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Early Tracking Behavior in Small-field Quintessence Models *

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Abstract We study several quintessence models which are exotic at $Q = 0$, and use a simple constraint $Q \geq H/2\pi$ to check when they enter the tracking regime, disregarding the details of inflation. We find that it can also give strong constraints for $V = V_0 Q^{-\alpha}$, which has to enter the tracking regime after $\ln z \sim 10$, while for the supergravity model $V = V_0 Q^{-\alpha} \exp(kQ^2/2)$, the constraint is much weaker. For the exponential form of inverse power-law potential $V = V_0 \exp(\lambda/Q)$, it exhibits no constraints.

Key words: cosmology: theory

1 INTRODUCTION

Recent observations of type Ia supernova (SN) survey (Garnavich et al. 1998; Perlmutter et al. 1998) and cosmic microwave background (CMB) anisotropies (de Bernardis et al. 2002; Lee et al. 2001) strongly show evidence for a cosmological constant or quintessence. In the general case, quintessence can have a time-dependent equation of state, $\omega_Q(t) = p_Q/\rho_Q$ (Wetterich 1988; Peebles & Ratra 1988), which is invoked to explain the coincidence problem. Some researchers reanalyzed the cosmological data of CMB, SN, large scale structure and gravitational lens statistics (Bean & Melchiorri 2002; Baccigalupi et al. 2002; Hannestad & Mörtsell 2002; Chae et al. 2002), and confirmed that quintessence is slightly preferred over the cosmological constant.

An important class of quintessence models are known as tracking models. By coupling a scalar field to matter one can obtain tracking solutions (Zlatev et al. 1998; Steinhardt et al. 1999) for the time dependence of the dark energy density so that it always follows the dominant energy density component fairly independently of initial conditions. Recently, Malquarti & Liddle (2002) used a stochastic inflation model to constrain the initial quintessence value after inflation. They have shown that for an inverse power-law form $V = V_0 Q^{-\alpha}$ satisfying current

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observations, the initial Q was so large that it could not enter the tracking regime until the matter-domination epoch. This has put the tracking behavior in considerable jeopardy for such quintessence models.

In the present paper, we will study several quintessence models which are exotic at $Q = 0$ after inflation. Quintessence being almost massless, one has $\delta Q \sim H/2\pi$ (see Liddle et al. 1993 for details) after inflation. The simplest constraint is $Q \geq \delta Q$, i.e., $Q \geq H/2\pi$. We shall use this constraint to find the time of entry into the tracking regime, disregarding the details of the inflation. In the following section, we will mainly discuss the early tracking behavior of three small-field quintessence models: inverse power-law potential, $V = V_0 Q^{-\alpha}$, supergravity, $V = V_0 Q^{-\alpha}\exp(kQ^2/2)$, and exponential form of inverse power-law, $V = V_0\exp(\lambda/Q)$.

2 MODELS AND TRACKING SOLUTIONS

We shall consider models of quintessence in a flat cosmological background. The ratio of energy density to the critical density today is Ω_Q for the Q-field and Ω_m for the matter density where $\Omega_m + \Omega_Q = 1$. We also define a background equation-of-state ω_B, $\omega_B = 1/3$ in the radiation-dominated epoch and 0 in the matter-dominated epoch. We use dimensionless units where the Planck mass is $M_p = 1$.

The equation of motion for the Q-field is

$$\ddot{Q} + 3H\dot{Q} + V' = 0, \tag{1}$$

where $V' = \ddot{Q}\frac{1-\omega_Q}{1+\omega_Q}$ and

$$H^2 = \left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi}{3}(\rho_Q + \rho_B), \tag{2}$$

a is the Robertson-Walker scale factor, $\rho_B = \rho_m + \rho_r$, ρ_m and ρ_r are the matter and radiation energy density, respectively. Early in the radiation-dominated epoch, we have $H \approx 1/2t$. Assuming ω_Q is a constant, we have

$$\ddot{Q} + \frac{3(1+\omega_Q)}{4t}\dot{Q} = 0. \tag{3}$$

We can easily obtain the solution of this equation, $\dot{Q} = Ct^{-3(1+\omega_Q)/4}$, C is a constant, so

$$\Omega_Q = \left(\frac{\dot{Q}}{H}\right)^2 \propto t^{-\frac{3}{2}(1+\omega_Q)+2} \propto a^{-3(1+\omega_Q)+4}. \tag{4}$$

The above equation gives roughly the evolution of the Q-field. We can obtain the Q energy density by another method. According to the present cosmological density ρ_0, we can write

$$\rho_Q = \rho_0\left(\frac{a_{\rm eq}}{a_0}\right)^{-3(1+\omega_1)}\left(\frac{a}{a_{\rm eq}}\right)^{-3(1+\omega_2)}, \tag{5}$$

or

$$\ln\rho_Q = \ln\rho_0 + 3(\omega_1 - \omega_2)\ln(z_{\rm eq}+1) + 3(1+\omega_2)\ln(z+1)\,, \tag{6}$$

where z is the redshift, the subscript eq denotes the epoch of matter-radiation equality, and ω_1 and ω_2 are the equation of state of the Q-field during matter- and radiation-dominated epochs. In the analytical formulae Eqs. (5) and (6), we assume ω_1 is a constant and that today is matter-dominated, which has led to considerable uncertainties when compared to the exact numerical case.

An important function is $\Gamma \equiv V''V/(V')^2$, whose properties determine whether tracking solutions exist. Taking the derivative of the equation-of-motion with respect to Q and combining with the equation-of-motion itself, we can obtain the tracking equation

$$\Gamma = 1 + \frac{\omega_B - \omega_Q}{2(1+\omega_Q)} - \frac{1+\omega_B - 2\omega_Q}{2(1+\omega_Q)}\frac{\dot{x}}{6+x} - \frac{2}{(1+\omega_Q)}\frac{\ddot{x}}{(6+\dot{x})^2}, \tag{7}$$

where $x \equiv (1+\omega_Q)/(1-\omega_Q), \dot{x} \equiv \mathrm{d}\ln x/\mathrm{d}\ln a$ and $\ddot{x} \equiv \mathrm{d}^2 \ln x/\mathrm{d}\ln a^2$.

In the following sections, we will discuss the tracking behavior of different quintessence models in detail. Moreover, we take the cosmological parameters derived from recent observational constraints throughout, $\Omega_m = 0.3, \omega_Q = -0.82$ and the Hubble constant $h = 0.65$.

2.1 Pure Inverse Power-law Models

The quintessence models of pure inverse power-law potentials are originally introduced by Ratra & Peebles (1988): $V = V_0 Q^{-\alpha}$. For the tracking solution,

$$\Gamma - 1 = \frac{\omega_B - \omega_Q}{2(1+\omega_Q)} = \frac{1}{\alpha}, \tag{8}$$

then $\omega_Q = (\alpha\omega_B - 2)/(\alpha + 2)$. With $\omega_1 = -2/(\alpha+2)$ ($\omega_B = 0$) and $\omega_2 = (\alpha - 6)/(3\alpha + 6)$ ($\omega_B = 1/3$), according to Eq. (6), we have

$$\ln\rho_Q = \ln\rho_0 - \frac{\alpha}{\alpha+2}\ln(z_{\mathrm{eq}}+1) + \frac{4\alpha+12}{\alpha+2}\ln(z+1). \tag{9}$$

For $\omega_Q = (\rho_Q - 2V)/\rho_Q$, we then have $V_0 Q^{-\alpha} = V = \rho_Q(1-\omega_Q)/2$, or $\ln V_0 - \alpha \ln Q = \ln \frac{1-\omega_Q}{2}\rho_Q$. When $Q_0 \sim 0$, taking the approximation $V_0 \simeq \rho_0$, we can obtain the final analytical form

$$\ln Q = (\ln\rho_0 - \ln\frac{1-\omega_Q}{2}\rho_Q)/\alpha. \tag{10}$$

We also have numerically computed the tracking behavior of the quintessence models according to Eqs. (1) and (2). In Fig. 1, the dashed ($\alpha = 1.4$) and dotted ($\alpha = 0.67$) lines show the evolution of the Q-field for the two values of α. We have the approximation $Q \propto (z+1)\exp[-(4\alpha+12)/\alpha(\alpha+2)]$ and $Q \sim 1$ today. For a smaller α, Q would be smaller in the earlier tracking regime, as shown in Fig. 1. Our fit with $\Omega_m = 0.3, \omega_Q = -0.82$ requires $\alpha \approx 0.67$ and $V_0^{1/4} = 1.8 \times 10^{-31}$. In Malquarti & Liddle (2002), their fit gives $0 \le \alpha \le 1$ (68% confidence), our fit is in agreement with theirs.

However, one can see that a small α would also make tracking behavior lose significance to some degree because it acts then like a cosmological constant. Theoretically the "best" tracking behavior predicts an energy density ρ_Q significantly higher than that of the cosmological constant, and ρ_Q is relatively close to but smaller than the energy density of radiation in its early stage when entering the tracking regime. For $V = V_0 Q^{-\alpha}$, a very small α would definitely predict $\omega_Q \sim -1$ when entering the tracking regime, which is hardly distinguishable from a cosmological constant. In any case, phenomenologically, there exists such a possibility for quintessence allowed by current observations. In Malquarti & Liddle (2002), the parameter is restricted to $\alpha \le 1$ at 68% confidence by current observations, our choice of $\alpha \approx 0.67$ is theoretically acceptable. In this sense we can see that the following two models show better tracking behaviors than does the pure inverse power law case.

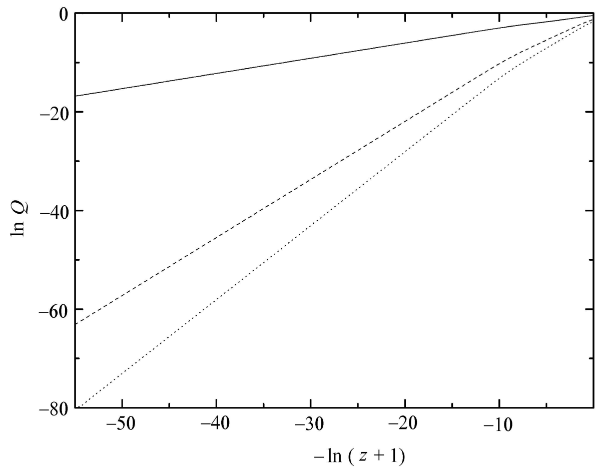

Fig. 1 Evolution of the Q-field with redshift. The solid line denotes the supergravity model, and the dashed and dotted lines display inverse power-law models with $\alpha = 1.4$ and $\alpha = 0.67$ respectively.

2.2 Supergravity Models

In this section we consider the supergravity version of the model considered previously with a superpotential of the form $V \propto Q^{-\alpha}$. We take the form $V = V_0 Q^{-\alpha} \exp(kQ^2/2)$ (Brax & Martin 2000), where $k = 8\pi$, and study its tracking behavior.

For tracking solutions, we have

$$\Gamma - 1 = \frac{(k + \alpha Q^{-2})V^2 + (kQ - \alpha Q^{-1})^2 V^2}{(kQ - \alpha Q^{-1})^2 V^2} - 1 = \frac{k + \alpha Q^{-2}}{(kQ - \alpha Q^{-1})^2}. \tag{11}$$

Because $k \ll \alpha Q^{-2} (Q \to 0)$, we have $\Gamma - 1 \simeq 1/\alpha$. It is similar to the former inverse power-law potentials. For $\omega_Q = -0.82$ and $\Omega_m = 0.3$, $\alpha = 11$ and $V_0^{1/4} = 1.93 \times 10^{-32}$ are expected.

In Fig. 1, the solid line shows the evolution of the Q-field with the redshift, and in Fig. 2 we also show the evolution of the equation-of-state ω_Q for both the inverse power-law potential and supergravity, taking the same parameter $\alpha = 11$. In the early radiation-dominated epoch, they have the same equations of state, while later when matter and quintessence (today) dominate, Q grows larger and the factor $\exp(kQ^2/2)$ takes effect today in the supergravity model, which makes the main contribution to the different behavior of the two models today. The pure inverse power law form $V = V_0 Q^{-11}$ is ruled out because it predicts $\omega_Q > -0.3$ and cannot give an accelerating universe today, whereas $V = V_0 Q^{-\alpha} \exp(kQ^2/2)$ is still not excluded. In the two models, we have given the analytical forms for the tracking behavior according to the Eqs. (9) and (10); we have also compared the analytical and numerical results of the supergravity version in Fig. 3.

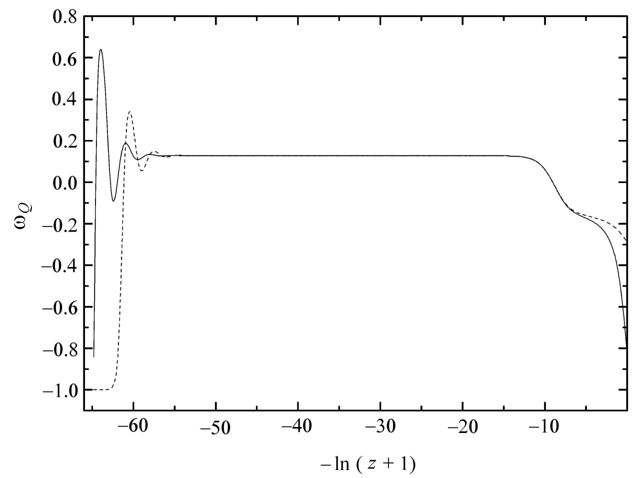

Fig. 2 Different tracking behaviors of the inverse power-law model (dashed) and the supergravity model (solid) taking $\alpha = 11$.

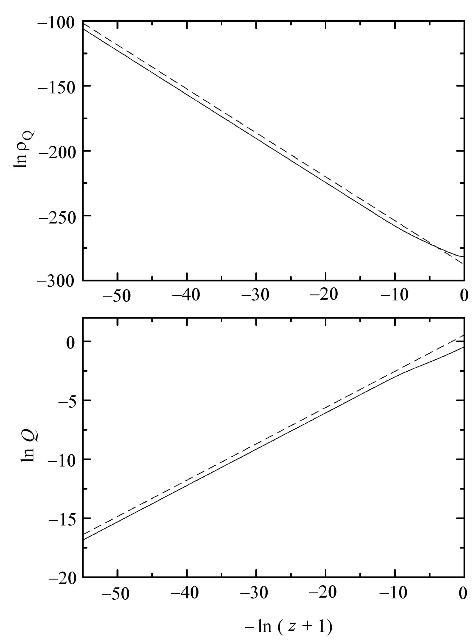

Fig. 3 Comparison between the analytical (solid) and numerical (dashed) tracking solutions in the supergravity model.

2.3 Exponential Form of Inverse Power-law Model

Here, we further check another inverse power-law potential with the form $V = V_0 e^{\lambda/Q}$. In our case, $\lambda \approx 0.3$ and $V_0^{1/4} = 1.85 \times 10^{-31}$. However, because the equation-of-state of the model ω_Q varies with t, it is relatively difficult to obtain analytical solutions. In Fig. 4, we have shown the evolution of the Q-field and equation-of-state ω_Q by numerical calculations, and the tracking behavior is different from the previous two models discussed above. The evolution curve of the Q-field always rises with time. We are able to give some rough estimations because $Q \ll 1$ is also satisfied for large z. The form $e^{\lambda/Q}$ can be expanded to a $Q^{-\alpha}$ series with $\alpha \longrightarrow \infty$, hence $\omega_Q = (\alpha - 6)/(3\alpha + 6) \approx 1/3$. Its early behavior of Q and ρ_Q can also be explained as $\alpha \longrightarrow \infty$, where Q would be much larger than in the pure inverse power-law models from above analysis.

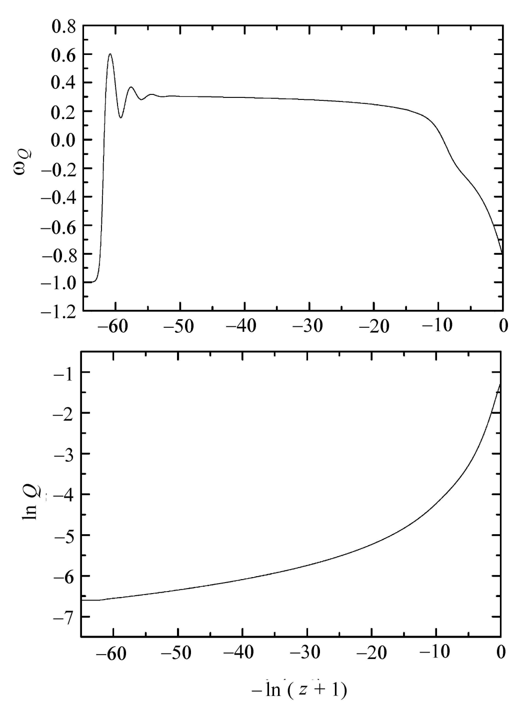

Fig. 4 Evolution of the equation-of-state ω_Q and Q-field with redshift in the exponential form of inverse power-law model. We found that no limit on the tracking time can be obtained here. See text for details.

For a comparison of the three models, we consider the very early behavior of the Q-field. In the very early time, the three models take the same form $V = V_0 Q^{-\alpha}$, with $\alpha = 0.67$, 11 and ∞, respectively.

In Fig. 5, we show our constraints on the tracking behavior of the quintessence models by taking the initial condition $Q_i \geq H/2\pi$, where $H \sim 10^{-5}$, and $\dot{Q}_i = 0$ because the kinetic energy of quintessence is diluted by inflation. We do not include the exponential form of inverse power-law model because we have no constraint on it, as can be seen from Fig. 4.[1] The solid line displays the evolution of the Q-field in the supergravity model, while the dashed line refers to the inverse power-law model. For $V = V_0 Q^{-\alpha}$, it can enter the tracking regime only after $\ln z \sim 10$. For the supergravity model $V = V_0 Q^{-\alpha}\exp(kQ^2/2)$, it requires $\ln(z+1) \leq 43$.

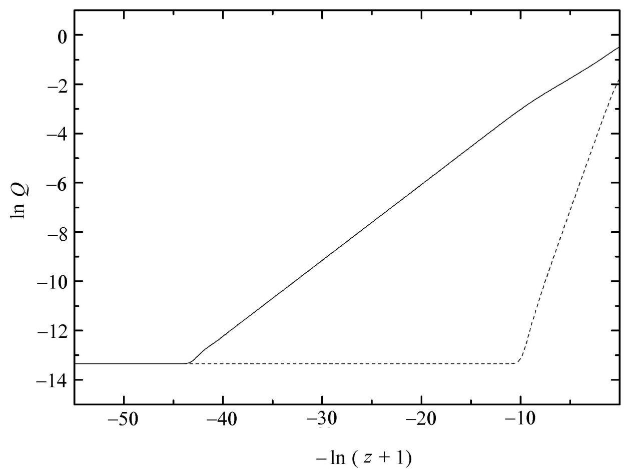

Fig. 5 Evolution of the Q-field with redshift in both inverse power-law (dashed) and supergravity (solid) models. We take the tracking condition $Q_i \geq H/2\pi$, where $H \sim 10^{-5}, \dot{Q}_i = 0$, and find that the time of the supergravity case enters the tracking regime is much earlier than in the pure inverse power-law model.

3 CONCLUSIONS AND DISCUSSION

We have analyzed the dynamical evolution and tracking solutions of three quintessence models. We used a simple constraint $Q_i \geq H/2\pi$ to study the tracking behavior, and found that it can also give a strong constraint on the pure inverse power-law model which enters the tracking regime at a late stage $\ln z \sim 10$. The key fact is that for such pure inverse power-law model, α has to be very small in order to fit current observations. For the supergravity model and the exponential form of inverse power-law model, the exponential form takes a positive effect, making them have $\omega_Q \sim -1$ today and satisfy the CMB and SN constraints, while in the early stage of evolution, they take the pure inverse-power law form with a much larger α, and little constraint is exhibited with $Q_i \geq H/2\pi$ for entering the tracking regime.

It is notable that theoretically the three models above show significantly different behaviors as $Q \longrightarrow \infty$. However, Q, being slow rolling and of the order of unity today, cannot get very large in the future for above models, which we have also made a numerical check. In this paper we mainly deal with the early tracking behaviors which can also be analytically presented to some degree, the future behaviors are not shown.

[1] In fact, the assumption that quintessence is massless and slow rolling requires $Q \gg H/2\pi$ for this model, but the constraint is not strong enough either when the Q-field enters the tracking regime. For the other two models, the constraint is weaker than $H/2\pi$.

Tracking behavior typical of pure inverse power-law models begins only at quite a late stage of evolution, as well as after nucleosynthesis and possibly after decoupling as presented by Malquarti & Liddle (2002) and the present paper. Tracking is a promising key to solve the coincidence problem, so tracking quintessence would lose significance if it has to enter the tracking regime extremely late. Therefore, the supergravity and exponential form of inverse power law models show better tracking behaviors in our analysis.

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