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INVITED REVIEWS

The Kinematic Theory of Solar Dynamo *

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Abstract Generation of the Sun's magnetic fields by self-inductive processes in the solar electrically conducting interior, the solar dynamo theory, is a fundamentally important subject in astrophysics. The kinematic dynamo theory concerns how the magnetic fields are produced by kinematically possible flows without being constrained by the dynamic equation. We review a number of basic aspects of the kinematic dynamo theory, including the magnetohydrodynamic approximation for the dynamo equation, the impossibility of dynamo action with the solar differential rotation, the Cowling's anti-dynamo theorem in the solar context, the turbulent alpha effect and recently constructed three-dimensional interface dynamos controlled by the solar tachocline at the base of the convection zone.

Key words: Solar magnetic fields — kinematic dynamos — magnetohydrodynamics

1 INTRODUCTION

Sunspots represent exceptionally high concentration of magnetic flux typically of the order of $O(10^3)$ G. Mainly confined within heliographic latitudes $\pm 35^\circ$, the sunspots are highly variable, both spatially and temporally. Observations show that sunspots usually appear in pairs of opposite magnetic polarities commonly separated by $O(10^5\,\mathrm{km})$ and are nearly aligned with the line of constant heliographic latitude with a slight tilt about 10°. A remarkable feature is that the average annual number of sunspot has a nearly 11 yr periodicity, which is usually referred to as the sunspot cycle. The striking regularity of the solar magnetic variation is reflected in the celebrated butterfly diagram (Maunder 1913). As a bipolar sunspot group varies in the 11-yr cycle, its polarity obeys the laws of sunspot polarity: the leading spot in the northern hemisphere reveres its sense from one 11-cycle to the next; the leading spot in the southern hemisphere is opposite in polarity to that in the northern hemisphere. It thus takes two sunspot cycles or 22 yr for the Sun to repeat its magnetic state: this is known as the Hale cycle. This characteristic feature indicates that sunspots are associated with a strong, deep-seated toroidal magnetic field that is coherent in the whole spherical system and that has dipolar equatorial symmetry. Because of the strikingly high coherency and regularity, it

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has been generally accepted that the solar cycle represents magnetohydrodynamic processes taking place in the deep solar interior (Parker 1955, 2001; Moffatt 1978; Weiss 1994). The solar dynamo theory concerns the study of the generation and variation of the Sun's magnetic field by self-inductive processes in its electrically conducting interior.

Great strides have been made towards the understanding of the quantitative features of solar global magnetic activities through observational and theoretical studies over recent years. In particular, interest in the solar dynamo problem has been heightened by modern helioseismology that probes the solar internal structure using the frequency splitting of acoustic modes (Schou 1991). Theoretical considerations of convection in rotating spherical shells suggest that the differential rotation generated by convection rolls should be nearly independent of the coordinate parallel to the rotation axis (Busse 1970; Zhang 1992; Zhang & Schubert 2002). However, helioseismological observations paint a quite different and intriguing picture (Schou 1998): the solar differential rotation observed on the Sun's surface persists all the way into the base of the convection zone. An important achievement in the solar dynamo theory over the past decade is perhaps the general recognition that the transition layer between the convection zone and the radiative core of the Sun, the solar tachocline, plays an essential role in the solar magnetohydrodynamic processes (Spiegel & Zahn 1992; Parker 1993; Gough et al. 1996; Dikpati & Charbonneau 1999).

It has been suggested that the solar tachocline is a strongly stably stratified layer with a thickness up to about 10% of the solar radius (Kosovichev 1996). The tachocline offers an ideal location for the generation and storage of the Sun's strong azimuthal magnetic fields, B_ϕ. If the strong azimuthal fields are stored in the convection zone, they would be expelled by magnetic buoyancy on a timescale that is too short ($O(1)$ month) compared to the solar cycle ($O(10)\,\mathrm{yr}$). Suppose that a strong toroidal field tube immerses at some depth in the convection zone. The magnetohydrostatic equilibrium of the tube with its surroundings demands that

$$p_\mathrm{i}(\rho_\mathrm{i}) + \frac{B_\phi^2}{\mu} = p_\mathrm{o}(\rho_\mathrm{o}),$$

where $p_\mathrm{i}, \rho_\mathrm{i}$ and $p_\mathrm{o}, \rho_\mathrm{o}$ denote the pressure and density inside and outside the magnetic tube, respectively. In consequence, for a sufficiently strong magnetic field, we must have

$$p_\mathrm{i}(\rho_\mathrm{i}) < p_\mathrm{o}(\rho_\mathrm{o}).$$

From the condition that the magnetic tube is also in the thermodynamic equilibrium with its surroundings, we must conclude that

$$\rho_\mathrm{i} < \rho_\mathrm{o},$$

which means magnetic buoyancy and radially upward flows (Parker 1979). In other words, the large-scale magnetic activities observed in the Sun's surface may be interpreted as a result of the rising and emerging of the tachocline-seated, strong toroidal magnetic fields driven by magnetic buoyancy (Weiss 1994).

The magnetic field rising from the tachocline has to be sufficiently strong, typically of the order $(10^4 - 10^5)\,\mathrm{G}$. If the strength of the toroidal magnetic field is weaker than the equipartition value ($B_\mathrm{eq} = O(10^4)\,\mathrm{G}$), the field would be distorted and twisted by turbulent convection. Moreover, the rising tube of a weaker field would be deflected towards higher latitudes by the solar Coriolis force (Choudhuri & Gilman 1987). If the toroidal field is too strong, $B_\phi > 10^5\,\mathrm{G}$, the buoyant tube would rise radially without a substantial tilt (D'Silva & Choudhuri 1993). It has been widely believed, however, that the toroidal magnetic fields with ($B_\phi = O(10^4 - 10^5)\,\mathrm{G}$)

are unlikely to be sustained by conventional convection-driven dynamos (Glatzmaier 1985; Zhang & Busse 1989; Roberts & Soward 1992).

The requirement of a strong toroidal magnetic field in the deep Sun and the existence of the solar tachocline lead naturally to the concept of the interface dynamo first proposed by Parker (1993), in which the generation of a weak poloidal magnetic field and a strong toroidal magnetic field takes place in separate fluid regions with discontinuous magnetic diffusivities across the interface. Parker's interface dynamo concept depicts an attractive picture of generating a strong toroidal magnetic field in the vicinity of the tachocline while avoiding the dilemma relating to the alpha quenching in the convection zone.

There have been a number of important extensions, with different emphases, of the Parker's interface dynamo model. One extension is to focus on the further understanding of the fundamental generation mechanism of interface dynamos. MacGregor & Charbonneau (1997) considered a different interface dynamo in which the shear flow and $\alpha-$effects are spatially localized in the form of a delta-function at a moderate distance on either side of the interface. Because the shear flow and α-effects are spatially separated, the effect of magnetic diffusion plays a more important role than that in the Parker model. By introducing the action of the Lorentz force using the Malkus-Proctor mechanism, Tobias (1997) investigated the nonlinear modulation of Parker's interface dynamo (see also Brandenburg et al. 1989; Tobias et al. 1995; Ponty et al. 2001). The Cartesian interface dynamo of Parker's has been also extended to the case of spherical geometry (for example, Charbonneau & MacGregor 1997; Markiel & Thomas 1999; Dikpati & Charbonneau 1999). In a linear spherical interface dynamo model, Charbonneau & MacGregor (1997) found a class of dynamo solutions that reply on the latitudinal shear and that are distinct from the usual interface modes controlled by the radial shear. However, Markiel & Thomas (1999) showed that this class of dynamo solutions is invalid and results from an incorrect magnetic field boundary condition imposed at the interface between the core and the tachocline. The result of Markiel & Thomas (1999) demonstrated that the magnetic boundary condition can play a critical role in determining the key features of an interface dynamo and that the radial shear in the tachocline dominates the process of the magnetic field generation even though the latitudinal shear is present (see also Schubert & Zhang 2000, 2001).

This article attempts to give a review on the kinematic aspect of the solar dynamo theory. Clearly, it is impossible to cover all the research activities concerning the problem in detail. We shall concentrate on the fundamental aspects of the kinematic dynamo theory starting with the MHD approximation and the dynamo equation. We shall also briefly discuss a recently constructed three-dimensional solar dynamo model (Zhang et al. 2003).

2 THE KINEMATIC SOLAR DYNAMO PROBLEM

2.1 The MHD Approximation

The kinematic theory of the solar dynamo is described by the dynamo equation which determines the effect of a moving electric conductor on the electromagnetic field (Moffatt 1978; Parker 1979; Weiss 1994; Zhang & Schubert 2000). The dynamo equation is derived from Maxwell's equations under the MHD (magnetohydrodynamic) approximation. The complete set of Maxwell's equations consists of the four equations: Faraday's law

$$\frac{\partial}{\partial t}\boldsymbol{B} = -\nabla \times \boldsymbol{E}, \tag{1}$$

where $\boldsymbol{E}$ is the electric field and $\boldsymbol{B}$ is the magnetic field; Ampère's law

$$\nabla \times \boldsymbol{B} = \mu \boldsymbol{J} + (\mu\epsilon)\frac{\partial}{\partial t}\boldsymbol{E}, \tag{2}$$

where $\boldsymbol{J}$ is the electric current, μ is the magnetic permeability and ϵ is the permittivity of free space; Gauss' law

$$\nabla \cdot \boldsymbol{E} = \left(\frac{\rho}{\epsilon}\right), \tag{3}$$

where ρ is the electric charge density; and finally, the solenoidal condition of the magnetic field

$$\nabla \cdot \boldsymbol{B} = 0, \tag{4}$$

which states that there exist no magnetic monopoles. The second term in the right-hand side of Ampère's law represents Maxwell's displacement current. Note that the speed of light, C_l, is given by

$$C_l = \frac{1}{\sqrt{\mu\epsilon}} = 2.998 \times 10^8 \text{m s}^{-1}.$$

It is significant to notice that the amplitudes of both the magnetic and electric fields in Maxwell's equations are associated with a frame of reference in which we measure them. Suppose that $\boldsymbol{B}$ and $\boldsymbol{E}$ are the amplitudes measured in an inertial frame of reference. Suppose also that a fluid in the Sun moves with velocity $\mathcal{U} = |\boldsymbol{u}|$ relative to the frame. In the moving frame with $\mathcal{U}$, the amplitudes of the magnetic and electric field are denoted by $\boldsymbol{B}'$ and $\boldsymbol{E}'$. The relationship between $(\boldsymbol{B}', \boldsymbol{E}')$ and $(\boldsymbol{B}, \boldsymbol{E})$ is described by the Lorentz transformation

$$\boldsymbol{B}' = \left(1 + \frac{1}{2}\delta^2 + ...\right)(\boldsymbol{B} - \epsilon\mu \boldsymbol{u} \times \boldsymbol{E}) - \left(\frac{1}{2}\delta^2 + \frac{3}{8}\delta^4 + ...\right)\left[\frac{(\boldsymbol{u}\cdot\boldsymbol{B})\boldsymbol{u}}{\mathcal{U}^2}\right], \tag{5}$$

$$\boldsymbol{E}' = \left(1 + \frac{1}{2}\delta^2 + ...\right)(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}) - \left(\frac{1}{2}\delta^2 + \frac{3}{8}\delta^4 + ...\right)\left[\frac{(\boldsymbol{u}\cdot\boldsymbol{E})\boldsymbol{u}}{\mathcal{U}^2}\right]. \tag{6}$$

In the Lorentz transformation,

$$\delta = \frac{\mathcal{U}}{C_l}$$

is related to the relativistic factor. In a conducting fluid of electric conductivity $\sigma(\boldsymbol{r})$, where $\boldsymbol{r}$ is the position vector, moving with velocity $\boldsymbol{u}$ in the presence of an electric charge density ρ, the relation between the electric current $\boldsymbol{J}$ and the electric field $\boldsymbol{E}'$ is given by Ohm's law

$$\boldsymbol{J} = \sigma(\boldsymbol{r})\boldsymbol{E}' + \rho\boldsymbol{u}. \tag{7}$$

A non-relativistic approximation can be made to simplify the governing equations for the solar dynamo problem. Let us denote a typical length scale of $\boldsymbol{B}$ by $\mathcal{L}$ and a typical amplitude of $\boldsymbol{B}$ (or $\boldsymbol{E}$) by $\mathcal{B}$ (or $\mathcal{E}$). We assume that the typical timescale for variation of the magnetic field is given by $\mathcal{L}/\mathcal{U}$. From Faraday's law, we can obtain an estimate for the scaling relation between $\mathcal{E}$ and $\mathcal{B}$

$$\mathcal{E} = O(\mathcal{U}\mathcal{B}).$$

The relative order of the terms in Ampère's law is

$$\frac{|(\mu\epsilon)\partial\boldsymbol{E}/\partial t|}{|\nabla \times \boldsymbol{B}|} = O\left(\frac{\mu\epsilon\mathcal{E}\mathcal{U}}{\mathcal{B}}\right) = O\left(\frac{\mathcal{U}^2}{C_l^2}\right) = O\left(\delta^2\right). \tag{8}$$

If we take typical velocity $\mathcal{U}$ as the velocity in the photosphere near a sunspot, which is about $\mathcal{U} = 10^4$ m s^{-1}, then we have

$$\frac{|\mu\epsilon\partial \boldsymbol{E}/\partial t|}{|\nabla \times \boldsymbol{B}|} = O\left(10^{-11}\right) \ll 1\,. \tag{9}$$

This implies that Maxwell's displacement current can be safely neglected in the solar dynamo problem, that is, the magnetohydrodynamic (MHD) approximation is valid when $\delta^2 \ll 1$. Ampère's law under the MHD approximation becomes

$$\nabla \times \boldsymbol{B} = \mu \boldsymbol{J}\,. \tag{10}$$

Now consider the relative orders of magnitude in the Lorentz transformation,

$$\frac{|(\epsilon\mu)\boldsymbol{u} \times \boldsymbol{E}|}{|\boldsymbol{B}|} = O\left(\frac{|\epsilon\mu\mathcal{U}\mathcal{E}|}{\mathcal{B}}\right) = O\left(\frac{\mathcal{U}^2}{C_l^2}\right) = O\left(\delta^2\right)\,, \tag{11}$$

$$\frac{|\boldsymbol{E}|}{|\boldsymbol{u} \times \boldsymbol{B}|} = O\left(\frac{|\mathcal{U}\mathcal{E}|}{\mathcal{B}}\right) = O(1)\,. \tag{12}$$

It follows that, in the first approximation to order δ^2, the Lorentz transformation (5)–(6) is reduced to

$$\boldsymbol{B}^{'} = \boldsymbol{B}\,, \tag{13}$$

$$\boldsymbol{E}^{'} = \boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\,. \tag{14}$$

It gives rise to Ohm's law in the form

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right) + \rho\boldsymbol{u}\,. \tag{15}$$

However, when $\delta^2 \ll 1$, the term $\rho\boldsymbol{u}$ is much smaller than the current $\boldsymbol{J}$ because

$$\frac{|\rho\boldsymbol{u}|}{|\boldsymbol{J}|} = O\left(\frac{|\mu\rho\boldsymbol{u}|}{|\nabla \times \boldsymbol{B}|}\right) = O\left(\delta^2\right)\,. \tag{16}$$

Here we have used Gauss' law in the scaling analysis. To order δ^2, Ohm's law becomes

$$\boldsymbol{J} = \sigma\left(\boldsymbol{E} + \boldsymbol{u} \times \boldsymbol{B}\right)\,. \tag{17}$$

Under the MHD approximation, all the terms of $O(\delta^2)$ are neglected and the corresponding equations are dramatically simplified. The resulting approximate equations are usually adopted in the study of magnetic field generation in the solar interior. Evidently, they cannot be used in describing other solar physical processes such as electromagnetic radiation.

2.2 The Dynamo Equation and Energy Equation

For the solar dynamo problem, it is convenient to eliminate the electric field $\boldsymbol{E}$ and the electric current $\boldsymbol{J}$, leading to the well-known dynamo equation or the induction equation,

$$\frac{\partial}{\partial t}\boldsymbol{B} = \nabla \times \left(\boldsymbol{u} \times \boldsymbol{B}\right) - \frac{1}{\mu\sigma}\left(\nabla \times \nabla \times \boldsymbol{B}\right)\,, \tag{18}$$

where the velocity $\boldsymbol{u}$ represents a kinematically possible flow satisfying

$$\nabla \cdot \boldsymbol{u} = 0\,. \tag{19}$$

A set of boundary conditions for $\boldsymbol{B}$ and $\boldsymbol{E}$ must be satisfied at an interface between different spherical layers in the Sun, for example, between the convection zone and the exterior

$$< \boldsymbol{B} >= 0\,, \tag{20}$$

$$< \hat{\boldsymbol{r}} \times \boldsymbol{E} >= 0\,, \tag{21}$$

$$< \hat{\boldsymbol{r}} \cdot \boldsymbol{J} >= 0\,, \tag{22}$$

where $\hat{\boldsymbol{r}}$ is the unit radial vector and $< . >$ denotes the jump across the interface.

To understand the physical significance of each term in the dynamo equation, we multiply (18) by the magnetic field

$$\frac{\partial}{\partial t}\left(\frac{1}{2}|\boldsymbol{B}|^2\right) = \nabla \cdot [(\boldsymbol{u} \times \boldsymbol{B}) \times \boldsymbol{B} + \lambda \boldsymbol{B} \times (\nabla \times \boldsymbol{B})] - \lambda \left|\nabla \times \boldsymbol{B}\right|^2 - (\boldsymbol{u} \times \boldsymbol{B}) \cdot (\nabla \times \boldsymbol{B})\ , \tag{23}$$

where magnetic diffusivity $\lambda = 1/(\mu\sigma)$. Integrate (23) over the whole body of the Sun, we obtain an energy equation

$$\frac{\partial}{\partial t}\int_V \left(\frac{1}{2\mu}|\boldsymbol{B}|^2\right) \mathrm{d}V = \int_S \hat{\boldsymbol{r}} \cdot \boldsymbol{P} \mathrm{d}S - \frac{\lambda}{\mu}\int_V \left|\nabla \times \boldsymbol{B}\right|^2 \mathrm{d}V - \int_V \boldsymbol{F} \cdot \boldsymbol{u}\, \mathrm{d}V, \tag{24}$$

where $\boldsymbol{P} = \boldsymbol{E} \times \boldsymbol{B}/\mu$ is the Poynting vector and $\boldsymbol{F} = \boldsymbol{J} \times \boldsymbol{B}$ is the Lorentz force. The first term on the right-hand side represents the electromagnetic flux out of the Sun, the second term denotes the total Ohmic losses which irreversibly dissipate the electromagnetic energy, and the work done by the solar internal flow against the Lorentz force is given by the third term.

Physically speaking, if a dynamo is capable of operating in the Sun, the total work done by the conducting flow must be positive and greater than the Ohmic losses, i.e.,

$$\frac{|\boldsymbol{F} \cdot \boldsymbol{u}|}{\lambda|\nabla \times \boldsymbol{B}|^2/\mu} = O\left(\frac{\mathcal{UL}}{\lambda}\right) = O(R_{\mathrm{m}}) > O(1)\,, \tag{25}$$

where R_{m} is the magnetic Reynolds number. In the solar interior, observations suggest that $R_{\mathrm{m}} \gg 1$ (Parker 1979).

2.3 The Kinematic Dynamo Problem

The theory of solar kinematic dynamo is one in that the observed flow velocity in the Sun's interior is assumed to be known and its effect in the form of a growing magnetic field is investigated. Alternatively, the kinematic dynamo problem may be regarded as an instability problem in which the system becomes unstable to infinitesimal magnetic disturbances.

Mathematically, the solar kinematic dynamo problem may be defined as follows. Let V denote the Sun's interior with S as its bounding spherical surface. Its internal magnetic diffusivity is denoted by $\lambda(\boldsymbol{r})$. Let V_{e} be the vacuum exterior extending to infinity. For the velocity $\boldsymbol{u}$ within the Sun, the magnetic field $\boldsymbol{B}$ inside the Sun is described by

$$\frac{\partial}{\partial t}\boldsymbol{B} = \nabla \times (\boldsymbol{u} \times \boldsymbol{B}) - \nabla \times [\lambda(\boldsymbol{r})\nabla \times \boldsymbol{B}]\ , \tag{26}$$

$$\nabla \cdot \boldsymbol{u} = 0\,, \tag{27}$$

$$\nabla \cdot \boldsymbol{B} = 0\,. \tag{28}$$

In the exterior V_{e}, there exist no flows or electric currents, the magnetic field $\boldsymbol{B}_{\mathrm{e}}$ is described by

$$\nabla \times \boldsymbol{B}_{\mathrm{e}} = 0\,, \tag{29}$$

$$\nabla \cdot \boldsymbol{B}_{\mathrm{e}} = 0\,. \tag{30}$$

The kinematic solar dynamo seeks solutions of (26)–(30) under the following conditions

$$\boldsymbol{B} = \boldsymbol{B}_{\mathrm{e}} \quad \text{on } S\,, \tag{31}$$

$$\hat{\boldsymbol{r}} \cdot \boldsymbol{u} = 0 \quad \text{on } S\,. \tag{32}$$

Solar dynamo action occurs if the magnetic energy of the system attains a non-zero value

$$\int_{V+V_{\mathrm{e}}} \frac{1}{2\mu} |\boldsymbol{B}|^2 \mathrm{d}V \neq 0, \qquad \text{as} \quad t \to \infty \tag{33}$$

for a given initial condition of the magnetic field.

In contrast to the kinematic dynamo theory in which the velocity $\boldsymbol{u}$ satisfies only (27), the dynamic dynamo theory constrains the velocity $\boldsymbol{u}$ further by the equation of motion

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\frac{1}{\rho_{\mathrm{o}}} \nabla p + \frac{1}{\rho_{\mathrm{o}} \mu} (\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \nu \nabla^2 \boldsymbol{u} + \boldsymbol{f}\,, \tag{34}$$

where p is the pressure, ρ_{o} is the fluid density, ν is the kinematic viscosity, $\boldsymbol{f}$ represents external forces such as buoyancy forces, and $\boldsymbol{\Omega}$ denotes the angular velocity of the Sun. In comparison with a well advanced state of the kinematic theory, the dynamic problem is much more complicated and is still poorly understood.

3 FLOWS IN THE SUN AND THE BASIC MHD PROCESSES

3.1 The Flow Structure within the Sun

There exist three major zones in the solar interior (see Figure 1). The convection zone extends from the visible surface down about 20% of its radius ($R_S = 7 \times 10^{10}$ cm). Helioseismic inversion reveals a weak differential rotation in the convection envelope: the equatorial region is rotating about 30% faster than the polar region. The Sun's core is a radiative region which rotates almost uniformly (for example, Schou et al. 1998; Gough & McIntyre 1998). The tachocline lies between the convection zone and the solid-body rotating radiative core (Schou 1991; Spiegel & Zahn 1992; Gough et al. 1996) and is widely believed to be the seat where the solar dynamo processes operate to generate its strong toroidal magnetic field with a strength up to $O(10^5)$ G, about 10 times the equipartition field strength in the convection zone (for example, Parker 1993; Weiss 1994).

The helioseismic studies (for example, Schou 1998) allow the determination of the variation of angular velocity $\boldsymbol{\Omega}$ as a function of the radius r and co-latitude θ in the Sun's interior. It has been suggested that the velocity $\boldsymbol{u}$ in the Sun may be written as

$$\begin{aligned}
\boldsymbol{u} &= \boldsymbol{\Omega}_c \times \boldsymbol{r}, \quad 0 \le r < r_i = 0.66 R_S; \\
\boldsymbol{u} &= [\boldsymbol{\Omega}_c + (\boldsymbol{\Omega}_S(\theta) - \boldsymbol{\Omega}_c) f(r)] \times \boldsymbol{r} = [f(r)\Delta(\boldsymbol{\Omega}_S)] \times \boldsymbol{r}, \quad r_i \le r < r_t = 0.71 R_S; \\
\boldsymbol{u} &= \boldsymbol{\Omega}_S(\theta) \times \boldsymbol{r} + \boldsymbol{U}_p(r,\theta) + \hat{\boldsymbol{u}}(r,\theta,\phi), \quad r_t \le r < r_o = R_S,
\end{aligned} \tag{35}$$

where $\hat{\boldsymbol{u}}$ represents small-scale turbulent flow, which is critically important to the solar dynamo action as we will discuss later, $\boldsymbol{U}_p$ represents the axisymmetric meridional circulation and

$$|\boldsymbol{\Omega}_c| = 430\,\text{nHz}, \quad |\boldsymbol{\Omega}_S(\theta)| = \left(455.8 - 51.2\cos^2\theta - 84.0\cos^4\theta\right)\,\text{nHz}.$$

Here $\boldsymbol{\Omega}_c$ is the rotation rate of the radiative core and (r, θ, ϕ) are spherical polar coordinates with $\theta = 0$ at the axis of rotation. Moreover we have in Equation (35)

$$f(0.66R_s) = 0, \quad f(0.71R_S) = 1, \quad \left|\frac{\mathrm{d}f(r)}{\mathrm{d}r}\right| \gg \left|\frac{1}{R_s}\frac{\mathrm{d}|\boldsymbol{\Omega}_S(\theta)|}{\mathrm{d}\theta}\right|.$$

It should be noted that a differential rotation can be readily maintained even in a stably stratified layer (Zhang & Schubert 1996, 2001). In the following, we shall discuss the role of each component of the flow in the solar dynamo processes.

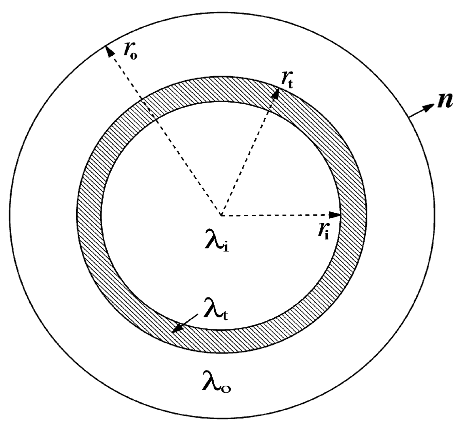

Fig. 1 Geometry of a three-dimensional, four-zone, interface dynamo model: $0 < r \le r_i$, the uniformly rotating, electrically conducting core with magnetic diffusivity λ_i; $r_i \le r \le r_t$, the differentially rotating tachocline with magnetic diffusivity λ_t; $r_t \le r \le r_0$, the convection zone with magnetic diffusivity λ_o; and $r > r_0$, the exterior with a large magnetic diffusivity λ_e.

3.2 The Effect of the Differential Rotation $\boldsymbol{\Omega}(r, \theta)$

The differential rotation, an axisymmetric flow with circular streamlines around the axis of the Sun's rotation, distorts a poloidal magnetic field and generates a toroidal magnetic field. In the solar context, it is perhaps the most important ingredient in producing an oscillatory dynamo solution. To appreciate the effect of the differential rotation on the solar dynamo action, we express an axisymmetric magnetic field in spherical geometry

$$\boldsymbol{B} = \boldsymbol{B}_P + \boldsymbol{B}_T = \nabla \times \left[A(r,\theta)\hat{\boldsymbol{\phi}}\right] + B_\phi(r,\theta)\hat{\boldsymbol{\phi}}\,, \tag{36}$$

where $\hat{\boldsymbol{\phi}}$ is a unit vector in the $\phi-$direction and $\boldsymbol{B}_P$ and $\boldsymbol{B}_T$ denote the poloidal and toroidal components of the axisymmetric magnetic field, respectively. By noting that

$$-\hat{\boldsymbol{\phi}} \cdot (\nabla \times \nabla \times \boldsymbol{B}) = \mathcal{D}^2 B_\phi = \left(\nabla^2 - \frac{1}{r^2\sin^2\theta}\right) B_\phi\,, \tag{37}$$

the $\phi-$component of the steady ($\partial B_\phi/\partial t = 0$) dynamo equation may be written as

$$-(r\sin\theta)\boldsymbol{B}_P \cdot \nabla|\boldsymbol{\Omega}_S(\theta)| = \lambda \mathcal{D}^2 B_\phi\,. \tag{38}$$

Equation (38) describes an important physical mechanism: a non-zero gradient $\nabla|\boldsymbol{\Omega}_S(\theta)|$ (differential rotation) on a poloidal field $\boldsymbol{B}_P$ can generate a toroidal magnetic field B_ϕ. The process that produces a toroidal field B_ϕ from a poloidal/meridional field $\boldsymbol{B}_P$ by the differential rotation $\boldsymbol{\Omega}(r,\theta)$ is known as the omega-effect. By this mechanism, the strong differential rotation in the tachocline can generate, from a poloidal field produced in the convection zone, an extremely strong toroidal magnetic field.

3.3 The Effect of Turbulent Flows $\hat{\boldsymbol{u}}$

While the differential rotation $\boldsymbol{\Omega}(r,\theta)$ and meridional circulation $\boldsymbol{U}_P$ in the Sun may be inferred from observations and are treated as known functions, the turbulent convection $\hat{\boldsymbol{u}}$ in the Sun's convection zone, which produces the random disturbance of magnetic fields, is poorly understood. Analytical or observational descriptions of the detailed feature of the turbulence are notoriously difficult and remain an unsolved problem. A widely accepted picture of turbulent flows is that strong nonlinear inertial effects tend to cascade the kinetic energy of the larger scales to the smaller scales, where it is transferred by viscosity into thermal energy. To avoid the intrinsic difficulties in the strong nonlinearity, the solar dynamo theory usually assumes that all statistical properties of the turbulent convection are known. Both the flow $\boldsymbol{u}$ and the magnetic field $\boldsymbol{B}$ in the convection zone are then divided into axisymmetric and non-axisymmetric parts

$$\boldsymbol{u} \;=\; \boldsymbol{U}_p(r,\theta) + \boldsymbol{\Omega}_S(\theta) \times \boldsymbol{r} + \hat{\boldsymbol{u}}(r,\theta,\phi)\,, \tag{39}$$

$$\boldsymbol{B} \;=\; \boldsymbol{B}_0(r,\theta) + \hat{\boldsymbol{B}}(r,\theta,\phi)\,, \tag{40}$$

where $\hat{\boldsymbol{u}}$ and $\hat{\boldsymbol{B}}$ have the properties

$$\int_0^{2\pi} \hat{\boldsymbol{B}}(r,\theta,\phi)\mathrm{d}\phi = 0, \qquad \int_0^{2\pi} \hat{\boldsymbol{u}}(r,\theta,\phi)\mathrm{d}\phi = 0\,. \tag{41}$$

Substitution of (39)–(40) into the dynamo equation and taking an average over longitude ϕ yield an equation for the mean magnetic field $\boldsymbol{B}_0$

$$\frac{\partial}{\partial t}\boldsymbol{B}_0 = \nabla \times (\boldsymbol{U}_p + \boldsymbol{\Omega}_S \times \boldsymbol{r}) \times \boldsymbol{B}_0 + \nabla \times \int_0^{2\pi} \left(\hat{\boldsymbol{u}} \times \hat{\boldsymbol{B}}\right) \mathrm{d}\phi + \lambda_o \nabla^2 \boldsymbol{B}_0\,. \tag{42}$$

Its corresponding non-axisymmetric $\hat{\boldsymbol{B}}$ is governed by

$$\frac{\partial}{\partial t}\hat{\boldsymbol{B}} = \nabla \times (\boldsymbol{U}_p + \boldsymbol{\Omega}_S \times \boldsymbol{r}) \times \hat{\boldsymbol{B}} + \nabla \times (\hat{\boldsymbol{u}} \times \boldsymbol{B}_0) + \lambda_o \nabla^2 \hat{\boldsymbol{B}} + ... \tag{43}$$

An essential question is whether the interaction of small-scale flow $\hat{\boldsymbol{u}}$ and magnetic fluctuation $\hat{\boldsymbol{B}}$ can maintain the large-scale $\boldsymbol{B}_0$. Equation (43) suggests that there exists a linear relationship between $\hat{\boldsymbol{B}}$ and $\boldsymbol{B}_0$ so that we may make the following expansion

$$\int_0^{2\pi} \left(\hat{\boldsymbol{u}} \times \hat{\boldsymbol{B}}\right) \mathrm{d}\phi = \alpha_{ij} B_{0j} + \beta_{ijk} \frac{\partial B_{0j}}{\partial x_k} + ..., \tag{44}$$

where α_{ij} and β_{ijk} are pseudo-tensors dependent upon the properties of turbulent convection. The relationship (44), which is called the alpha-effect, represents a cornerstone of the modern kinematic dynamo theory, first discovered by Parker (1955) and later developed by Steenbeck & Krause (1966). If we assume further that the turbulent flow in the solar convection zone is isotropic, we can write

$$\alpha_{ij} = \alpha_0 \delta_{ij}, \qquad \beta_{ijk} = -\beta \epsilon_{ijk}\,, \tag{45}$$

which gives

$$\int_0^{2\pi} \left(\hat{\boldsymbol{u}} \times \hat{\boldsymbol{B}}\right) \mathrm{d}\phi = \alpha_0 \boldsymbol{B}_0 - \beta \nabla \times \boldsymbol{B}_0\,. \tag{46}$$

The mean-field dynamo equation in the solar convection zone then becomes

$$\frac{\partial}{\partial t}\boldsymbol{B}_0 = \nabla \times (\boldsymbol{U}_p + \boldsymbol{\Omega}_S \times \boldsymbol{r}) \times \boldsymbol{B}_0 + \nabla \times \alpha_0 \boldsymbol{B}_0 + (\lambda_o + \beta)\nabla^2 \boldsymbol{B}_0\,, \tag{47}$$

where β is the turbulent magnetic diffusivity. The value of β in the convection zone may be estimated by (for example, Moffatt 1978)

$$\beta = 0.1 \langle \hat{\boldsymbol{u}} \rangle \hat{L},$$

where $\langle \hat{\boldsymbol{u}} \rangle$ is the root-mean-square turbulent velocity in the convection zone and $\hat{L}$ is the correlation length of the turbulence. Doppler shift measurements indicate that $\langle \hat{\boldsymbol{u}} \rangle$ is of the order of $1\,\mathrm{km\,s^{-1}}$ in the granulation on scales of $O(10^3\,\mathrm{km})$, which yields

$$\beta = 0.1 \times 10^3\,\mathrm{km}^2\,\mathrm{s}^{-1} = 10^8 \mathrm{m}^2\,\mathrm{s}^{-1} = 10^{12}\,\mathrm{cm}^2\,\mathrm{s}^{-1}\,.$$

This corresponds to the solar magnetic decay time τ_S,

$$\tau_S = \frac{R_S^2}{\beta} = \frac{(7 \times 10^8\,\mathrm{m})^2}{10^8\,\mathrm{m}^2\,\mathrm{s}^{-1}} = O(10\,\mathrm{yr})\,,$$

which is consistent with the observed decade variation of the solar magnetic fields.

In contrast to hydrodynamic turbulence, the theory of magnetohydrodynamic turbulence involves transferring the magnetic energy of small scale back to the top (large-scale) of the energy spectrum via the alpha-effect. It is the alpha-effect that completes the solar dynamo generation cycle by producing a poloidal field from a toroidal field.

4 NECESSARY CONDITIONS FOR THE SOLAR DYNAMO

4.1 Differential Rotation $\boldsymbol{\Omega}(r, \theta)$ cannot Maintain the Solar Dynamo

By completely ignoring the Lorentz force on the velocity $\boldsymbol{u}$, the kinematic solar dynamo represents the mathematically simplest problem. However, obtaining an exponentially growing solution for a given $\boldsymbol{u}$, i.e., a working solar dynamo, can be mathematically complicated. In fact, most of the earlier research on the dynamo theory focused almost exclusively on proving anti-dynamo theorems.

Denote the magnetic field in the radiative core, the tachocline, the convection zone and the exterior $\boldsymbol{B}_i$, $\boldsymbol{B}_t$, $\boldsymbol{B}_o$, $\boldsymbol{B}_e$, respectively. The simplest solar dynamo problem seems to neglect both the meridional circulation $\boldsymbol{U}_P$ and turbulent convection $\hat{\boldsymbol{u}}$ in the convection zone. The dynamo equation in each region within the Sun becomes,

$$\frac{\partial}{\partial t}\boldsymbol{B}_i = \nabla \times (\boldsymbol{\Omega}_c \times \boldsymbol{r}) \times \boldsymbol{B}_i + \lambda_i \nabla^2 \boldsymbol{B}_i, \quad \text{in } V_i : 0 < r < r_i; \tag{48}$$

$$\frac{\partial}{\partial t}\boldsymbol{B}_t = \nabla \times [f(r)\Delta(\boldsymbol{\Omega}_s) \times \boldsymbol{r}] \times \boldsymbol{B}_t + \lambda_t \nabla^2 \boldsymbol{B}_t, \quad \text{in } V_t : r_i < r < r_t; \tag{49}$$

$$\frac{\partial}{\partial t}\boldsymbol{B}_o = \nabla \times (\boldsymbol{\Omega}_S \times \boldsymbol{r}) \times \boldsymbol{B}_o + \lambda_o \nabla^2 \boldsymbol{B}_o, \quad \text{in } V_c : r_t < r < r_o; \tag{50}$$

$$0 = \nabla^2 \boldsymbol{B}_e, \text{ in } V_e : r_o < r < \infty. \tag{51}$$

Different regimes are coupled by the magnetic matching conditions (20)–(22). An interesting question is whether a purely differential rotation can produce a self-exciting dynamo action in the Sun. It was recognized by Bullard & Gellman (1954) (see also Backus 1958) that magnetic fields cannot be maintained by a purely differential rotation in a sphere. In this review, we extend the previous proof for a sphere to the spherical system of multiple layers with discontinuous magnetic diffusivities for the solar application.

It is mathematically convenient to expand a general three-dimensional magnetic field, for example $\boldsymbol{B}_t$ in the tachocline, in terms of the vector potentials g_t and h_t

$$\boldsymbol{B}_t = \nabla \times [\boldsymbol{r} g_t(r,\theta,\phi,t)] + \nabla \times \nabla \times [\boldsymbol{r} h_t(r,\theta,\phi,t)] \ , \tag{52}$$

where

$$\boldsymbol{r} \cdot \boldsymbol{B}_t = \mathcal{L} h_t = -\left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta} \frac{\partial}{\partial\phi^2} \right) h_t \, . \tag{53}$$

Note that the addition of any function $Y(r)$ to the scalar functions g_t and h_t in (52) has no effects on $\boldsymbol{B}_t$. In consequence, we may assume that g_t and h_t satisfy

$$\int_0^{2\pi} \left(\int_0^{\pi} g_t \sin\theta \mathrm{d}\theta \right) \mathrm{d}\phi = 0, \quad \int_0^{2\pi} \left(\int_0^{\pi} h_t \sin\theta \mathrm{d}\theta \right) \mathrm{d}\phi = 0 \, . \tag{54}$$

The magnetic fields in the other regions of the Sun can be also expanded in a similar way

$$\begin{aligned} \boldsymbol{B}_i &= \nabla \times (\boldsymbol{r} g_i) + \nabla \times \nabla \times (\boldsymbol{r} h_i) \ , \\ \boldsymbol{B}_c &= \nabla \times (\boldsymbol{r} g_o) + \nabla \times \nabla \times (\boldsymbol{r} h_o) \ , \\ \boldsymbol{B}_e &= \nabla \times \nabla \times (\boldsymbol{r} h_e) \ . \end{aligned}$$

Here we have assumed that the exterior is vacuum and that $g_e = 0$ in V_e.

The equations for the poloidal field component can be obtained by forming the scalar product of Equations (48)–(51) for each zone

$$\frac{\partial}{\partial t} \mathcal{L} h_i + (\boldsymbol{\Omega}_c \times \boldsymbol{r}) \cdot \nabla \mathcal{L} h_i = (\boldsymbol{B}_i \cdot \nabla) [\boldsymbol{r} \cdot (\boldsymbol{\Omega}_c \times \boldsymbol{r})] + \lambda_i \nabla^2 \mathcal{L} h_i \, , \tag{55}$$

$$\frac{\partial}{\partial t} \mathcal{L} h_t + (f(r) \Delta(\boldsymbol{\Omega}_S) \times \boldsymbol{r}) \cdot \nabla \mathcal{L} h_t = (\boldsymbol{B}_t \cdot \nabla) \{ \boldsymbol{r} \cdot [f(r) \Delta(\boldsymbol{\Omega}_S)] \times \boldsymbol{r} \} + \lambda_t \nabla^2 \mathcal{L} h_t \, , \tag{56}$$

$$\frac{\partial}{\partial t} \mathcal{L} h_o + (\boldsymbol{\Omega}_S \times \boldsymbol{r}) \cdot \nabla \mathcal{L} h_o = (\boldsymbol{B}_o \cdot \nabla) [\boldsymbol{r} \cdot (\boldsymbol{\Omega}_S \times \boldsymbol{r})] + \lambda_o \nabla^2 \mathcal{L} h_o \, , \tag{57}$$

$$0 = \nabla^2 \mathcal{L} h_e \, . \tag{58}$$

Evidently, the first term in the right-hand side of equations (55–57) vanishes. Multiplying Equation (55) by $\mathcal{L} h_i$ gives

$$\frac{\partial}{\partial t} \frac{1}{2\lambda_i} (\mathcal{L} h_i)^2 + \frac{1}{2\lambda_i} \nabla \cdot \left[(\mathcal{L} h_i)^2 (\boldsymbol{\Omega}_c \times \boldsymbol{r}) \right] = \nabla \cdot [(\mathcal{L} h_i) \nabla (\mathcal{L} h_i)] - |\nabla \mathcal{L} h_i|^2 \, . \tag{59}$$

Integrating it over the Sun's radiative core V_i and making use of the condition $\hat{\boldsymbol{r}} \cdot (\boldsymbol{\Omega}_c \times \boldsymbol{r}) = 0$, we have

$$\frac{\partial}{\partial t} \int_{V_i} \frac{1}{2\lambda_i} (\mathcal{L} h_i)^2 = -\int_{V_i} \nabla^2 |\mathcal{L} h_i|^2 \, \mathrm{d}V + \int_{\partial V_i} \left[(\mathcal{L} h_i) \frac{\mathcal{L} h_i}{\partial r} \right] \mathrm{d}S \, , \tag{60}$$

where ∂V_i denotes the interface between the core and the tachocline. By the same procedure, we can derive similar equations for the other zones in the Sun. The summation of all the equations obtained from different zones gives

$$\frac{\partial}{\partial t}\left[\int_{V_i}\frac{(\mathcal{L}h_i)^2}{2\lambda_i}\mathrm{d}V+\int_{V_t}\frac{(\mathcal{L}h_t)^2}{2\lambda_t}\mathrm{d}V+\int_{V_o}\frac{(\mathcal{L}h_o)^2}{2\lambda_o}\mathrm{d}V\right]$$
$$= -\left[\int_{V_i}|\nabla\mathcal{L}h_i|^2\,\mathrm{d}V+\int_{V_t}|\nabla\mathcal{L}h_t|^2\,\mathrm{d}V+\int_{V_o}|\nabla\mathcal{L}h_o|^2\,\mathrm{d}V+\int_{V_e}|\nabla\mathcal{L}h_e|^2\,\mathrm{d}V\right]. \quad (61)$$

All the boundary integrals are cancelled out by using the following boundary or matching conditions

$$\begin{aligned}
\mathcal{L}h_i &= \mathcal{L}h_t,\ \frac{\partial\mathcal{L}h_i}{\partial r}=\frac{\partial\mathcal{L}h_t}{\partial r},\ \text{ at } r=r_i;\\
\mathcal{L}h_t &= \mathcal{L}h_o,\ \frac{\partial\mathcal{L}h_t}{\partial r}=\frac{\partial\mathcal{L}h_o}{\partial r},\ \text{ at } r=r_t;\\
\mathcal{L}h_o &= \mathcal{L}h_e,\ \frac{\partial\mathcal{L}h_o}{\partial r}=\frac{\partial\mathcal{L}h_e}{\partial r},\ \text{ at } r=r_o;\\
\mathcal{L}h_e &= O(r^{-2})\to 0,\ \text{ as } r\to\infty.
\end{aligned} \quad (62)$$

It follows that the poloidal fields in all the zones starting with any initial condition must decay ultimately to zero in time.

Consider now the toroidal component of the magnetic field. In the radiative core, we have

$$\frac{\partial}{\partial t}\nabla\times(\boldsymbol{r}g_i)=\nabla\times[(\boldsymbol{\Omega}_c\times\boldsymbol{r})\times(\nabla\times\boldsymbol{r}g_i)]+\lambda_i\nabla^2[\nabla\times(\boldsymbol{r}g_i)], \quad (63)$$

which can be written as

$$\boldsymbol{r}\times\nabla\left[\frac{\partial g_i}{\partial t}+(\boldsymbol{\Omega}_c\times\boldsymbol{r})\cdot\nabla g_i-\lambda_i\nabla^2 g_i+F(r)\right]=0, \quad (64)$$

where $F(r)$ is an arbitrary function of r. In the following derivation, we need the identity

$$Q\nabla^2 Q=\nabla\cdot\left[\frac{Q}{r}\nabla(rQ)\right]-\left[\frac{1}{r}\hat{\boldsymbol{r}}\cdot\nabla(rQ)\right]^2-|\nabla_H Q|^2,$$

where ∇_H is the gradient on a unit spherical surface

$$\nabla_H=\frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial}{\partial\theta}+\frac{\hat{\boldsymbol{\phi}}}{r\sin\theta}\frac{\partial}{\partial\phi}.$$

We first uncurl (64) and then multiply the resulting equation with g_i using the above identity

$$\frac{1}{2}\frac{\partial g_i^2}{\partial t}+\frac{1}{2}\nabla\cdot\left[g_i^2(\boldsymbol{\Omega}\times\boldsymbol{r})\right]$$
$$=\lambda_i\left\{\nabla\cdot\left[\frac{g_i}{r}\nabla(rg_i)\right]-\left[\frac{1}{r}\hat{\boldsymbol{r}}\cdot\nabla(rg_i)\right]^2-|\nabla_H g_i|^2+g_iF(r)\right\}. \quad (65)$$

Integrating it over the radiative core in V_i, we obtain

$$\frac{1}{2}\frac{\partial}{\partial t}\int_V g_i^2\mathrm{d}V=\int_{\partial V_i}\lambda_i\left(\frac{g_i}{r}\hat{\boldsymbol{r}}\cdot\nabla(rg_i)\right)\mathrm{d}S-\int_{V_i}\left\{\left[\frac{1}{r}\hat{\boldsymbol{r}}\cdot\nabla(rg_i)\right]^2+|\nabla_H g_i|^2\right\}\mathrm{d}V, \quad (66)$$

where we have used the property (54). Summing all the similar equations in the different zones gives

$$\frac{1}{2}\frac{\partial}{\partial t}\left(\int_{V_i} g_i^2 \mathrm{d}V + \int_{V_t} g_t^2 \mathrm{d}V + \int_{V_o} g_o^2 \mathrm{d}V\right) = -\int_{V_i}\left\{\left[\frac{1}{r}\hat{\boldsymbol{r}}\cdot\nabla(rg_i)\right]^2 + |\nabla_H g_i|^2\right\}\mathrm{d}V.$$

$$-\int_{V_t}\left\{\left[\frac{1}{r}\hat{\boldsymbol{r}}\cdot\nabla(rg_t)\right]^2 + |\nabla_H g_t|^2\right\}\mathrm{d}V - \int_{V_o}\left\{\left[\frac{1}{r}\hat{\boldsymbol{r}}\cdot\nabla(rg_o)\right]^2 + |\nabla_H g_o|^2\right\}\mathrm{d}V\,. \qquad (67)$$

Here we have made use of the marching or boundary conditions

$$\begin{aligned} g_i &= g_t, \;\; \lambda_i\hat{\boldsymbol{r}}\cdot\nabla(rg_i) = \lambda_t\hat{\boldsymbol{r}}\cdot\nabla(rg_t) \;\text{ at }\; r = r_i\,, \\ g_t &= g_o, \;\; \lambda_t\hat{\boldsymbol{r}}\cdot\nabla(rg_t) = \lambda_o\hat{\boldsymbol{r}}\cdot\nabla(rg_o) \;\text{ at }\; r = r_t\,, \\ g_o &= 0, \quad \text{at }\; r = r_o\,. \end{aligned} \qquad (68)$$

The last condition in (68) is related to the vacuum boundary condition. As a result, the integrals of g_i^2, g_t^2 and g_o^2 vanish as $t \to \infty$, so that

$$g_i = g_t = g_o = 0.$$

The conclusion is that any profile of a purely differential rotation in the Sun, whether spatially continuous or discontinuous, cannot sustain the solar magnetic fields.

4.2 $\boldsymbol{\Omega}(r,\theta)$ with $\boldsymbol{U}_p(r,\theta)$ cannot Sustain an Axisymmetric Solar Dynamo

The next question is whether the combination of differential rotation $\boldsymbol{\Omega}(r,\theta)$ and meridional circulation $\boldsymbol{U}_p(r,\theta)$ can produce a self-exciting axisymmetric dynamo in the Sun. After the solar dynamo proposal first put forward by Larmor (1919), it was Cowling (1934) who proved that steady axisymmetric poloidal field cannot be maintained by dynamo processes. Cowling's theorem was later extended to the more general form that any axisymmetric magnetic field in a sphere cannot be self-sustained (Braginsky 1964). This paper extends the Cowling theorem in a sphere to the situation of multiple spherical layers for the Sun.

If we neglect the alpha-effect in the convection zone, an axisymmetric magnetic field in the different zones satisfies

$$\frac{\partial}{\partial t}\boldsymbol{B}_i(r,\theta,t) = \nabla\times[(\boldsymbol{\Omega}_c\times\boldsymbol{r})\times\boldsymbol{B}_i] + \lambda_i\nabla^2\boldsymbol{B}_i, \quad \text{in } V_i : 0 \le r < r_i; \qquad (69)$$

$$\frac{\partial}{\partial t}\boldsymbol{B}_t(r,\theta,t) = \nabla\times\{[(f(r)\Delta(\boldsymbol{\Omega}_s)\times\boldsymbol{r})]\times\boldsymbol{B}_t\} + \lambda_t\nabla^2\boldsymbol{B}_t, \quad \text{in } V_t : r_i < r < r_t; \qquad (70)$$

$$\frac{\partial}{\partial t}\boldsymbol{B}_o(r,\theta,t) = \nabla\times\{[(\boldsymbol{\Omega}_S\times\boldsymbol{r}) + \boldsymbol{U}_p]\times\boldsymbol{B}_o\} + \lambda_o\nabla^2\boldsymbol{B}_o, \quad \text{in } V_o : r_t < r < r_o; \qquad (71)$$

$$0 = \nabla^2\boldsymbol{B}_e, \quad \text{in } V_e : r_o < r < \infty. \qquad (72)$$

It is mathematically helpful to express the axisymmetric field in the form, for example,

$$\boldsymbol{B}_o = \nabla\times\left[\left(\frac{\Phi_o}{r\sin\theta}\right)\hat{\boldsymbol{\phi}}\right] + (r\sin\theta\Psi_o)\,\hat{\boldsymbol{\phi}},$$

which satisfies the solenoidal condition automatically, where Φ_o and Ψ_o are poloidal and toroidal flux functions.

First, we look at the poloidal component of the dynamo equation, for example, in the convection zone

$$\nabla \times \left\{ \frac{\partial}{\partial t} \left[\left(\frac{\Phi_o}{r \sin\theta} \right) \hat{\boldsymbol{\phi}} \right] + \left[\frac{1}{r \sin\theta} \boldsymbol{U}_P \cdot \nabla \Phi_o \right] \hat{\boldsymbol{\phi}} - \lambda_o \mathcal{D}^2 \left(\frac{\Phi_o}{r \sin\theta} \right) \hat{\boldsymbol{\phi}} + \nabla F(r, \theta) \right\} = 0, \quad (73)$$

where

$$\mathcal{D}^2 \left(\frac{\Phi_o}{r \sin\theta} \right) = \frac{1}{r \sin\theta} \left[\nabla^2 \Phi_o - \frac{2}{r \sin\theta} \nabla (r \sin\theta) \cdot \nabla \Phi_o \right].$$

We first uncurl (73) and then form the scalar product of ϕ-component of the equation. We multiply the resulting equation by Φ_o

$$\frac{1}{2\lambda_o} \left[\frac{\partial}{\partial t} \Phi_o^2 + \nabla \cdot \left(\Phi_o^2 \boldsymbol{U}_P \right) \right] = \nabla \cdot \left(\Phi_o \nabla \Phi_o \right) - |\nabla \Phi_o|^2 - \nabla \cdot \left\{ \left[\frac{(\sin\theta) \hat{\boldsymbol{r}} + (\cos\theta) \hat{\boldsymbol{\theta}}}{r \sin\theta} \right] \Phi_o^2 \right\}. \quad (74)$$

Integrating it over the convection zone V_o and making use of the velocity boundary condition $\hat{\boldsymbol{r}} \cdot \boldsymbol{U}_P = 0$, we obtain

$$\frac{\partial}{\partial t} \left(\int_{V_o} \frac{1}{2\lambda_o} \Phi_o^2 \mathrm{d}V \right) = - \int_{V_o} |\nabla \Phi_o|^2 \, \mathrm{d}V + \left(\int_{\partial V_o} - \int_{\partial V_t} \right) \left[\Phi_o \frac{\partial \Phi_o}{\partial r} - \frac{\Phi_o^2}{r} \right] \mathrm{d}S \,, \quad (75)$$

where ∂V_t and ∂V_o denote the spherical surfaces at $r = r_t$ and $r = r_o$. By the same procedure we can derive similar equations for the other zones. The summation of these gives

$$\begin{aligned} &\frac{\partial}{\partial t} \left[\int_{V_o} \frac{\Phi_i^2}{2\lambda_o} \mathrm{d}V + \int_{V_t} \frac{\Phi_t^2}{2\lambda_t} \mathrm{d}V + \int_{V_o} \frac{\Phi_o^2}{2\lambda_o} \mathrm{d}V \right] \\ = &- \left[\int_{V_i} |\nabla \Phi_i|^2 \, \mathrm{d}V + \int_{V_t} |\nabla \Phi_t|^2 \, \mathrm{d}V + \int_{V_o} |\nabla \Phi_o|^2 \, \mathrm{d}V + \int_{V_e} |\nabla \Phi_e|^2 \, \mathrm{d}V \right]. \quad (76) \end{aligned}$$

All the boundary integrals at the various interfaces are either cancelled out or vanish by using the matching or boundary conditions

$$\begin{aligned} \Phi_i &= \Phi_t, \ \frac{\partial \Phi_i}{\partial r} = \frac{\partial \Phi_t}{\partial r}, \ \text{at} \ r = r_i; \\ \Phi_t &= \Phi_o, \ \frac{\partial \Phi_t}{\partial r} = \frac{\partial \Phi_o}{\partial r}, \ \text{at} \ r = r_t; \\ \Phi_o &= \Phi_e, \ \frac{\partial \Phi_o}{\partial r} = \frac{\partial \Phi_e}{\partial r}, \ \text{at} \ r = r_o; \\ \Phi_e &= O(r^{-1}) \to 0, \ \text{as} \ r \to \infty. \quad (77) \end{aligned}$$

Equation (76) implies that the poloidal fields in all the zones starting with any initial condition must decay ultimately to zero in time.

The equation for the toroidal component of magnetic fields can be obtained by the $\phi-$component of the dynamo equation, for example, in the convection zone,

$$\frac{\partial \Psi_o}{\partial t} + \boldsymbol{U}_P \cdot \nabla \Psi_o = \left\{ \nabla \times \left[\left(\frac{\Phi_o}{r \sin\theta} \right) \hat{\boldsymbol{\phi}} \right] \right\} \cdot \nabla |\boldsymbol{\Omega}_S| + \frac{\lambda_o}{r \sin\theta} \hat{\boldsymbol{\phi}} \cdot \nabla^2 \left(r \sin\theta \Psi_o \hat{\boldsymbol{\phi}} \right). \quad (78)$$

It is crucial to note that the poloidal magnetic field $\Phi_0 \to 0$, as suggested by (76). In consequence, there is no source term for the toroidal flux Ψ_o. We also note the identity

$$\frac{Q}{r \sin\theta} \hat{\boldsymbol{\phi}} \cdot \nabla^2 \left(r \sin\theta Q \hat{\boldsymbol{\phi}} \right) = \nabla \cdot \left[\frac{Q}{r} \nabla (rQ) \right] - |\nabla Q|^2 + \nabla \cdot \left[\left(\frac{Q^2 \cot\theta}{r} \right) \hat{\boldsymbol{\theta}} \right].$$

Multiplying (78) with Ψ_o and making use of the above identity, we obtain

$$\begin{aligned} &\frac{1}{2}\ \frac{\partial \Psi_o^2}{\partial t} + \frac{1}{2}\nabla \cdot \left(\boldsymbol{U}_P \Psi_o^2\right) \\ &= \lambda_o \left\{ \nabla \cdot \left[\frac{\Psi_o}{r} \nabla (r\Psi_o) \right] - |\nabla \Psi_o|^2 + \nabla \cdot \left[\left(\frac{\Psi_o^2 \cot\theta}{r} \right) \hat{\boldsymbol{\theta}} \right] \right\}. \end{aligned} \tag{79}$$

Integration of the equation over the convection zone in V_o gives

$$\frac{1}{2}\frac{\partial}{\partial t}\int_{V_o} \Psi_o^2 \mathrm{d}V = -\lambda_o \int_{V_o} |\nabla \Psi_o|^2 \,\mathrm{d}V + \left(\int_{\partial V_t} - \int_{\partial V_o} \right) \frac{1}{r} \left[\lambda_o \Psi_o \hat{\boldsymbol{r}} \cdot \nabla (r\Psi_o) \right] \mathrm{d}S\,. \tag{80}$$

The summation of the integral equations obtained from different zones cancels all the surface integrals, leading to

$$\begin{aligned} &\frac{1}{2}\frac{\partial}{\partial t}\left(\int_{V_i} \Psi_i^2 \mathrm{d}V + \int_{V_t} \Psi_t^2 \mathrm{d}V + \int_{V_o} \Psi_o^2 \mathrm{d}V \right) \\ = &-\left[\lambda_i \int_{V_i} |\nabla \Psi_i|^2 \,\mathrm{d}V + \lambda_t \int_{V_t} |\nabla \Psi_t|^2 \,\mathrm{d}V + \lambda_o \int_{V_o} |\nabla \Psi_o|^2 \,\mathrm{d}V \right]. \end{aligned} \tag{81}$$

In the derivation, we have used the following marching or boundary conditions

$$\begin{aligned} \Psi_i &= \Psi_t, \quad \lambda_i \hat{\boldsymbol{r}} \cdot \nabla (r\Psi_i) = \lambda_t \hat{\boldsymbol{r}} \cdot \nabla (r\Psi_t) \quad \text{at} \quad r = r_i; \\ \Psi_t &= \Psi_o, \quad \lambda_t \hat{\boldsymbol{r}} \cdot \nabla (r\Psi_t) = \lambda_o \hat{\boldsymbol{r}} \cdot \nabla (r\Psi_o) \quad \text{at} \quad r = r_t; \\ \Psi_o &= 0, \quad \text{at} \;\; r = r_o\,. \end{aligned} \tag{82}$$

It follows that the integrals of Ψ_i^2, Ψ_t^2 and Ψ_o^2 vanish as $t \to \infty$, so that

$$\Psi_i = \Psi_t = \Psi_o = 0.$$

In the Sun's interior characterized by a radially discontinuous variation of the magnetic diffusivity, any combination of differential rotation $\boldsymbol{\Omega}(r,\theta)$ and meridional circulation $\boldsymbol{U}_P$ cannot sustain an axisymmetric solar magnetic field. It is these anti-dynamo theories that make the alpha-effect, which is neglected in (71), a critically important ingredient in the solar dynamo theory.

5 PARKER'S CARTESIAN INTERFACE SOLAR DYNAMO

Parker (1993) proposed a two-dimensional, linear, Cartesian interface dynamo that operates in the tachocline coupling with the alpha-effect in the convection zone. The interface concept has since been at the heart of modern solar dynamo theory. In Parker's model, the surface $z = 0$ gives an interface between the solar tachocline ($z < 0$) and the solar convection ($z > 0$). In the upper region with a large eddy magnetic diffusivity λ^+, turbulent convective motions produce an α-effect, generating a weak magnetic field $\boldsymbol{B}^+$ (Parker 1955; Moffatt 1978). In the lower region with a reduced eddy magnetic diffusivity λ^-, a uniform shear in the $y-$direction, $G = \mathrm{d}u_y/\mathrm{d}z$, generates a strong magnetic field $\boldsymbol{B}^-$ in the form of a dynamo wave confined to propagate along the $x-$direction at the interface between the two regions. Similar to an axisymmetric magnetic field in spherical geometry, Parker (1993) expressed two-dimensional Cartesian magnetic fields in the form

$$\boldsymbol{B}^+ = B^+ \hat{\boldsymbol{j}} + \nabla \times \hat{\boldsymbol{j}} A^+, \quad \boldsymbol{B}^- = B^- \hat{\boldsymbol{j}} + \nabla \times \hat{\boldsymbol{j}} A^-\,, \tag{83}$$

where $(\hat{\boldsymbol{i}}, \hat{\boldsymbol{j}}, \hat{\boldsymbol{k}})$ are unit vectors in the Cartesian coordinates. The dynamo Equation (26) in the upper region $z > 0$ becomes

$$\left[\frac{\partial}{\partial t} - \lambda^+ \nabla^2\right] B^+ = 0\,, \tag{84}$$

$$\left[\frac{\partial}{\partial t} - \lambda^+ \nabla^2\right] A^+ = \alpha_0 B^+\,. \tag{85}$$

In the lower region $z < 0$, we have

$$\left[\frac{\partial}{\partial t} - \lambda^- \nabla^2\right] B^- = G\frac{\partial A^-}{\partial z}\,, \tag{86}$$

$$\left[\frac{\partial}{\partial t} - \lambda^- \nabla^2\right] A^- = 0\,. \tag{87}$$

There are four boundary conditions at the interface $z = 0$: the continuity of the three components of the magnetic field and the normal component of the electric current

$$B^+ = B^-, \quad A^+ = A^-, \quad \frac{\partial A^+}{\partial z} = \frac{\partial A^-}{\partial z}, \quad \lambda^+\frac{\partial A^+}{\partial z} = \lambda^-\frac{\partial A^-}{\partial z}\,. \tag{88}$$

It is obvious that Equations (84)–(87) are linear and allow the plane wave solution, for example,

$$\begin{aligned} B^+ &= C^+ \exp\left(\sigma t - S^+ z\right) \cos(\omega t + kx - Q^+ z)\,, \\ B^- &= C^- \exp\left(\sigma t + S^- z\right) \cos(\omega t + kx + Q^- z + \Gamma)\,, \end{aligned}$$

where S^+ and S^- are positive so that B^+ or B^- vanishes at $|z| \to \infty$. Substitution of the wave solution into equations (84)–(87) leads to the dispersion relation. Moreover, the boundary conditions (88) at the interface give rise to a relationship between B^+ and B^-. A particularly important case concerns the properties of the dynamo wave solution in the limit $(\lambda^-/\lambda^+) \to 0$. In this limit, Parker (1993) found that

$$\frac{|B^-|_{\max}}{|B^+|_{\max}} = O\left(\frac{\lambda^+}{\lambda^-}\right)^{1/2}\,. \tag{89}$$

Though Parker's model is not a realistic model for the Sun, it perhaps contains the most fundamental ingredients of a real solar dynamo. The interface dynamo model describes a highly attractive picture of how to generate a strong toroidal magnetic field in the solar tachocline whose strength greatly exceeds the equipartition field obtained from conventional convection-driven dynamo models.

6 A 3-D NONLINEAR INTERFACE SOLAR DYNAMO

We have shown that the alpha-effect represents an essential element in the action of the solar dynamo. However, there is important physics missing from the linear relationship (46) between the alpha-effect and the large-scale magnetic field $\boldsymbol{B}_o$. The relationship (46) becomes un-physical when the generated magnetic field is so strong as to have the Reynolds stresses of the turbulent flow comparable to the magnetic stresses,

$$\frac{1}{2}\rho_o|\hat{\boldsymbol{u}}|^2 = \frac{1}{2\mu}|\boldsymbol{B}_o|^2\,. \tag{90}$$

In this case, the Lorentz forces, as suggested by Equation (34), must affect the convection and modify the alpha-effect. One indeed expects that the strength of the α-effect would be suppressed when the kinetic energy of the flow is comparable with the magnetic energy. A modified alpha-effect expression in the convection zone may be in the form

$$\alpha(r,\theta,\phi) \sim \frac{\alpha_o}{(1+|\boldsymbol{B}_o/B_{\rm eq}|^2)}, \quad r_t < r \le r_o\,, \tag{91}$$

where $B_{\rm eq}$ is the equipartition field. This nonlinear alpha effect has been widely used, for example, by Choudhuri et al. (1995) and Küker et al. (2001) (see Brandenburg 1994, for detailed discussion). A major advantage of this formulation is that it allows simulation of many essential dynamo processes without reference to the difficult dynamics of strong nonlinear interaction between the flow and the Lorentz forces. The formulation (91) is usually referred to as $\alpha-$quenching, which introduces nonlinearity into the kinematic solar dynamo model.

With the $\alpha-$quenching nonlinearity, the interface solar dynamo problem must be treated numerically. The following discusses briefly a new 3-D interface solar dynamo model based on a finite element method (Zhang et al. 2003). If we take the reference of frame that rotates with $\boldsymbol{\Omega}_c$, then the magnetic field $\boldsymbol{B}_i$ in the radiative core is governed by the equations

$$\frac{\partial \boldsymbol{B}_i}{\partial t} + \lambda_i \nabla\times\nabla\times\boldsymbol{B}_i = 0\,, \tag{92}$$

$$\nabla\cdot\boldsymbol{B}_i = 0\,. \tag{93}$$

The magnetic field $\boldsymbol{B}_i$ cannot be generated in this uniformly rotating sphere. On the top of the radiative core, a strong differential rotation, $\Omega_0 f(r)\Delta(\boldsymbol{\Omega}_S(\theta))$, where Ω_0 is the amplitude of the differential rotation, is confined in the tachocline $r_i < r < r_t$. It shears the weak poloidal magnetic field, which is generated in the convection zone and penetrates into the tachocline, into a strong magnetic field (for example, Roberts 1972).

Global instabilities in rotating spherical systems in the presence of a toroidal magnetic field or a differential rotation have been extensively studied. It was shown by Gilman & Fox (1997) that even very simple differential rotation can become unstable to azimuthal wavenumber $m = 1$ disturbances in the presence of a moderate magnetic field (see also Zhang 1995; Zhang & Busse 1995; Cally 2001; Miesch 2001). In a fully three-dimensional stability analysis for a toroidal magnetic field in rotating spherical systems, Zhang, Liao & Schubert (2003) demonstrate analytically that the magnetohydrodynamic system is unstable to the $m = 1$ perturbation. It is therefore likely that the strong shear flow in the tachocline, as a result of hydrodynamic or magnetic instabilities, is non-axisymmetric.

To examine the effect of non-axisymmetric flow on the solar interface dynamo, consider a three-dimensional flow in the tachocline

$$\frac{\partial \boldsymbol{B}_t}{\partial t} = \nabla\times[\boldsymbol{u}(r,\phi,\theta)\times\boldsymbol{B}_t] - \lambda_t\nabla\times\nabla\times\boldsymbol{B}_t\,, \tag{94}$$

$$\nabla\cdot\boldsymbol{B}_t = 0\,. \tag{95}$$

In the fully turbulent convection zone in the region $r_t < r < r_o$, a weak magnetic field $\boldsymbol{B}_o$ is generated by an α-effect with eddy magnetic diffusivity λ_o, where $\lambda_t/\lambda_o \ll 1$. A nonlinear $\alpha-$dynamo in the convection zone is described by

$$\frac{\partial \boldsymbol{B}_o}{\partial t} = \alpha_o\nabla\times\left\{\sin^2\theta\cos\theta\sin\left[\pi\frac{(r-r_t)}{(r_o-r_t)}\right]\frac{1}{(1+|\boldsymbol{B}_o/B_{\rm eq}|^2)}\right\} + \lambda_o\nabla^2\boldsymbol{B}_o\,, \tag{96}$$

$$\nabla \cdot \boldsymbol{B}_o = 0\,, \tag{97}$$

where α_o is a positive parameter. The weak effect of the radial-independent differential rotation in the convection zone is neglected in this model. The outer exterior to the convection zone is assumed to be almost electrically insulating with magnetic diffusivity λ_e. Its magnetic field $\boldsymbol{B}_e$ is then governed by

$$\frac{\partial \boldsymbol{B}_e}{\partial t} + \lambda_e \nabla \times (\nabla \times \boldsymbol{B}_e) = 0\,, \tag{98}$$

$$\nabla \cdot \boldsymbol{B}_e = 0\,. \tag{99}$$

By taking a sufficiently large magnetic diffusivity λ_e such that

$$\frac{\lambda_e}{\lambda_o} \gg 1,$$

the magnetic field $\boldsymbol{B}_e$ in the exterior represents an approximate potential field in the vacuum which is also part of the numerical dynamo solution.

There are five non-dimensional quantities that numerically characterize the interface dynamo: the magnetic diffusivity ratios β_i, β_t, β_m, the magnetic alpha Reynolds number R_α and the magnetic omega Reynolds number R_m, respectively, defined by

$$\beta_i = \frac{\lambda_i}{\lambda_{\;o}}, \quad \beta_t = \frac{\lambda_t}{\lambda_{\;o}}, \quad \beta_m = \frac{\lambda_e}{\lambda_{\;o}},$$

$$R_\alpha = \frac{(r_o - r_i)\alpha_0}{\lambda_o}, \quad R_m = \frac{(r_o - r_i)^2 \Omega_0}{\lambda_o}.$$

The governing equations are solved subject to a number of matching and boundary conditions at the interfaces. That all components of the magnetic field and the tangential component of the electrical field are continuous at the three interfaces of the four zones, $r = r_i, r_t$ and r_0, yields

$$\begin{aligned}
(\boldsymbol{B}_i - \boldsymbol{B}_t) &= 0, \quad \text{at } r = r_i;\\
\boldsymbol{r} \times [\beta_i \nabla \times \boldsymbol{B}_i + R_m\,(\boldsymbol{u} \times \boldsymbol{B}_t) - \beta_t \nabla \times \boldsymbol{B}_t] &= 0, \quad \text{at } r = r_i;\\
(\boldsymbol{B}_t - \boldsymbol{B}_o) &= 0, \quad \text{at } r = r_t;\\
\boldsymbol{r} \times [R_m\,(\boldsymbol{u} \times \boldsymbol{B}_t) - \beta_t \nabla \times \boldsymbol{B}_t - R_\alpha \alpha \boldsymbol{B}_o + \nabla \times \boldsymbol{B}_o] &= 0, \quad \text{at } r = r_t;\\
(\boldsymbol{B}_e - \boldsymbol{B}_o) &= 0, \quad \text{at } r = r_o;\\
\boldsymbol{r} \times (\beta_e \nabla \times \boldsymbol{B}_e + R_\alpha \alpha \boldsymbol{B}_o - \nabla \times \boldsymbol{B}_o) &= 0, \quad \text{at } r = r_o\,.
\end{aligned} \tag{100}$$

For the boundary condition at the outer bounding surface of the dynamo solution domain, $r = r_m$, an appropriate approximation must be made. Since there are no sources at infinity, i.e.,

$$\boldsymbol{B}_e = O(r^{-3}), \quad \text{as } r \to \infty, \tag{101}$$

we may approximate the magnetic field boundary condition at $r = r_m$ by

$$\boldsymbol{B}_e = 0, \quad \text{at } r = r_m \text{ with } (r_m/r_o)^3 \gg 1\,. \tag{102}$$

Equations (92)–(99) together with the matching and boundary conditions (100) and (102) define a nonlinear spherical interface dynamo problem. For given parameters of the model such as R_α

and R_m, numerical solutions of the nonlinear dynamo are sought using a fully three-dimensional finite element method.

Figure 2 (see Plate I) displays the butterfly diagram of a simulated interface dynamo solution, showing contours of the azimuthal magnetic field at the interface r_t plotted against time, for $R_m = 200, R_\alpha = 30$ with $\beta_i = \beta_t = 0.1$. The results are very robust in the sense that we have performed many more simulations in various parameter regimes, for example, smaller values of β_i, and we have always found qualitatively the same features. The effect of the tachocline produces a nonlinear dynamo wave with a period of about 20 years. The interface dynamo solution is non-axisymmetric, shows dipolar symmetry and propagates equatorward. Moreover, the generated magnetic field mainly concentrates in the vicinity of the interface between the tachocline and the convection zone. More significantly, the strength of the toroidal magnetic field is dramatically amplified by the strong radial shear in the tachocline, reaching a maximum strength about 10^5 G. Such strong toroidal magnetic fields in the tachocline would be susceptible to magnetic buoyancy instabilities leading to a quick eruption of the field into the surface of the Sun in the form of sunspots.

7 CONCLUDING REMARKS

The spatial and temporal regularities shown in the variation of the Sun's magnetic field and evidence for the existence of the solar tachocline lead to a coherent picture of the solar interface dynamo first proposed by Parker (1993). The central idea is that a weak poloidal field generated in the turbulent convection zone can be dramatically amplified by the radial shear in the tachocline via the omega-effect. Recent observations, such as the Michelson Doppler Imager on the SOHO spacecraft and the Global Network Group project, provide further challenging issues in the solar dynamo theory (Howe et al. 2000).

While the kinematic theory of the solar dynamo is now well developed, we are just beginning to understand the dynamic problem. There are many important issues in the solar magnetic behaviors which require a dynamical dynamo theory to explain. Tobias (1997) showed that a simple $\alpha\omega$ Cartesian interface dynamo, which includes the dynamic feedback of the generated magnetic field, is capable of producing the modulation of the basic magnetic cycle and recurrent grand minima (see also Brandenburg et al. 1989). The existence of the solar torsional oscillations with about a period of 11 yr was suggested by the helioseismic data (Schou et al. 1998). It remains unclear whether the oscillating motions penetrate all the way into the tachocline and whether the fashion of the oscillation changes at different depth of the Sun (Howe et al. 2000). How the Lorentz forces affect flows in the convection zone would be a primary key in the understanding of the solar torsional oscillation.

Modelling and predicting the solar magnetic activities are obviously enormously important for all mankind. Further investigation of the solar dynamo problem would heavily reply on the modern computational power of massively parallel computers. The fully dynamical solar dynamo problem involving the solution of the Navier-Stokes Equation (34) together with the dynamo Equation (26) and other equations is too complicated to be studied analytically. Many fundamental questions about the Sun, such as why the Sun has a tachocline, why the Sun's radiative core rotates rigidly and why the differential rotation in the convection zone is nearly independent of r, can be answered only by constructing realistic dynamical solar dynamos together with modern observations aided by deep analytical understandings of the problem.

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References

Backus G. E., 1958, Ann. Phys. (NY), 4, 372
Braginsky S. I., Sov. Phys. JETP, 20, 726
Brandenburg A., Krause F. et al., 1989, A&A, 213, 411
Brandenburg A., 1994, In: M. R. E Proctor, A. D. Gilbert, eds., Lectures on Solar and Planetary Dynamos, England: Cambridge University Press
Bullard E. C., Gellman H., 1954, Phil. Trans. R. Soc. London, Ser. A, 247, 213
Busse F. H., 1970, J. Fluid Mech., 44, 441
Cally P. S., 2001, Solar Phys., 199, 231
Charbonneau P., MacGregor K. B., 1997, ApJ, 486, 502
Choudhuri A. R., Gilman P. A., 1987, ApJ, 316, 788
Choudhuri A. R., Schüssler M., Dikpati M., 1995, A&A, 303, L29
Cowling T. G., 1934, MNRAS, 140, 39
Dikpati M., Charbonneau P., 1999, ApJ, 518, 508
Dikpati M., Gilman P., 2001, ApJ, 559, 428
D'Silva S., Choudhuri A. R., 1993, A&A, 272, 621
Howe R., Christesen-Dalsgaard J., Hill F., Komm R. W., Larsen R. W., Schou J., Thompson M. J., Toomre J., 2000, Science, 287, 2456
Gilman P. A., Fox P. A., 1997, ApJ, 484, 439
Glatzmaier G. A., 1985, ApJ, 192, 300
Gough D. O., Leibacher J. W., Scherrer P. H., Toomre J., 1996, Science, 272, 1281
Gough D. O., McIntyre M. E., 1998, Nature, 394, 755
Kosovichev A. G., 1996, ApJ, 469, L61
Küker M., Rüdifer G., Schultz M., 2001, A&A, 374, 301
Larmor J., 1919, Brit. Assn. Adv. Sci. Rep., 159
Markiel J. A., Thomas J. H., 1999, ApJ, 523, 827
MacGregor K. B., Charbonneau P., 1997, ApJ, 486, 484
Maunder E. W., 1913, MNRAS, 74, 112
Miesch S. M., 2001, ApJ, 562, 1058
Moffatt H. K., 1978, Magnetic Field Generation in Electrically Conducting Fluids, Cambridge, England: Cambridge University Press
Roberts P. H., 1972, Phil. Trans. R. Soc. London, Ser. A, 272, 663
Roberts P. H., Soward A. M., 1992, Annu. Rev. Fluid Mech., 24, 459
Parker E. N., 1955, ApJ, 122, 293
Parker E. N., 1979, Cosmical Magnetic Fields: their origin and activity, Clarendon Press
Parker E. N., 1993, ApJ, 408, 707
Parker E. N., 2001, Chin. J. Astron. Astrophys. (ChJAA), 1(2), 99
Ponty Y., Gilbert A. D., Soward A. M., 2001, J. Fluid Mech., 435, 261
Schou J., 1991, In: D. Gough, R. Toomre, eds., Proc. Conf. at Inst. Theoret. Physics (Santa Barbra, CA), Berlin: Springer, 81
Schou J., Antia H. M., Basu S. et al., 1998, ApJ, 505, 390
Schubert G., Zhang K., 2000, ApJ, 532, L149
Schubert G., Zhang K., 2001, ApJ, 557, 930
Spiegel E. A., Zahn J. P., 1992, A&A, 265, 106

Steenbeck M., Krause F., 1966, Z. Naturforsch., 21a, 1285
Tobias S. M., Weiss N. O., Kirk V., 1995, MNRAS, 273, 1150
Tobias S. M., 1997, A&A, 322, 1007
Weiss G., 1994, In: MRE Proctor, AD Gilbert, eds., Lectures on Solar and Planetary Dynamos, pp. 219–244, Cambridge, England: Cambridge University Press
Zhang K., 1992, J. Fluid Mech., 236, 535
Zhang K., 1995, Proc R. Soc. Lond. Ser A, 448, 245
Zhang K., Busse F., 1989, Geophys. Astrophys. Fluid Dyn., 49, 97
Zhang K., Busse F., 1995, J. Fluid Mech., 304, 363
Zhang K., Schubert G., 1996, Science, 273, 941
Zhang K., Schubert G., 2000, Ann. Rev. Fluid Mech, 2000, 32, 411
Zhang K., Schubert G., 2001, Science, 290, 1944
Zhang K., Schubert G., 2002, ApJ, 572, 461
Zhang K., Liao X., Schubert G., 2003, ApJ (in press)
Zhang K., Chan K., Zou J., Liao X., Schubert G., 2003, (submitted)

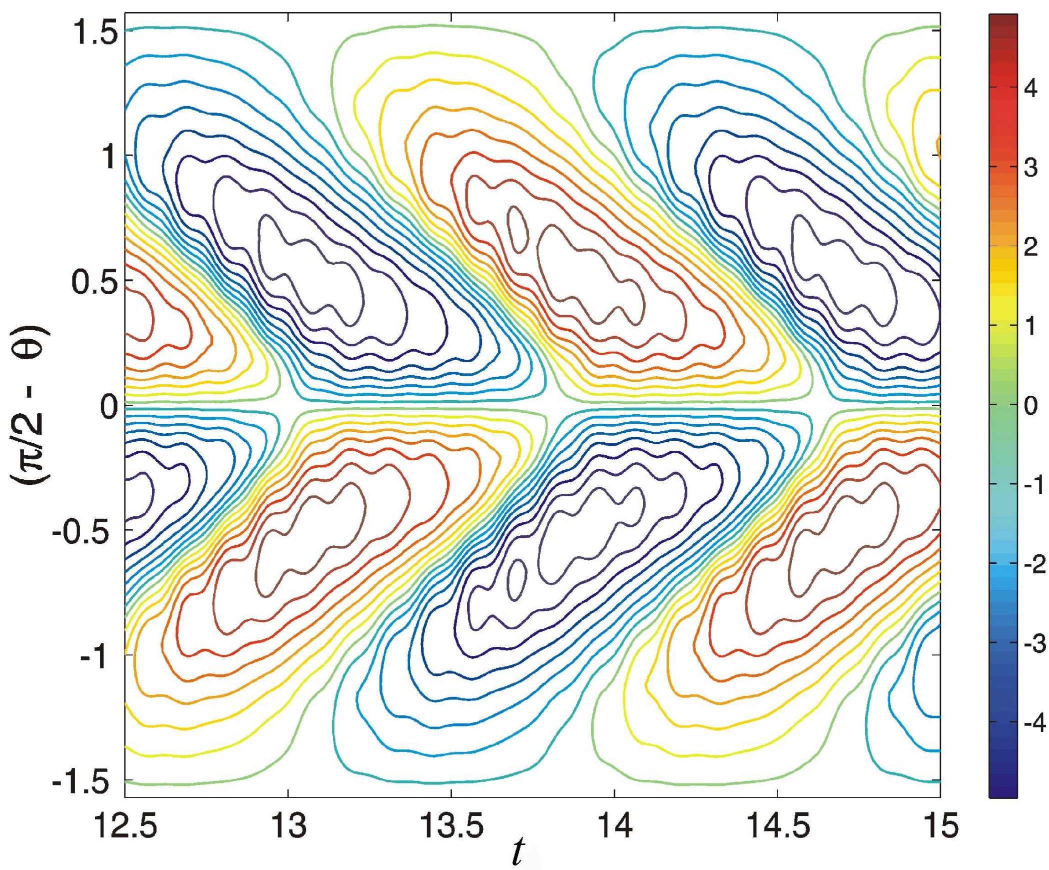

Fig. 2 A non-axisymmetric interface dynamo with $R_m = 200, R_\alpha = 30$, showing a butterfly diagram for the dynamo solution at $\phi = 0$ with B_ϕ evaluated at the interface between the tachocline and convection zone.