the two parameters is 1. However, the chaotic layers become slightly thinner for any energies over the energy threshold. On the other hand, chaos gets gradually weaker, and even disappears, as both Λ and λ grow and at least one of them is larger than 1.

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