Chin. J. Astron. Astrophys. Vol. 7 (2007), No. 6, 743–756
(*http://www.chjaa.org*)

Chinese Journal of
Astronomy and
Astrophysics

A New Formulation for General Relativistic Force-Free Electrodynamics and Its Applications *

Cong Yu

National Astronomical Observatories / Yunnan Observatory, Chinese Academy of Sciences, Kunming 650011; *yccit@yahoo.com.cn*

Received 2007 February 7; accepted 2007 October 31

Abstract We formulate the general relativistic force-free electrodynamics in a new $3 + 1$ language. In this formulation, when we have properly defined electric and magnetic fields, the covariant Maxwell equations could be cast in the traditional form with new vacuum constitutive constraint equations. The fundamental equation governing a stationary, axisymmetric force-free black hole magnetosphere is derived using this formulation which recasts the Grad-Shafranov equation in a simpler way. Compared to the classic 3+1 system of Thorne and MacDonald, the new system of 3+1 equations is more suitable for numerical use for it keeps the hyperbolic structure of the electrodynamics and avoids the singularity at the event horizon. This formulation could be readily extended to non-relativistic limit and find applications in flat spacetime. We investigate its application to disk wind, black hole magnetosphere and solar physics in both flat and curved spacetime.

Key words: method: analytical — magnetic fields — accretion — jet — relativity

1 INTRODUCTION

It is broadly known that magnetic fields (the magnetosphere) around accreting black holes are very important in explaining many observed features of galactic black holes and active galactic nuclei (AGNs). The impetus of studying neutron star and black hole magnetospheres stems from our anxiety to know the central power engine in pulsars, AGNs and double radio sources (e.g. Begelman, Blandford & Rees 1984; Punsly 2001). Given the complexity of the full relativistic MHD equation, many early efforts involving nonrelativistic approximation have been made in this regard. Goldreich & Julian (1969) found that a rotating magnetized neutron star must possess a magnetosphere with charge-separated plasma. They showed that if a pulsar is surrounded by vacuum, a much stronger electric force than the gravitational force will be set up along the magnetic filed and dynamical equilibrium of the surface charge layer can not be achieved. Blandford & Payne (1982) explored the self-similar MHD solutions that describe winds launched magneto-centrifugally from a thin magnetized accretion disk. Heyvaerts & Norman (1989) established that stationary axisymmetric magnetized wind will collimate along the symmetry axis at large distance from the source. The collimation by magnetic field has been further investigated both non-relativistically (Vlahakis & Tsinganos 1998; Tsinganos & Bogovalov 2000) and relativistically (Chiueh, Li & Begelman 1991). Tsinganos & Bogovalov (2002) proposed a two-component model consisting of a relativistic wind-type outflow from a central source and a non-relativistic wind from the surrounding disk. Sulkanen & Lovelace (1990) found jet solutions in the force-force limit of the general Grad-Shafranov equation which governs the behavior of ideal MHD flow around an aligned magnetized rotating neutron star. Contopoulos et al. (1999) criticized the jet solutions, showing that a possibly unique solution is a quasi-spherical wind without jets but with

* Supported by the National Natural Science Foundation of China.

an equatorial current sheet. Their numerical treatment was later adopted by Ogura & Kojima (2003), who explored the magnetosphere by a numerical solution with higher spatial resolution and obtained essentially the same results. Later, much attention was paid to some key subtle issues regarding the separatrix between the open and closed field line region as well as the intersection of this separatrix with the equatorial plane by Uzdensky (2003). Recently, inspired by numerical simulation, Narayan et al. (2006) found self-similar wind solutions.

To incorporate the general relativistic effects, Blandford & Znajek (1977) used a covariant equations of electrodynamics and dealt with the components of the electromagnetic field tensor and four potential, and found that magnetic fields threading the horizon of a Kerr black hole can extract rotational energy of the hole powering luminous double radio lobes. In theoretical studies of the BZ effects, authors have usually adopted the framework of force-free electrodynamics. The plasma is considered to have such a low density that it is dynamically unimportant while still having the ability to carry the charges and currents without much dissipation. Under the assumptions of time stationarity and axisymmetry, the main equation governing the magnetosphere behavior reduces to the Grad-Shafranov equation. In an attempt to bridge the gap between the language of black hole electrodynamics and that of flat spacetime electrodynamics, a 3+1 decomposition of the Maxwell equation was first made by Landau & Lifshitz (1975). Later, a new 3+1 formulation was proposed by Thorne & MacDonold (1982), who adopted the metric used in the Arnowitt-Deser-Misner (ADM) formalism. In this formalism, McDonald & Thorne (1982) cast the equations of black hole electrodynamics into more or less traditional form in terms of the spatial vectors of electric and magnetic field as measured by a fiducial observer (FIDO). Recently, Sakai & Shibata (2003) and Komissarov (2004), in order to make covariant language more tractable, developed different 3+1 languages to interpret the general relativity effects. Subsequently, Menon & Dermer (2005) used such new formulation to find new type of analytic solutions.

With this formulation, quite a few of the problems could be discussed. The astrophysical studies, which can get more insights from black hole electrodynamics, are wide-ranging. Studies of the BZ-process with field configuration given by recent numerical simulation have been carried out by Komissarov (2002, 2004). The force-free electrodynamics is closely relevant to studies of accretion disk with regard to dynamo action, hydromagnetic winds, jet formation and collimation in AGNs as well as microquasars. Park & Vishniac (1989) discussed the problem in the non-stationary regime, Okamoto (1992) investigated the evolution of black hole magnetosphere. Lee et al. (2000) discussed the BZ effects in GRBs and recently Kim et al. (2005) systematically investigated the differences between pulsar and black hole magnetospheres. Problems such as collimation of jets (Fendt & Camenzind 1995), AGN dynamo processes (Khanna & Camenzind 1996), efficiency of magnetized thin disk (Gammie 1999), centrifugally driven MHD-wind in AGN (Camenzind 1986), super- and trans-fast MHD winds/accretion from rapid rotators (Takahashi 1991; Takahashi & Shibata 1998), magnetized torus in hypernovae and black hole-neutron star coalescence (van Putten 2003) as well as energy and angular momentum transport between the magnetic field and the matter could be explored in such regimes when plasma dynamic effects are taken into account (Takahashi et al. 1990). Furthermore, we could consider the shock effects when plasma accretes onto Kerr black holes (Takahashi et al. 2002, 2006). Ghosh (2000) classified solutions in the Schwarzschild black hole magnetosphere. Unlike previous exploration on open magnetic field configuration, Uzdensky (2004, 2005) recently focused on closed magnetic field solutions in the context of magnetically linked black hole-disk system in both the Schwarzschild and the Kerr black hole magnetospheres.

Such a formulation also finds application in solar physics. It is well known that the force-free magnetic field plays a very significant role in solar eruptive phenomena, e.g., coronal mass ejections. Pioneering modelling on solar force-free field in spherical coordinates was carried out by Low (1986). Later partially open magnetic field in solar corona was taken into account (Low & Smith 1993). Recent calculations show that energy storage could give rise to corona mass ejection (Low & Zhang 2002, 2004; Hu 2004; Zhang & Low 2004, 2005).

Throughout this paper, we adopt the $(- + ++)$ signature for the spacetime and assume the Greek indices to range from 0 to 3 (in the spacetime components) and the Latin symbols to range from 1 to 3 (in the spatial components), and the Einstein summation convention is used.

2 NEW FORMULATION OF ELECTRODYNAMICS IN ABSOLUTE SPACE

In this section, we first briefly state the essential equations of electrodynamics in an absolute three dimensional space. Komissarov (2004) explained how these equations are derived. The construction of absolute space is facilitated by noting that an arbitrary spacetime metric can be written in the form

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(\alpha^2 - \beta^2)dt^2 + 2\beta_i dx^i dt + \gamma_{ij}dx^i dx^j \,, \tag{1}$$

where $g_{\mu\nu}$ is the general spacetime metric, α the lapse function (also called gravitational redshift factor), $\boldsymbol{\beta}_i$ the shift vector and γ_{ij} the components of spatial metric. The functions x^i serve as coordinates for our space-like hypersurfaces defined by constant values of t. In this paper we consider only coordinates that satisfy $\partial_t g_{\mu\nu} = 0$. We can think of electric and magnetic fields ($\boldsymbol{E}$ and $\boldsymbol{B}$) as objects existing in our absolute space Σ. The covariant Maxwell equations are (e.g. Jackson 1975)

$$\nabla^*_\nu F^{\mu\nu} = 0 \,, \quad \nabla_\nu F^{\mu\nu} = I^\mu \,, \tag{2}$$

where $F^{\mu\nu}$ is the Maxwell tensor of the electromagnetic field, ${}^*F^{\mu\nu}$ the Faraday tensor, I^μ the four vector of the electric current and ∇_μ the covariant derivative associated with the general metric $g_{\mu\nu}$. Note that ${}^*F^{\mu\nu} = e^{\mu\nu\rho\sigma}F_{\rho\sigma}/2$ and $F^{\mu\nu} = -e^{\mu\nu\rho\sigma}\,{}^*F_{\rho\sigma}/2$, where $e^{\mu\nu\rho\sigma} = (-g)^{1/2}[\mu\nu\rho\sigma]$ and $[\mu\nu\rho\sigma]$ is completely antisymmetric Levi-Civita symbol and g is the determinant of the metric $g_{\mu\nu}$, i.e., $g = \mathrm{Det}(g_{\mu\nu})$. When the electromagnetic quantities are properly defined, the covariant Maxwell equations in the curved spacetime could be written down in a familiar form. The following is the definitions of the electromagnetic quantities (Komissarov 2004):

$$E_i = F_{it} \,, \quad D^i = \alpha F^{ti} \,, \quad B^i = \alpha^* F^{it} \,, \quad H_i =^* F_{it}, \tag{3}$$

$$E^i = \gamma^{ij}E_j \,, \quad D_i = \gamma_{ij}D^j \,, \quad B_i = \gamma_{ij}B^j \,, \quad H^i = \gamma^{ij}H_j. \tag{4}$$

Here both the covariant and contravariant definitions of the electric field $\boldsymbol{E}$, electric displacement $\boldsymbol{D}$, magnetic induction $\boldsymbol{B}$ and magnetic field $\boldsymbol{H}$ are explicitly given. Note that the lower and upper indices of the Maxwell and Farady tensors are related by the general metric $g_{\mu\nu}$ via $F^{\mu\nu} = g^{\xi\mu}g^{\sigma\nu}F_{\xi\sigma}$, where $g^{\xi\mu}$ is the inverse of $g_{\xi\mu}$ and γ^{ij} is the inverse of γ_{ij}. The time evolution equations for $\boldsymbol{E}$ and $\boldsymbol{B}$ in the presence of a charge density $\rho_e = \alpha I^t$ and electric current density vector $\boldsymbol{J}(J^k = \alpha I^k)$, in our absolute (curved) space endowed with a metric γ_{ij}, are given by the following set of Maxwell's equations:

$$\nabla \cdot \boldsymbol{B} = 0, \quad \partial_{\mathrm{t}}\boldsymbol{B} + \nabla \times \boldsymbol{E} = 0, \tag{5}$$

$$\nabla \cdot \boldsymbol{D} = \rho_e, \quad -\partial_{\mathrm{t}}\boldsymbol{D} + \nabla \times \boldsymbol{H} = \boldsymbol{J}. \tag{6}$$

Note that $\boldsymbol{E}$, $\boldsymbol{B}$, $\boldsymbol{D}$, $\boldsymbol{H}$, and $\boldsymbol{J}$ are, in general, time-dependent vectors in the three dimensional absolute space Σ. Also, ∇ is the covariant derivative defined by the spatial metric γ_{ij} on Σ. As usual, the curl of a vector field is defined by the expression $(\nabla \times \boldsymbol{A})^i = \epsilon^{ijk}\partial_j A_k$, where ϵ^{ijk} is the completely antisymmetric pseudo-tensor such that $\epsilon^{ijk} = \frac{1}{\sqrt{\gamma}}[ijk]$, and $\gamma = \mathrm{Det}(\gamma_{ij})$ (e.g. Weinberg 1972). It is easily seen that Maxwell's equations imply the continuity equation $\partial_{\mathrm{t}}\rho_e + \nabla\cdot\boldsymbol{J} = 0$. Contrary to its flat space counterparts, even in regions of negligible electric and magnetic susceptibilities, $\boldsymbol{E} \neq \boldsymbol{D}$, and $\boldsymbol{B} \neq \boldsymbol{H}$. Instead, they satisfy the following important relations,

$$\boldsymbol{E} = \alpha\boldsymbol{D} + \boldsymbol{\beta}\times\boldsymbol{B} \,, \quad \boldsymbol{H} = \alpha\boldsymbol{B} - \boldsymbol{\beta}\times\boldsymbol{D} \,. \tag{7}$$

Or, equivalently and more clearly in component form,

$$E^i = \alpha D^i + \epsilon^{ijk}\boldsymbol{\beta}_j B_k \,, \quad H^i = \alpha B^i - \epsilon^{ijk}\boldsymbol{\beta}_j D_k \,. \tag{8}$$

All effects of gravity are hidden in the above constitutive equations and in the spatial metric γ_{ij}. If we establish the following correspondences, $\boldsymbol{B}$(this paper) $\rightarrow\ \widetilde{\boldsymbol{B}}$ (MT82) and $\boldsymbol{D}$(this paper) $\rightarrow\ \widetilde{\boldsymbol{E}}$ (MT82), the equations (2.17a)–(2.17d) in MacDonald & Thorne (1982) can be readily identified (see Appendix C for details). We find that this new formulation is more compact and seems more simpler than in Thorne & MacDonald (1982) and MacDonald & Thorne (1982). This new formulation is conspicuously similar to the equations of electrodynamics in a non-vacuum medium. Thorne & MacDonald (1982)'s formulation was derived under the particular Boyer-Lindquist coordinates. This formulation has been used successfully in a wide range of studies in black hole electrodynamics, but the Boyer-Lindquist coordinates have one evident disadvantage - singularity at the event horizon. This new formulation does not suffer from this disadvantage for it can be used in the lesser known Kerr-Schild coordinates, in which the singularity at event horizon is naturally removed. The new formulations avoid the Lie derivatives and keep the equations as hyperbolic conservative equations. Then the modern high resolution Godunov-type numerical scheme (e.g. Yu 2006) can be readily applied to this new formulation without much modification.

3 PROPERTIES OF STATIONARY, AXISYMMETRIC FORCE FREE MAGNETOSPHERES

The condition that the magnetosphere is force free brings about enough structure into Maxwell's equations to enable the introduction of a streaming function that will help us visualize the field structure in geometric terms. It is traditional to use spheroidal spatial coordinates given by $x^i = (r, \theta, \varphi)$ such that $\boldsymbol{m} = \partial_\varphi$. Now any vector field $\boldsymbol{A}$ can be decomposed into a poloidal and a toroidal component ($\boldsymbol{A}_{\rm p}$ and $\boldsymbol{A}_{\rm t}$), such that $\boldsymbol{A} = \boldsymbol{A}_{\rm p} + \boldsymbol{A}_{\rm t}$, where $\boldsymbol{A}_{\rm p} = A^r\partial_r + A^\theta\partial_\theta$ and $\boldsymbol{A}_{\rm t} = A^\varphi\partial_\varphi$. Then the assumptions of stationarity and axisymmetry imply that $\partial_\varphi g_{\mu\nu} = 0 = \partial_{\rm t} g_{\mu\nu}$ and naturally leads to $\boldsymbol{E}_{\rm t} = 0$. In our absolute space framework, then, the force free condition reduces to

$$\boldsymbol{E} \cdot \boldsymbol{J} = 0\,, \qquad \rho_e \boldsymbol{E} + \boldsymbol{J} \times \boldsymbol{B} = 0\,. \tag{9}$$

These conditions imply that,

$$\boldsymbol{E}_{\rm p} \cdot \boldsymbol{B}_{\rm p} = 0\,, \tag{10}$$

$$\boldsymbol{E}_{\rm p} \cdot \boldsymbol{J}_{\rm p} = 0\,, \quad \boldsymbol{B}_{\rm p} \parallel \boldsymbol{J}_{\rm p}\,. \tag{11}$$

Equation (10) together with $\boldsymbol{E}_{\rm t} = 0$ implies that there exists a vector $\boldsymbol{\omega} = \Omega\partial_\varphi$ such that

$$\boldsymbol{E} = -\,\boldsymbol{\omega} \times \boldsymbol{B}\,. \tag{12}$$

From the vanishing of the curl of E under the stationarity condition (Eq. (5)), one finds that $\boldsymbol{B} \cdot \nabla\Omega = 0$, and can see that the similar relation, $\boldsymbol{B} \cdot \nabla H_\varphi = 0$, also holds.

3.1 Fields and Currents in the 3+1 Formulation

To simplify the calculations, we shall assume that the spatial coordinates are orthogonal, and that the shift vector β is purely toroidal, i.e., $\boldsymbol{\beta} = (0, 0, \beta_\varphi)$. The Kerr solution written in Boyer-Lindquist coordinates can be written in this form. Surfaces of constant Ω are referred to as poloidal surfaces (note not to be confused with the poloidal components of a vector). From $\boldsymbol{B} \cdot \nabla\Omega = 0$ it is clear that B is tangential to the poloidal surfaces. Since Ω does not have any φ dependence, it is clear that the poloidal magnetic field $B_{\rm p}$ could be written as

$$\boldsymbol{B}_{\rm p} = \frac{Q}{\sqrt{\gamma}}(-\Omega_{,\theta}\partial_r + \Omega_{,r}\partial_\theta)\,, \tag{13}$$

where, for the moment, Q is an arbitrary function. This must be so because in the two dimensional subspace given by $\Omega = \mathrm{const}$, there is a unique vector (modulo magnitude) which is perpendicular to $\nabla\Omega$. The condition that $\boldsymbol{B}$ is divergence-free means that Q satisfies

$$Q_{,r}\,\Omega_{,\theta} \;=\; Q_{,\theta}\,\Omega_{,r}\,. \tag{14}$$

Consequently, Q is a poloidal function (a function that is constant on poloidal surfaces). In comparison with Blandford & Znajek (1977), here $Qd\Omega \equiv -dA_\varphi$. The electric field is immediately calculated from Equation (12) and, as expected, is the gradient of a scalar function:

$$\boldsymbol{E}_{\rm p} = Q\Omega d\Omega\,. \tag{15}$$

From the first half of Equation (7), we see that

$$\boldsymbol{D} = \boldsymbol{D}_{\rm p} = \frac{Q}{\alpha}(\Omega + \beta^\varphi)d\Omega\,. \tag{16}$$

Similarly, the expression for $H_{\rm p}$ can be calculated from the second half of Equation (7), giving

$$\boldsymbol{H}_{\rm p} = (\alpha^2 - \beta^2 - \beta_\varphi \Omega)\frac{\boldsymbol{B}_{\rm p}}{\alpha}\,. \tag{17}$$

The electric charge is determined by the divergence of the $D_{\rm p}$ (Eq. (6)). Explicitly,

$$\sqrt{\gamma}\rho_e = \partial_r \Big[\frac{Q}{\alpha\sqrt{\gamma}}(\gamma_{\varphi\varphi}\Omega + \beta_\varphi)\gamma_{\theta\theta}\Omega_{,r}\Big] + \partial_\theta \Big[\frac{Q}{\alpha\sqrt{\gamma}}(\gamma_{\varphi\varphi}\Omega + \beta_\varphi)\gamma_{rr}\Omega_{,\theta}\Big]\,. \tag{18}$$

The toroidal component of the electric current density vector can be obtained from the derivatives of the components of $\boldsymbol{H}_{\rm p}$:

$$\begin{aligned}\sqrt{\gamma}J^\varphi &= H_{\theta,r} - H_{r,\theta} \\ &= \partial_r \Big[\frac{Q}{\alpha\sqrt{\gamma}}(\alpha^2 - \beta^2 - \beta_\varphi\Omega)\gamma_{\theta\theta}\Omega_{,r}\Big] + \partial_\theta \Big[\frac{Q}{\alpha\sqrt{\gamma}}(\alpha^2 - \beta^2 - \beta_\varphi\Omega)\gamma_{rr}\Omega_{,\theta}\Big]\,. \end{aligned} \tag{19}$$

It is clear from the above discussion that the poloidal fields and, consequently, the toroidal current J^φ are uniquely described by the poloidal functions Ω and Q. On the other hand, the toroidal fields and the poloidal currents can be determined from the poloidal function H_φ. In particular, from Equation (7), it is clear that $H_\varphi = \alpha B_\varphi$. Maxwell's equation (Eq. (6)) implies

$$\sqrt{\gamma}\boldsymbol{J}_{\rm p} = H_{\varphi,\theta}\partial_r - H_{\varphi,r}\partial_\theta\,. \tag{20}$$

Thus we see that fields and currents separate into two distinct categories: objects that are determined by Ω and Q, and those that are determined by H_φ. Apart from the fact that H_φ is a poloidal function (by definition, Ω is), it is not yet clear as to how these two functions are dynamically related, and this issue will be probed in the following section.

3.2 Grad-Shafranov Equations of Black Hole Magnetospheres

For the fields and currents given in the previous section, the first half of Equation (9) is trivially satisfied. Since the toroidal component of the electric field vanishes, it is easily checked from the second half of Equation (9) that $(\boldsymbol{J} \times \boldsymbol{B})_\varphi = 0$. Thus the only remaining requirement for a force-free solution is

$$\rho_e \boldsymbol{E}_{\rm p} + (\boldsymbol{J} \times \boldsymbol{B})_{\rm p} = 0\,. \tag{21}$$

The implications of this equation can be made clear by projecting the equation onto $\boldsymbol{E}_{\rm p}$ and $\boldsymbol{B}_{\rm p}$. When projected onto $\boldsymbol{B}_{\rm p}$, an identity is trivially satisfied, giving no constraints. On the contrary, projecting the second half of Equation (9) onto $\boldsymbol{E}_{\rm p}$ gives the following important relation:

$$\rho_e \boldsymbol{E}^2 + [(\boldsymbol{J}_{\rm p} \times \boldsymbol{B}_{\rm t}) + (\boldsymbol{J}_{\rm t} \times \boldsymbol{B}_{\rm p})] \cdot \boldsymbol{E}_{\rm p} = 0\,. \tag{22}$$

After some manipulations, Equation (22) gets a compact form,

$$\frac{1}{2Q}\frac{dH_\varphi^2}{d\Omega} = \alpha(\rho_e \Omega \gamma_{\varphi\varphi} - J_\varphi). \tag{23}$$

This is the final constraint equation, which in Kerr metric can be identified as the commonly used Grad-Shafranov equation (see Appendix A). If Ω and Q are chosen such that the right hand side of the above equation is a poloidal function, then H_φ will continue to be poloidal. Then the poloidal functions Ω, Q and H_φ uniquely determine all the currents and fields. It is important to realize that Ω is not to be thought of as a potential: physically relevant quantities like the electric field depend on Ω directly, and are not invariant transformations of the type $\Omega \to \Omega + \mathrm{const}$. The charge density ρ and the toroidal current J_φ are functions of Ω and Q (see Eqs. (18) and (19)).

Inserting Equations (18) and (19) into Equation (23), our constraint equation gives

$$\frac{\sqrt{\gamma}}{\alpha\gamma_{\varphi\varphi}}\frac{1}{2Q}\frac{dH_\varphi^2}{d\Omega} = \Big[\Omega\partial_r\left(\frac{Q}{\alpha\sqrt{\gamma}}(\gamma_{\varphi\varphi}\Omega+\beta_\varphi)\gamma_{\theta\theta}\Omega_{,r}\right)+\Omega\partial_\theta\left(\frac{Q}{\alpha\sqrt{\gamma}}(\gamma_{\varphi\varphi}\Omega+\beta_\varphi)\gamma_{rr}\Omega_{,\theta}\right)+ \partial_r\left(\frac{Q}{\alpha\sqrt{\gamma}}(\beta^2-\alpha^2+\beta_\varphi\Omega)\gamma_{\theta\theta}\Omega_{,r}\right)+\partial_\theta\left(\frac{Q}{\alpha\sqrt{\gamma}}(\beta^2-\alpha^2+\beta_\varphi\Omega)\gamma_{rr}\Omega_{,\theta}\right)\Big]. \tag{24}$$

Therefore, Equations (23) and (24) written in the 3+1 formalism, are equivalent to equation (3.14) of Blandford & Znajek (1977). In the Boyer-Lindquist coordinates of the Kerr spacetime, this equation can be put in a more familiar Grad-Shafranov form (McDonald & Thorne 1982; Ghosh 2000; Lee et al. 2000; Uzdensky 2005; for detailed derivation see Appendix A, where $I = H_\varphi$ and $d\Psi = -Qd\Omega$),

$$\nabla\cdot\left\{\frac{\alpha}{\varpi^2}\left[1-\frac{(\Omega-\omega)^2\varpi^2}{\alpha^2}\right]\nabla\Psi\right\}+\frac{(\Omega-\omega)}{\alpha}\frac{d\Omega}{d\Psi}|\nabla\Psi|^2 = -\frac{1}{\alpha\varpi^2}\mathrm{II}'(\Psi), \tag{25}$$

where

$$\boldsymbol{B}_\mathrm{p} = \nabla\Psi\times\nabla\phi\,, \quad \boldsymbol{B}_\mathrm{t} = \frac{\mathrm{I}}{\alpha}\nabla\phi\,. \tag{26}$$

$$\boldsymbol{D}(r,\theta) = \boldsymbol{D}_\mathrm{p} = -\frac{\Omega-\omega}{\alpha}\nabla\Psi\,, \quad D_\phi = 0. \tag{27}$$

The prime here denotes derivative with respect to Ψ. We could see that the quantity Ψ plays the same role as A_ϕ in Blandford & Znajek (1977) and McDonald & Thorne (1982). Here

$$\begin{aligned} \alpha = \frac{\rho\sqrt{\Delta}}{\Sigma}\,, \quad \varpi = \frac{\Sigma}{\rho}\sin\theta\,, \\ \Delta = r^2-2Mr+a^2\,, \quad \rho^2 = r^2+a^2\cos^2\theta\,, \\ \Sigma^2 = (r^2+a^2)^2-\Delta a^2\sin^2\theta\,, \\ \gamma_{rr} = \frac{\rho^2}{\Delta}\,, \quad \gamma_{\theta\theta} = \rho^2\,, \quad \gamma_{\phi\phi} = \varpi^2\,, \\ \beta^r = 0\,, \quad \beta^\theta = 0\,, \quad \omega = -\beta^\phi = 2aMr/\Sigma^2\,. \end{aligned} \tag{28}$$

4 APPLICATIONS TO ACCRETION DISK WIND, SOLAR PHYSICS AND BLACK HOLE MAGNETOSPHERE

4.1 Force-Free Self Similar Disk Wind and Solar Force-Free Magnetic Field in Flat Spacetime

Given the complexity of the full relativistic MHD equations, investigators have adopted various nonrelativistic approximations (e.g., Okamoto 1974; Contopoulos & Lovelace 1994; Narayan et al. 2006). There are several ways to approximate the relativistic winds. One is to use the full MHD equations (e.g. Vlahakis & Konigl 2003). Another approach is to ignore the pressure or internal energy and consider just the inertia of the cold fluid (Contopoulos 1994). A further simplification is the force-free approximation, which even ignores the gas inertia. In this approximation, the only role of the plasma is to supply charges and currents to sustain the electromagnetic fields. Force-free electrodynamics is the simplest and neatest model of magnetized outflows. If magnetic fields play an essential role in disk winds and jets, then the force-free approximation would capture most of the essential physics. Recently, force-free electrodynamics has been given a formulation in terms of a system of time-dependent equations (Komissarov 2002, 2004; McKinney 2006). In cylindrical coordinates of flat spacetime, Equation (25) reduces to (e.g. Okamoto 1974; Contopoulos 1995),

$$(1-r^2)\left(\frac{\partial^2\Psi}{\partial r^2}+\frac{\partial^2\Psi}{\partial z^2}\right)-\frac{1+r^2}{r}\frac{\partial\Psi}{\partial r} = -\mathrm{II}'(\Psi). \tag{29}$$

4.1.1 Michel's Monopole Solution and Blandford's Paraboloidal Solution

In the simplified case where the right side of Equation (29) vanishes, i.e., only the poloidal components exist, two analytical solutions can be found. The first is

$$\Psi(r,z) = 1 - \frac{z}{\sqrt{r^2+z^2}}\,. \tag{30}$$

This is the monopole solution found by Michel (1973). Here, the three components of the magnetic field are

$$B_r(r,z) = \frac{r}{(r^2+z^2)^{3/2}}\,, \tag{31}$$

$$B_\phi(r,z) = -\frac{\Omega}{c}\frac{r}{(r^2+z^2)}\,, \tag{32}$$

$$B_z(r,z) = \frac{z}{(r^2+z^2)^{3/2}}\,. \tag{33}$$

The second solution is

$$\Psi(r,z) = \sqrt{r^2+z^2} - z\,. \tag{34}$$

This is the self-similar paraboloidal solution (Blandford 1976). Here the three components of the magnetic field are

$$B_r(r,z) \;=\; \frac{1}{r} - \frac{z}{r(r^2+z^2)^{1/2}}\,, \tag{35}$$

$$B_\phi(r,z) \;=\; \frac{1}{r} - \frac{z}{r(r^2+z^2)^{1/2}}\,, \tag{36}$$

$$B_z(r,z) \;=\; \frac{1}{(r^2+z^2)^{1/2}}\,. \tag{37}$$

Recently, Narayan et al. (2006), inspired by results of numerical simulation, reinvestigated this problem. They found that there exist other self-similar force-free wind solutions, which suggest that disk rotation plays a strong role in jet acceleration.

4.1.2 Normal and inverted Quiescent Prominence in Solar Physics

Recent calculations show that both the normal and inverted characteristic magnetic configurations are capable of storing enough magnetic energy to overcome the Aly limit for opening an initially closed magnetic field (Zhang & Low 2004). As a preliminary investigation, we construct prominences of these two field configurations. The corona is regarded as axisymmetric outside a sphere and the prominence is taken to be a cold plasma inside a purely azimuthal magnetic flux rope, held in equilibrium by the prominence's weight and by an external poloidal magnetic field rigidly anchored to the base of the modelled corona. The magnetic flux Ψ, which determines the poloidal magnetic field, is composed of three parts. $\Psi = \Psi_{\rm dipole} + \Psi_{\rm I} - \Psi_{\rm image}$, where $\Psi_{\rm dipole}$ is the classical dipole field of form $B_0 \sin^2\theta/r$, $\Psi_{\rm I}$ is the magnetic flux generated by a circular line current of intensity I_0 lying in the equator, with radius $a_0 > 1$ and centered at $r = 0$. The image flux is subtracted to make the combination $\Psi_{\rm I} - \Psi_{\rm image}$ vanish at the photosphere. The detailed expressions of $\Psi_{\rm I}$ and $\Psi_{\rm image}$ are taken from Jackson (1975) :

$$\Psi_{\rm I}(r,\theta) = 2I_0 a_0 r \sin\theta \int_0^\pi \frac{\cos\varphi d\varphi}{(a_0^2 + r^2 - 2a_0 r\sin\theta\cos\varphi)^{1/2}}\,, \tag{38}$$

$$\Psi_{\rm image}(r,\theta) = 2I_0 a_0 r \sin\theta \int_0^\pi \frac{\cos\varphi d\varphi}{(a_0^2 r^2 + 1 - 2a_0 r\sin\theta\cos\varphi)^{1/2}}\,. \tag{39}$$

The inverted and normal prominence fields could be constructed in parallel. When B_0 and I_0 are of the same sign, we obtain the inverted configuration; the opposite sign gives the normal configuration. Figure 1 illustrates these two different magnetic field configurations.

It is found that the magnetic topology is very important for the energy storage. Detailed calculations show that the normal field configurations reach more readily energetic states that give rise to CME expulsions than the inverted field configurations (Zhang & Low 2004).

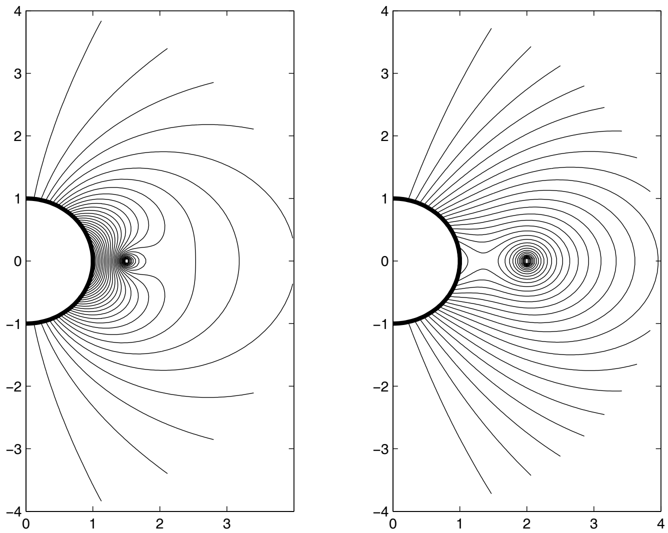

Fig. 1 Two types (normal and inverted) of prominence magnetic field configuration. Thick solid line shows the solar photosphere.

4.2 Force-Free Black Hole Magnetosphere

4.2.1 Schwarzschild Black Hole Magnetosphere

As another example of application, we would study in this section the black hole magnetospheres in which the current potential $I(\psi)$ is of such form that the last term on the right hand side of Equation (25) vanishes. In this case, the poloidal and toroidal field components are somewhat decoupled. For Schwarzschild black holes, $\Omega = \omega = 0$ and Equation (25) in Schwarzschild coordinates can be explicitly written as:

$$r^2 \frac{\partial}{\partial r}\left[\left(1 - \frac{2M}{r}\right)\frac{\partial \psi}{\partial r}\right] + \sin\theta \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \theta}\right) = 0\,. \tag{40}$$

Separable solutions of the above equation are of the form,

$$\Psi(r,\theta) = R(r)\Theta(\theta)\,. \tag{41}$$

Substituting Equation (41) into Equation (40), we obtain

$$\frac{d}{d\theta}\left(\frac{1}{\sin\theta}\frac{\partial \Theta}{\partial \theta}\right) = -\lambda \frac{\Theta}{\sin\theta}\,, \tag{42}$$

$$\frac{\partial}{\partial r}\left[\left(1 - \frac{2M}{r}\right)\frac{\partial R}{\partial r}\right] = \lambda \frac{R}{r^2}\,, \tag{43}$$

where λ is the separation constant. The lowest order solutions are the special case with $\lambda = 0$, which can be obtained by setting $\lambda = 0$ in the above two equations. The solutions are then

$$\Theta(\theta) = \mathrm{a}\cos\theta + \mathrm{b}\,, \tag{44}$$

$$R(r) = \mathrm{c}(r + 2M\ln(r - 2M)) + \mathrm{d}\,, \tag{45}$$

where a, b, c, d are constants. The solution with $\mathrm{c} = 0$ is the Schwarzschild monopole, while the solution with $\mathrm{a} + \mathrm{b} = 0$ and $\mathrm{d} = 0$ is separable Schwarzschild paraboloid, explicitly it is $\Psi = \big(r + 2\ln(r -$

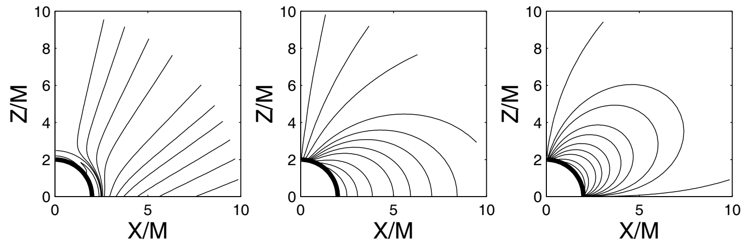

Fig. 2 Poloidal magnetic field lines of the separable Schwarzschild paraboloid (left), the Schwarzschild dipole ($m = 1$, middle) and the Schwarzschild quadrupole ($m = 2$, right). The horizon is shown by the thick solid line. All lengths are in natural units.

$2M)\big)(1 - \cos\theta)$. The order of the solution is denoted by the ordinal number m, related to the constant λ by $\lambda = m(m+1)$. Then Equations (42) and (43) become (Ghosh 2000):

$$(1 - x^2)\frac{d^2\Theta}{dx^2} + m(m+1)\Theta = 0\,, \tag{46}$$

$$(1 - z^2)\frac{d^2R}{dz^2} - 2\frac{dR}{dz} + m(m+1)R = 0\,, \tag{47}$$

where $x = \cos\theta$ and $z = r/M - 1$. The solutions of the Θ equation (46) are Gegenbauer polynomials $\mathcal{C}^{(0,2)}_{m-1}(z)$. The solutions of the radial Equation (47) are Jacobi polynomials $\mathcal{P}^{(0,2)}_{m-1}(z)$ and Jacobi functions of the second kind $\mathcal{Q}^{(0,2)}_{m-1}(z)$. We give three examples of the poloidal magnetic field configuration in Figure 2. For $M = 0$, the Schwarzschild metric reduces to the polar spherical coordinates commonly used in solar physics. Using a spectral method, Flyer et al. (2004) explored numerical solutions of the Grad-Shafranov equation, $\mathcal{L}\Psi = -\gamma\Psi^n$, in polar spherical coordinates, where $\mathcal{L} = \partial^2/\partial r^2 + (1/r^2)\partial^2/\partial\theta^2 - (\cot\theta/r^2)\partial/\partial\theta$. In our formulation, we can obtain this equation by simply taking $M = 0$.

4.2.2 *Force-Free Black Hole Magnetosphere in Kerr Metric*

Magnetospheres in the Kerr metric are complex, which we shall now investigate using a simplified approach. Because vacuum is unstable to the cascade production of electron-positron pairs when the magnetic field is strong enough, a surrounding magnetosphere will be established and the energy and angular momentum will be extracted electromagnetically (Blandford & Znajek 1977). Making use of this new formulation, we re-derive Equation (25), which governs a stationary axisymmetric magnetosphere. Blandford & Znajek (1977) found two types of perturbative solutions: i) split monopole magnetic field $A_\phi(r,\theta) = -C\cos\theta$, ii) paraboloidal magnetic field $A_\phi(r,\theta) = \frac{C}{2}\big[r(1-\cos\theta) + 2M(1+\cos\theta)(1+\ln(1+\cos\theta))\big]$. Based on these, they then explicitly explained the energy and angular momentum extraction from a rotating hole by a mechanism directly analogous to that of Goldreich & Julian (1969). Ghosh & Abramowicz (1997) found a new solution, which has asymptotically vertical field lines with specified slopes on the equatorial plane. Adding the separable solutions to the paraboloidal solutions above would produce further non-separable solutions, but this is of no fundamental importance. High order non-separable solutions would be more important for practical purposes. Very recently, Menon & Dermer (2005) followed the same 3+1 formulation in absolute space and found solutions that generalize the Blandford-Znajek monopole solution for a slowly rotating black hole to one with arbitrary angular momentum. They found through their solution that energy and angular momentum extraction occur mostly in the equatorial plane.

5 CONCLUSIONS AND FUTURE WORK

We have presented a new framework for describing general relativistic force-free electrodynamics. With a proper definition of electromagnetic quantities, the covariant Maxwell equations can be recast in a form similar to, and more general than the classical equations of electrodynamics. With this formulation, we have summarized some of the basic properties of stationary, axisymmetric force-free black hole magnetospheres, and re-derived the Grad-Shafranov equations that govern their behavior. As simple applications of the complicated Grad-Shafranov equation, we have re-investigated Michel's monopole solution and Blandford's paraboloidal solution in flat spacetime. We also explored in detail the Schwarzschild black hole force-free magnetosphere and examined briefly the force-free magnetosphere in the Kerr metric. Due to the complexity of force-free magnetosphere, only a small number of analytic solutions can be found. This motivated us to seek numerical solution to the underlying, highly nonlinear Grad-Shafranov equation. Furthermore, the hyperbolic nature of this formulation (Komissarov 2002) makes it more suitable for the numerical work, which can make use of current state-of-the-art numerical techniques to the evolution of force-free field (e.g. Toro 1997; LeVeque 1998). The next step of our work is to evolve the force-free field around a neutron star or black hole numerically using the high-resolution shock-capturing scheme of Yu (2006). Direct numerical simulation would definitely provide us more insights into the properties of the magnetosphere of black holes.

Acknowledgements The author thanks the anonymous referee for the valuable comments and suggestions that further improve this paper. This work was funded by the National Natural Science Foundation of China (NSFC, 10703012).

Appendix A: DERIVATION OF GRAD-SHAFRANOV EQUATION

This appendix aims at transforming our new form of constraint Equations (23) and (24) into the widely-used Grad-Shafranov equation form in the literature. The left side of Equation (24) is (note $I = H_\varphi$ and $d\Psi = -Qd\Omega$)

$$\frac{\sqrt{\gamma}}{\alpha\gamma_{\varphi\varphi}}\frac{1}{2Q}\frac{dH_\varphi^2}{d\Omega} = -\sqrt{\gamma}\frac{I}{\alpha\varpi^2}\frac{dI}{d\Psi}\,. \tag{A.1}$$

Thus Equation (24) becomes

$$\begin{aligned} -\sqrt{\gamma}\frac{I}{\alpha\varpi^2}\frac{dI}{d\Psi} \; = & \\ & \Big[-\frac{Q}{\alpha\sqrt{\gamma}}(\gamma_{\varphi\varphi}\Omega+\beta_\varphi)\gamma_{\theta\theta}(\Omega_{,r})^2 - \frac{Q}{\alpha\sqrt{\gamma}}(\gamma_{\varphi\varphi}\Omega+\beta_\varphi)\gamma_{rr}(\Omega_{,\theta})^2 + \\ & \partial_r(\frac{Q}{\alpha\sqrt{\gamma}}(\beta^2-\alpha^2+2\beta_\varphi\Omega+\gamma_{\varphi\varphi}\Omega^2)\gamma_{\theta\theta}\Omega_{,r}) + \\ & \partial_\theta(\frac{Q}{\alpha\sqrt{\gamma}}(\beta^2-\alpha^2+2\beta_\varphi\Omega+\gamma_{\varphi\varphi}\Omega^2)\gamma_{rr}\Omega_{,\theta})\Big]\,. \end{aligned} \tag{A.2}$$

Making use of the following relation (this equation can be derived using the relations in Eq. (28)),

$$\beta^2-\alpha^2+2\beta_\varphi\Omega+\gamma_{\varphi\varphi}\Omega^2 = \varpi^2(\Omega-\omega)^2-\alpha^2\,, \tag{A.3}$$

we can find that the last two terms in Equation (A.2) are

$$\begin{aligned} & \Big[\partial_r\Big(\frac{Q}{\alpha\sqrt{\gamma}}(\beta^2-\alpha^2+2\beta_\varphi\Omega+\gamma_{\varphi\varphi}\Omega^2)\gamma_{\theta\theta}\Omega_{,r}\Big)+ \\ & \partial_\theta\Big(\frac{Q}{\alpha\sqrt{\gamma}}(\beta^2-\alpha^2+2\beta_\varphi\Omega+\gamma_{\varphi\varphi}\Omega^2)\gamma_{rr}\Omega_{,\theta}\Big)\Big] = \\ & \Big[\partial_r\Big(\frac{\alpha}{\varpi^2}\sqrt{\gamma_{\theta\theta}}\sqrt{\gamma_{\varphi\varphi}}\Big(1-\frac{\varpi^2(\Omega-\omega)^2}{\alpha^2}\Big)\frac{\Psi_{,r}}{\sqrt{\gamma_{rr}}}\Big)+ \end{aligned}$$

$$\partial_\theta\left(\frac{\alpha}{\varpi^2}\sqrt{\gamma_{rr}}\sqrt{\gamma_{\varphi\varphi}}\left(1-\frac{\varpi^2(\Omega-\omega)^2}{\alpha^2}\right)\frac{\Psi_{,\theta}}{\sqrt{\gamma_{\theta\theta}}}\right)\Bigg] =$$

$$\left[\sqrt{\gamma}\times\nabla\cdot\left\{\frac{\alpha}{\varpi^2}\left(1-\frac{(\Omega-\omega)^2\varpi^2}{\alpha^2}\right)\nabla\Psi\right\}\right] . \quad (A.4)$$

Using the above equation, Equation (A.2) becomes

$$-\frac{I}{\alpha\varpi^2}\frac{dI}{d\Psi} = \nabla\cdot\left\{\frac{\alpha}{\varpi^2}\left(1-\frac{(\Omega_F-\omega)^2\varpi^2}{\alpha^2}\right)\nabla\Psi\right\}+$$

$$\left[-\frac{Q}{\alpha\gamma}(\gamma_{\varphi\varphi}\Omega+\beta_\varphi)\gamma_{\theta\theta}(\Omega_{,r})^2-\frac{Q}{\alpha\gamma}(\gamma_{\varphi\varphi}\Omega+\beta_\varphi)\gamma_{rr}(\Omega_{,\theta})^2\right] . \quad (A.5)$$

Note that $(\gamma_{\varphi\varphi}\Omega+\beta_\varphi)=\varpi^2(\Omega-\omega)$. As a result, the last two terms in Equation (A.5) can be rearranged as

$$-\frac{Q}{\alpha\gamma_{rr}}(\Omega-\omega)(\Omega_{,r})^2-\frac{Q}{\alpha\gamma_{\theta\theta}}(\Omega-\omega)(\Omega_{,\theta})^2 . \quad (A.6)$$

On the other hand we know that

$$\frac{(\Omega-\omega)}{\alpha}\frac{d\Omega}{d\Psi}|\nabla\Psi|^2 = -\frac{(\Omega-\omega)}{\alpha}\frac{1}{Q}\left(\frac{(\partial_r\Psi)^2}{\gamma_{rr}}+\frac{(\partial_\theta\Psi)^2}{\gamma_{\theta\theta}}\right) =$$

$$-\frac{(\Omega-\omega)}{\alpha}\frac{1}{Q}\left(\frac{Q^2(\Omega_{,r})^2}{\gamma_{rr}}+\frac{Q^2(\Omega_{,\theta})^2}{\gamma_{\theta\theta}}\right) = -\frac{(\Omega-\omega)}{\alpha}Q\left(\frac{(\Omega_{,r})^2}{\gamma_{rr}}+\frac{(\Omega_{,\theta})^2}{\gamma_{\theta\theta}}\right) . \quad (A.7)$$

Substituting Equation (A.7) into Equation (A.5), we obtain the final Grad-Shafranov equation commonly used in the literature

$$\nabla\cdot\left[\frac{\alpha}{\varpi^2}\left\{1-\frac{(\Omega-\omega)^2\varpi^2}{\alpha^2}\right\}\nabla\Psi\right]+\frac{(\Omega-\omega)}{\alpha}\frac{d\Omega}{d\Psi}|\nabla\Psi|^2 = -\frac{1}{\alpha\varpi^2}\mathrm{II}'(\Psi) , \quad (A.8)$$

where the prime denotes the derivative with respect to Ψ.

Appendix B: NONRELATIVISTIC LIMIT AND RELATION TO CLASSICAL FORCE-FREE FIELD

In many circumstances such as in the solar corona and in star formation regions, gravity is not strong so that general relativity does not play a significant role. Then we can take the nonrelativistic limit to recover the classical force-free field. In the nonrelativistic limit (indicated by the arrow) , $M\rightarrow 0$ and $a\rightarrow 0$, and the relevant general relativity parameters reduce from Equation (28) to

$$\Delta\rightarrow r^2 , \quad \rho\rightarrow r , \quad \Sigma\rightarrow r^2 ,$$
$$\alpha\rightarrow 1 , \quad \varpi\rightarrow r\sin\theta , \quad (B.1)$$
$$\Omega\rightarrow 0 , \quad \omega\rightarrow 0 .$$

The general spatial metric γ_{ij} becomes the spatial metric of polar spherical coordinates,

$$\gamma_{rr}\rightarrow 1 , \quad \gamma_{\theta\theta}\rightarrow r^2 , \quad \gamma_{\phi\phi}\rightarrow r^2\sin^2\theta . \quad (B.2)$$

Furthermore, the gradient operator ∇ becomes that in polar spherical coordinates. In particular

$$\nabla\phi\rightarrow\frac{1}{r\sin\theta}\hat{\boldsymbol{e}}_\phi. \quad (B.3)$$

The axisymmetric magnetic field in polar spherical coordinates, widely used in the study of CME in solar physics and outflow in star formation region (e.g. Low 1986; Hu 2004), can be obtained by taking this nonrelativistic limit,

$$\boldsymbol{B}_{\rm p} = \nabla\Psi \times \nabla\phi \ \rightarrow \nabla\Psi \times \frac{\hat{\boldsymbol{e}}_\phi}{r\sin\theta} \ , \tag{B.4}$$

$$\boldsymbol{B}_\phi = \frac{\rm I}{\alpha}\nabla\phi \rightarrow \frac{I}{r\sin\theta}\hat{\boldsymbol{e}}_\phi \ . \tag{B.5}$$

Combining these two relations, we arrive at commonly used classical force-free field,

$$(B_r, B_\theta, B_\phi) = \frac{1}{r\sin\theta}\left(\frac{1}{r}\frac{\partial\Psi}{\partial\theta}, -\frac{\partial\Psi}{\partial r}, I\right). \tag{B.6}$$

From this equation we see that Ψ is the magnetic flux function, I the current density and it is a strict function of Ψ, i.e., $I = I(\Psi)$. For classical force-field used in solar physics, the magnetic field is described by equations

$$\nabla \times \boldsymbol{B} = \lambda \boldsymbol{B} \ , \quad \boldsymbol{B}\cdot\nabla\lambda = 0 \ . \tag{B.7}$$

Note here the parameter λ is the proportionality between the magnetic field and the curl of the magnetic field (usually this parameter is denoted as α in the literature, but to avoid confusion with the lapse function α in this paper, I used a different symbol λ instead). The proportionality parameter λ relates to the previous parameter by

$$\lambda = \frac{dI}{d\Psi} \ . \tag{B.8}$$

In the nonrelativistic limit (B.1) and (B.2), Equation (25) becomes

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\Psi}{\partial\theta}\right) + I\frac{dI}{d\Psi} = 0 \ . \tag{B.9}$$

Or more compactly

$$\frac{\partial^2\Psi}{\partial r^2} + \frac{1-\mu^2}{r^2}\frac{\partial^2\Psi}{\partial\mu^2} + I\frac{dI}{d\Psi} = 0 \ , \tag{B.10}$$

where $\mu = \cos\theta$.

Appendix C: RELATION TO MACDONALD & THORNE'S FORMULATION

To recover the formulation in MacDonald & Thorne (1982), we take $\boldsymbol{B}$ and $\boldsymbol{D}$ in our main text as $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{E}}$ respectively. With $\boldsymbol{J} = \alpha\boldsymbol{j} - \rho_e\boldsymbol{\beta}$ ($\boldsymbol{J}$ is the electric current in our main text, $\boldsymbol{j}$ the electric current in MacDonald & Thorne 1982, ρ_e the electric charge, α the lapse function and $\boldsymbol{\beta}$ the shift vector), Equations (5) and (6) will be

$$\nabla\cdot\widetilde{\boldsymbol{B}} = 0 \ , \quad \partial_t\widetilde{\boldsymbol{B}} - \mathcal{L}_\beta\widetilde{\boldsymbol{B}} + \nabla\times(\alpha\widetilde{\boldsymbol{E}}) = 0 \ , \tag{C.1}$$

$$\nabla\cdot\widetilde{\boldsymbol{E}} = \rho_e \ , \quad -\partial_t\widetilde{\boldsymbol{E}} + \nabla\times(\alpha\widetilde{\boldsymbol{B}}) + \mathcal{L}_\beta\widetilde{\boldsymbol{E}} = \alpha\boldsymbol{j} \ , \tag{C.2}$$

where the Lie derivatives of $\widetilde{\boldsymbol{B}}$ and $\widetilde{\boldsymbol{E}}$ along the shift vector $\boldsymbol{\beta}$ are defined as

$$\mathcal{L}_\beta\widetilde{\boldsymbol{B}} = (\boldsymbol{\beta}\cdot\nabla)\widetilde{\boldsymbol{B}} - (\widetilde{\boldsymbol{B}}\cdot\nabla)\boldsymbol{\beta} \ , \quad \mathcal{L}_\beta\widetilde{\boldsymbol{E}} = (\boldsymbol{\beta}\cdot\nabla)\widetilde{\boldsymbol{E}} - (\widetilde{\boldsymbol{E}}\cdot\nabla)\boldsymbol{\beta} \ .$$

One can identify that Equations (C.1) and (C.2) are as same as equations (2.17a)–(2.17d) in MacDonald & Thorne (1982).

References

Asano E., Uchida T., Matsumoto R., 2005, PASJ, 57, 409
Begelman M. C., Blandford R. D., Rees M. J., 1984, Rev. Mod. Phys., 56, 255
Blandford R. D., 1976, MNRAS, 176, 465
Blandford R. D., Payne D. G., 1982, MNRAS, 199, 883
Blandford R. D., Znajek R. L., 1977, MNRAS, 179, 433
Camenzind M., 1986, A&A, 156, 137
Contopoulos I., Kazanas D., Fendt C., 1999, ApJ, 511, 351
Chiueh T., Li Z. Y., Begelman M. C., 1991, ApJ, 377, 462
Contopoulos J., Lovelace R. V. E., 1994, ApJ, 429, 139
Fendt C., 1997, A&A, 319, 1025
Flyer N., Fornberg B., Thomas S., Low B. C., 2004, ApJ, 606, 1210
Gammie C. F., 1999, ApJ, 522, L57
Gammie C. F., McKinney J. C., Tóth G., 2003, ApJ, 589, 444
Ghosh P., 2000, MNRAS, 315, 89
Ghosh P., Abramowicz M. A., 1997, MNRAS, 292, 887
Goldreich P., Julian W. H., 1969, ApJ, 157, 869
Heyvaerts J., Norman C., 1989, ApJ, 347, 1055
Hu Y. Q., 2004, ApJ, 607, 1032
Jackson J. D., 1979, Classical Electrodynamics, New York: Wiley
Khanna R., Camenzind M., 1996, A&A, 307, 665
Kim H. S., Lee H. M., Lee C. H., Lee H. K., 2005, MNRAS, 358, 998
Komissarov S. S., 2002, MNRAS, 336, 759
Komissarov S. S., 2004, MNRAS, 350, 427
Landau L. D., Lifshitz E. M., 1975, The Classical Theory of Fields. New York: Pergamon
Lee H. K., Wijers R. A. M. J., Brown G. E., 2000, Phys. Rep., 325, 83
LeVeque R. J., 1998, In: O. Steiner, A. Gautschy, eds., Computational Methods for Astrophysical Fluid Flow, Berlin: Springer, p.1
Lovelace R. V. E., 2003, ApJ, 596, L159
Lovelace R. V. E., 2006, ApJ, 652, 1494
Low B. C., 1986, ApJ, 310, 953
Low B. C., Smith D. F., 1993, ApJ, 410, 412
Low B. C., Zhang M., 2002, ApJ, 564, L53
Low B. C., Zhang M., 2004, ApJ, 609, 1098
MacDonald D., Thorne K. S., 1982, MNRAS, 198, 345
McKinney J. C., 2006, MNRAS, 367, 1797
Menon G., Dermer C. D., 2005, ApJ, 635, 1197
Michel F. C., 1973, ApJ, 180, 207
Narayan R., McKinney J. C., Farmer A. J., MNRAS, 375, 548
Ogura J., Kojima Y., 2002, Prog. Theor. Phys., 109, 619
Okamoto I., 1974, MNRAS, 167, 457
Okamoto I., 1992, MNRAS, 254, 192
Park S. J., Vishniac E. T., 1989, ApJ, 337, 78
Punsly B., 2001, Black Hole Gravitohydromagnetics Berlin: Springer
Sakai N., Shibata S., 2003, ApJ, 584, 427
Sulkanen M. E., Lovelace R. V. E., 1990, ApJ, 350, 732
Takahashi M., Nitta S., Tatematsu Y., Tomimatsu A., 1990, ApJ, 363, 206
Takahashi M., 1991, PASJ, 43, 569
Takahashi M., Shibata S., 1998, PASJ, 50, 271
Takahashi M., Rilett D., Fukumura K., Tsuruta S., 2002, ApJ, 572, 950
Takahashi M., Goto J., Fukumura K., Rilett D., Tsuruta S., 2006, ApJ, 645, 1408
Thorne K. S., MacDonald D., 1982, MNRAS, 198, 339
Toro E. F., 1997, Riemann Solvers and Numerical Methods for Fluid Dynamics, Berlin: Springer
Tsinganos K., Bogovalov S., 2000, A&A, 356, 989

Tsinganos K., Bogovalov S., 2002, MNRAS, 337, 553
Uzdensky D. A., 2003, ApJ, 598, 446
Uzdensky D. A., 2004, ApJ, 603, 652
Uzdensky D. A., 2005, ApJ, 620, 889
van Putten M. H. P. M., Levinson A., 2003, ApJ, 584, 937
Vlahakis N., Konigl A., 2003, ApJ, 596, 1080
Vlahakis N., Tsinganos K., 1998, MNRAS, 298, 777
Weinberg S., 1972, Gravitation and Cosmology, New York: Wiley
Yu C., 2006, Chin. J. Astron. Astrophys. (ChJAA), 6, 680
Zhang M., Low B. C., 2004, ApJ, 600, 1043
Zhang M., Low B. C., 2005, ARA&A, 43, 103