Charged stellar model with three layers

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Abstract We establish new charged stellar models from the Einstein-Maxwell field equations for relativistic superdense objects outfitted with three layers. The core layer is described by a linear equation of state describing quark matter, the intermediate layer described by Bose-Einstein condensate equation of state for Bose-Einstein condensate matter and the envelope layers satisfying quadratic EoS for the neutron fluid. We have specified a new choice of the electric field and one of the metric potentials. It is interesting to note that the choice of electric field in this model can be set to vanish and we can regain earlier neutral models. Plots generated depict that the matter variables, gravitational potentials, and other physical conditions are consistent with astrophysical studies. The interior layers and exterior boundary are also matched.

Key words: charged anisotropic — stars: equations of state: layered — interior

1 INTRODUCTION

Formulation of stellar models in a core-envelope setting has become an area of interest to many researchers in astrophysics and other related subjects. The physical phenomena in such models are still complicated in the solutions that have been generated. This is due to the reason that the interior structure of the stellar body has several layers with distinct physical properties. It is a challenging question for researchers to identify the nature of the layers possessed by superdense stellar objects. The compact bodies with such properties have strong magnetic and gravitational fields. These include neutron stars, quark stars, gravastars, pulsars, white dwarfs, and quasars. It has been noted that superdense stellar objects possess high pressures and extremely density leading to the variation of physical properties. It has been shown that the overall masses and density of superdense objects are approximately $1.0 - 2.0 M_\odot$ and $10^{15} \text{g cm}^{-3}$ respectively (Pant et al. 2020; Bisht et al. 2021). According to Jasim et al. (2018), the extreme superdense stellar bodies may alter the spacetime geometry while the radii and masses remain between $11.0 - 15.0$ km and $1.4 - 2.0 M_\odot$ respectively.

The concept of the stellar object with a core and an envelope is embedded in the general theory of relativity discovered by Albert Einstein between 1907 and 1916. The theory has been applied to describe the structure of stellar bodies and understanding of the universe at large (Misner et al. 1973). In addition to Einstein’s theory, various theories of gravitation like vector-tensor theory, and scalar-tensor theory were developed with some modifications of Einstein’s theory. These theories may be used to formulate relativistic stellar models governed by a set of differential equations (Collins 2003). Stellar models may also be built with corrections from the Ricci and Riemman tensors in modified gravity.
The Einstein field equations are the most basic tools in the general theory of relativity. These equations depict the relationship between metric potentials, pressures, and energy density in the form of differential equations (Stephani et al. 2003). The field equations are purely geometric. It is possible to solve the field equations if some reasonable physical assumptions are imposed. Importantly, the field equations are used to investigate different structures, behaviours, and properties of stellar spheres relevant to astrophysical studies. They have drawn the attention and interest of researchers to develop relativistic stellar models in static or non-static spherically symmetric spacetime (Fulara & Sah 2018). A variety of mathematical approaches have been used to solve the system of the field equations to investigate the physical features of superdense stellar objects (Tiwari 2010).

The electric field introduced in the Einstein-Maxwell field equations has to capture the reality of physical properties that charged stellar objects in the universe should possess. Stellar spheres should loose electric charge in time to become uncharged. A realistic physical model for charged spheres should therefore have vanishing electric field in some limit. The presence of an electric field in a stellar object creates repulsion with the pressure gradient to prevent gravitational collapse. This phenomenon makes the object stable and works against collapse (Jasim et al. 2018). The effect of electric field on the critical mass, stability, and the redshift in single-layered models has been shown in the papers by Dev & Gleiser (2002); Ivanov (2002); Chaisi & Maharaj (2005); Chaisi & Maharaj (2006); Sunzu et al. (2014). The effect of the electric field on physical properties in two layered models have been discussed in the work developed by Mafa Takisa & Maharaj (2016). It is important to generate a model of three layers with electric field present for a better understanding of the model.

The importance of pressure anisotropy has been shown in different stellar models. According to Malaver (2018), the presence of pressure anisotropy in the interior of the stellar object may be due to the existence of a solid core, a particular phase transition, and electric charge. The detailed analysis of pressure anisotropy has also been highlighted in the studies performed by Bower & Liang (1974); Bijalwan (2011); Cosenza et al. (1981); Sokolov (1980); Uslov (1994); Komathiraj & Maharaj (2007); Ivanov (2010); Maharaj et al. (2014); Maurya et al. (2018) and Sunzu et al. (2019).

Models with two layers describing a core layer and the corresponding envelope layer for superdense stellar objects have been formulated. This has been illustrated in the paper by Sharma & Mukherjee (2002) who formulated a core-envelope model with an inner layer containing quark matter and the envelope layer which is less compact. The model which describes a parabolic density profile, developed in Negi et al. (1990), indicates the continuity of all variables at the interface between the core and the envelope. The paper developed by Thomas et al. (2005) describes superdense matter configuration which is characterized by an isotropic fluid for the core and the anisotropic fluid at the envelope layer. Other models with two layers include the works of Mafa Takisa & Maharaj (2016); Mafa Takisa et al. (2019); Pant et al. (2019); Gedela et al. (2019); Hansraj et al. (2016); Paul & Tikekar (2005) and Tikekar & Jotania (2009). It is also important to consider the physics of three-layered model with both electric field and anisotropy present.

Models with three interior layers are rarely found in general relativity. The recent models studied by Bisht et al. (2021) and Pant et al. (2020) indicate that there exists a sublayer between the core-envelope boundary. The models analysed the physical properties and features of the neutron star as the dense stellar object. However, the models generated was uncharged with the core obeying linear equation of state (EoS), and intermediate and envelope layers satisfying quadratic EoS. In the present model we include the electric field which is missing in Bisht et al. (2021) and Pant et al. (2020).

The motivation of this paper is to generate a new stellar model comprising three regions. We apply a separate equation of state in each region depending on the nature of the material it contains. It is reasonable to have a stellar model that incorporates three interior layers with distinct equations of state. The core layer is described by a linear equation of state sufficient to describe quark matter, the intermediate and envelope layers both satisfy different quadratic equations of state. In our model, we choose one of the gravitational potentials and the electric fields. We generate a new anisotropic charged model containing three regions which generalizes the neutral models developed by Bisht et al. (2021) and Pant et al. (2020).
2 BASIC MODEL EQUATIONS

We describe the interior with a static spherically symmetric spacetime with metric given in Schwarzschild coordinates \((x^i) = (t, r, \theta, \phi)\) as

\[
ds^2 = -e^{2\nu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

(1)

where \(\nu(r)\) and \(\lambda(r)\) defines the metric potentials. The exterior line element is given by the Reissner-Nordstrom spacetime as

\[
ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

(2)

where \(Q\) stands for the total charge and \(M\) represents the total mass of the sphere. For charged stellar objects, the energy momentum tensor is written as

\[
T_{ij} = \text{diag}\left(-\rho - \frac{1}{2} E^2, \ p_r - \frac{1}{2} E^2, \ p_t + \frac{1}{2} E^2, \ p_t + \frac{1}{2} E^2\right),
\]

(3)

where \(\rho\) defines the energy density, \(E\) is the electric field, \(p_r\) and \(p_t\) are radial and tangential pressures respectively. Setting \(G = c = 1\), the system of field equations become

\[
8\pi \rho = \frac{1}{r^2} (1 - e^{-2\lambda}) + \frac{2\lambda'}{r} e^{-2\lambda},
\]

(4a)

\[
8\pi p_r = \frac{1}{r^2} (1 - e^{-2\lambda}) + \frac{2\nu'}{r} e^{-2\lambda},
\]

(4b)

\[
8\pi p_t = e^{-2\lambda} \left(\nu'' + \nu'^2 - \nu' \lambda' + \frac{\nu' - \lambda'}{r} \right) + \frac{2\lambda'}{r} e^{-2\lambda},
\]

(4c)

\[
\sigma = \frac{1}{r^2} e^{-\lambda} (r^2 E')',
\]

(4d)

where primes (\('\)) represent derivatives with respect to radial coordinate (\(r\)).

We introduce the transformation variables from Durgapal & Bannerji (1983) given by

\[
x = r^2, \quad Z(x) = e^{-2\lambda}, \quad e^{2\nu} = A^2 y^2(x).
\]

(5)

Using (5), the system (4) becomes

\[
8\pi \rho = \frac{1}{x} \left(1 - Z\right) - 2 \frac{dZ}{dx} \frac{1}{2} E^2,
\]

(6a)

\[
8\pi p_r = -\frac{1}{x} (1 - Z) + 4Z \frac{1}{y} \frac{dy}{dx} + \frac{1}{2} E^2,
\]

(6b)

\[
8\pi p_t = 4xZ \frac{1}{y} \frac{d^2 y}{dx^2} + \left(4Z + 2x \frac{dZ}{dx}\right) \frac{1}{y} \frac{dy}{dx} + \frac{dZ}{dx} = \frac{1}{2} E^2,
\]

(6c)

\[
\Delta = 8\pi p_t - 8\pi p_r
\]

\[
= 4xZ \frac{1}{y} \frac{d^2 y}{dx^2} + \left(4xZ + 2x \frac{dZ}{dx} - 4Z\right) \frac{1}{y} \frac{dy}{dx} + \frac{dZ}{dx} + \frac{1}{x} (1 - Z) - E^2,
\]

(6d)

\[
\sigma = 2 \left(\frac{dE}{dx} + E\right) \sqrt{\frac{Z}{x}}.
\]

(6e)

The system (6) has eight unknown variables. To find a solution of the system (6), we need to specify any two parameters so as to find expressions for the others. In our model we choose to specify metric
potential $Z$ and electric field intensity $E^2$. We take the metric potential $Z$ used in the study of Pant et al. (2020). This is expressed in the form

$$Z(x) = e^{-2A} = 1 - hx + gx^2,$$

where $h$ and $g$ are the arbitrary real constants. The metric function $Z$ in equation (7) is real, continuous and regular at the centre which allow an extensive range of values of $h$ and $g$. It is in acceptable condition to avoid the singularity at the centre and within the stellar sphere. The system (6) is possible to solve once electric field is specified. The particular choice of electric field is specified in the form

$$E^2 = m x Z = m x (1 - hx + gx^2), \quad m \geq 0$$

where $m$ stands for arbitrary real constant. The choice of $E^2$ given in equation (8) vanishes at the centre ($x = 0$) of the stellar object, remains real and continuous throughout the interior points of the stellar body. It is physically reasonable and thus, consistent for the study of the charged stellar models. Substituting $m = 0$ in equation (8), we regain an uncharged model in Pant et al. (2020).

3 LAYERS OF THE STELLAR OBJECT

We have three interior regions namely; the core ($\eta$), the intermediate ($\iota$) and the envelope ($\epsilon$) as follows:

- Core layer (Region 1); $0 \leq r \leq R_\eta$,
- Intermediate layer (Region 2); $R_\eta \leq r \leq R_\iota$, and
- Envelope layer (Region 3); $R_\iota \leq r \leq R_\epsilon$.

For the classified boundaries, line element (1) become

$$ds^2|_1 = -e^{2\nu_\eta}dt^2 + e^{2\lambda_\eta}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

$$ds^2|_2 = -e^{2\nu_\iota}dt^2 + e^{2\lambda_\iota}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

$$ds^2|_3 = -e^{2\nu_\epsilon}dt^2 + e^{2\lambda_\epsilon}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2),$$

in the three regions.

3.1 The Core

We present the core layer to be quark matter satisfying the linear equation of state in the form

$$p_{r_\eta} = \alpha \rho_{r_\eta} - \beta,$$

where $\alpha$ and $\beta$ are arbitrary real constants. Combining (6a) and (10) yields

$$p_{r_\eta} = \alpha \left( \frac{1 - Z}{8\pi x} - \frac{1}{4\pi} \frac{dZ}{dx} - \frac{1}{16\pi} E^2 \right) - \beta.$$  

Equating (6b) and (11) gives

$$\frac{1}{y} \frac{dy}{dx} = (\alpha + 1) \left( \frac{1 - Z}{4xZ} \right) - \frac{\alpha}{2Z} \frac{dZ}{dx} - \frac{E^2(\alpha - 1)}{8Z} - 2\pi \frac{\beta}{Z}.$$  

Substituting (7) and (8) into (12) yields

$$\frac{1}{y} \frac{dy}{dx} = [h(2 - mx^2(\alpha - 1) - 6\alpha)mx(\alpha - 1)gx(-2 + mx^2(\alpha - 1) + 10\alpha) + 16\pi \beta] [8x(h - gx) - 8]^{-1}.$$  

(13)
Integrating (13) gives

\[ y = \exp \left[ \frac{1}{16} \left[ -m x^2 (\alpha - 1) + 4(h(\alpha - 1)) - 16\pi \beta \arctan \left( 2g x - h \right) \right] \right. \]
\[ \times \left[ \sqrt{4g - h^2} \right]^{-1} + \frac{1}{16} \left( 2(1 - 5\alpha) \log(1 - hx + gx^2) \right) \left( \sqrt{4g - h^2} \right)^{-1} + k_0 \] .

Putting (7), (8), (13) and (14) into the system (6), we obtain

\[ e^{2\lambda_0} = [1 - hx + gx^2]^{-1}, \]
\[ e^{2\nu_0} = A^2[\exp(Q_0(x))]^2, \]
\[ \rho_q = \left[ h(6 + mx^2 - x(m + g(10 + mx^2))) \right][16\pi]^{-1}, \]
\[ p_{r_q} = \alpha[h(6 + mx^2 - x(m + g(10 + mx^2)))][16\pi]^{-1} - \beta, \]
\[ p_{\nu_q} = Q_1(x) + Q_2(x) + Q_3(x) + Q_4(x) + Q_5(x) + Q_6(x), \]
\[ \Delta_\eta = p_{r_q} - p_{\nu_q}, \]
\[ \sigma_\eta = 2(mx(1 - hx + gx^2) + \frac{1}{2}(3 + [hx - 2][1 - hx + gx^2]^{-1}))\sqrt{[1 - hx + gx^2]^{-1}}, \]
\[ E_\eta^2 = mx(1 - hx + gx^2), \]

where for simplicit, we have set

\[ Q_0(x) = \left[ \frac{1}{16} \left[ -m x^2 (\alpha - 1) + 4(h(\alpha - 1)) - 16\pi \beta \arctan \left( 2g x - h \right) \right] \right. \]
\[ \times \left[ \sqrt{4g - h^2} \right]^{-1} + \frac{1}{16} \left( 2(1 - 5\alpha) \log(1 - hx + gx^2) \right) \left( \sqrt{4g - h^2} \right)^{-1} + k_0, \]
\[ Q_1(x) = [h^2 x(-m^2 x^4(\alpha - 1)^2 - 4mx^2(4 + 3(3\alpha - 4))) - 64(gx - 2\pi)] [128\pi(x(h - gx) - 1)]^{-1}, \]
\[ Q_2(x) = [2h(16 - 24\alpha + x(mx(12 + mx^2(\alpha - 1)^2 - 26\alpha + 6\alpha^2)))][128\pi(x(h - gx) - 1)]^{-1}, \]
\[ Q_3(x) = [2hgx^2(40 + mx^2(\alpha - 1)^2 - 92\alpha + 60\alpha^2 + 2mx^2(9 + \alpha(8\alpha - 21)))] \]
\[ \times[128\pi(x(h - gx) - 1)]^{-1}, \]
\[ Q_4(x) = [32h\pi(-4 + mx^2(\alpha - 1) + 6\alpha)\beta + 64g\pi x^2(1 - 5\alpha)\beta - 256\pi^2 \beta^2 x][128\pi(x(h - gx) - 1)]^{-1}, \]
\[ Q_5(x) = [x(-m^2 x^2(1 + gx^2)^2(\alpha - 1)^2 + 4g(40\alpha + gx^2(-13 + 5(6 - 5\alpha)\alpha)))] [128\pi(x(h - gx) - 1)]^{-1}, \]
\[ Q_6(x) = 4mx(1 + gx^2)(2 - 4\alpha + x(gx(5 + \alpha(5\alpha - 12)) + 8\pi(\alpha - 1)\beta)), \]

and \( k_0 \) stands for a constant of integration. The total mass of the stellar object for the core is given in the form

\[ M_\eta(r) = 4\pi \int r^2 \rho_\eta dr, \quad r^2 = x \]
\[ = \left[ x^2(h(6 + mx^2) - x(m + g(10 + mx^2))) \right][96\pi]^{-1}. \]

### 3.2 The Intermediate Region

We consider the middle layer to be less dense than the core. A modified Bose-Einstein condensate equation of state is seem to satisfy the intermediate layer characterised by Bose-Einstein condensate
matter to describe the physical features of the dense stellar object. Thus we describe this region by the equation

\[ p_r = \mu \rho^2 - a, \]  

(17)

where \( \mu \) and \( a \) are arbitrary real constants. Substituting (6a) into (17) we have

\[ p_r = \mu \left( \frac{1 - Z}{8\pi x} - \frac{1}{4\pi} \frac{dZ}{dx} - \frac{1}{16\pi} E^2 \right)^2 - a. \]  

(18)

From (6b) and (18) we get

\[ \frac{1}{y} \frac{dy}{dx} = \frac{2\pi \mu}{Z} \left( \frac{1 - Z}{8\pi x} - \frac{1}{4\pi} \frac{dZ}{dx} - \frac{1}{16\pi} E^2 \right)^2 - \frac{2\pi}{Z} \left( z - \frac{1}{8\pi x} \right) \]  

(19)

Putting (7) and (8) into (19) we get

\[ \frac{1}{y} \frac{dy}{dx} = \left[ -16\pi (16a\pi + h(2 - mx^2) + x(m - 2g + gm^2)) + (h(6 + mx^2) - x(m + g(10 + mx^2)))^2 \mu \right] \left[ \frac{128\pi (1 - hx + gx^2)}{128} \right]^{-1}. \]  

(20)

Integrating equation (20) gives

\[ y = \exp \left[ \frac{1}{3} m(20g + m)x^3 \mu - \frac{1}{4} hm^2 x^4 \mu + \frac{1}{5} gm^2 x^5 \mu - 2mx^2 (4\pi + 3h\mu) + 2(8\pi - 5h\mu) \log(1 - hx + gx^2) [128\pi]^{-1} + \left( -8h\pi - 128a^2 - 50g\mu \right) \right] \]  

(21)

Substituting (7), (8), (20) and (21) into the system (6) gives the matter variables

\[ e^{2\lambda} = [1 - hx + gx^2]^{-1}, \]  

(22a)

\[ e^{2\nu} = A^2 y^2(x) = A^2 [\exp(\tau_0(x))]^2, \]  

(22b)

\[ \rho = [h(6 + mx^2 - x(m + g(10 + mx^2)))] [16\pi]^{-1}, \]  

(22c)

\[ p_r = \mu \left( [h(6 + mx^2 - x(m + g(10 + mx^2)))] [16\pi]^{-1} \right)^2 - a, \]  

(22d)

\[ p_t = \tau_1(x) + \tau_2(x) + \tau_3(x) + \tau_4(x) + \tau_5(x), \]  

(22e)

\[ \Delta = p_t - p_r, \]  

(22f)

\[ \sigma = 2 \left( mx(1 - hx + gx^2) + \frac{1}{2} (3 + [hx - 2] [1 - hx + gx^2]^{-1}) \right) \sqrt{[1 - hx + gx^2] x^{-1}} \]  

(22g)

\[ E^2 = mx(1 - hx + gx^2), \]  

(22h)
where for simplicity we have set

\[
\tau_0(x) = \left[100g_0x + \frac{1}{3}m(20g + m)x^3 + \frac{1}{4}gm^2x^4 + \frac{1}{5}gm^2x^5 - 2mx^2(4\pi + 3h)\mu + 2(8\pi - 5h\mu)\log(1 - hx + gx^2)[(128\pi)^{-1} + 4(-8h\pi - 128a\pi^2 - 50gh) + 13h^2]\arctan\left(\sqrt{4g - h^2}\right)\right]^{-1} + k_2,
\]

\[
\tau_1(x) = \left[-4096(h - 2gx) - 2048m(1 - hx + gx^2)[(32768\pi)^{-1} + (6 + mx^2, 2 + mx^2 + x(2 - 2g + gx^2)]\right] x[(32768\pi)^{-1} (1 - hx + gx^2)^{-1},
\]

\[
\tau_2(x) = \left[32\pi(6 + 6hx + 8gx^2)(-16\pi(16a\pi + h(2 - mx^2) + x(m - 2g + mx^2))\right]
\times[(32768\pi)^{-1}(1 - hx + gx^2)^{-1},
\]

\[
\tau_3(x) = \left[x(-16\pi(16a\pi + h(2 - mx^2) + x(6 + mx^2)) + x(m + g(10 + mx^2))^{2}\mu\right]
\times[(32768\pi)^{-1}(1 - hx + gx^2)^{-1},
\]

\[
\tau_4(x) = \left[x(h - 2gx)(-16\pi(16a\pi + h(2 - mx^2) + x(6 + mx^2))^{2}\mu\right]
\times[(32768\pi)^{-1}(1 - hx + gx^2)^{-1} - x(m + g(10 + mx^2))]^{2}[512\pi^{2}(1 - hx + gx^2)^{-1},
\]

\[
\tau_5(x) = \left[2x(5m - 2hx + g(10 + 3mx^2))(-h(6 + mx^2) + x(m + g(10 + mx^2))µ\right]
\times[(32768\pi)^{-1} + x(-16\pi(5m - 2hx + g(3mx^2 - 2)))][32768\pi^{-1}]^{-1}.
\]

The total mass of the stellar object for intermediate layer is given by

\[
M_i(x) = 4\pi \int r^2 \rho \, dr, \quad r^2 = x
\]

\[
M_i = \int [x^2(h(6 + mx^2) - x(m + g(10 + mx^2)))] [96\pi]^{-1}.
\]

### 3.3 The Envelope

With consideration of the density profile, the extended envelope region may have less dense material than the core and the intermediate layers. We consider a general quadratic equation of state to be convenient to describe the outermost layer influenced by neutron fluid and Coulomb liquids. The quadratic EoS is taken to describe the envelope region so that to ensure the radially addressed pressure in the outer layer is less compact than the pressure in the inner layer and the intermediate layer [Mafa Takisa et al. (2019)]. This is written in the form

\[
p_{re} = b \rho^2 + d \rho - f,
\]

where \(b, d\) and \(f\) are the arbitrary real constants. From (6a) and (24) we have

\[
p_{re} = b \left(\frac{1 - Z}{8\pi x} - \frac{1}{4\pi} \frac{dZ}{dx} - \frac{1}{16\pi} E^2\right)^2 + d \left(\frac{1 - Z}{8\pi x} - \frac{1}{4\pi} \frac{dZ}{dx} - \frac{1}{16\pi} E^2\right) - f.
\]

Equating (6b) and (25) yields

\[
\frac{1}{\gamma} \frac{dy}{dx} = \frac{2\pi b}{Z} \left(\frac{1 - Z}{8\pi x} - \frac{1}{4\pi} \frac{dZ}{dx} - \frac{1}{16\pi} E^2\right)^2 + \frac{2\pi d}{Z} \left(\frac{1 - Z}{8\pi x} - \frac{1}{4\pi} \frac{dZ}{dx} - \frac{1}{16\pi} E^2\right) - \frac{2\pi}{Z} \left(f + \frac{Z - 1}{8\pi x} + \frac{1}{16\pi} E^2\right).
\]

\[
\frac{1}{\gamma} \frac{dy}{dx} = \frac{2\pi b}{Z} \left(\frac{1 - Z}{8\pi x} - \frac{1}{4\pi} \frac{dZ}{dx} - \frac{1}{16\pi} E^2\right)^2 + \frac{2\pi d}{Z} \left(\frac{1 - Z}{8\pi x} - \frac{1}{4\pi} \frac{dZ}{dx} - \frac{1}{16\pi} E^2\right) - \frac{2\pi}{Z} \left(f + \frac{Z - 1}{8\pi x} + \frac{1}{16\pi} E^2\right).
\]
Putting (7) and (8) into (26) the differential equation becomes

\[
\frac{1}{y} \frac{dy}{dx} = [16f \pi + (2g + 10d) + m + dm]x + (1 + d)gm^3 - h(2 + 6d + mx^2 + dm^2)] \\
\times [8(x(h - gx) - 1)]^{-1} - [b(h(6 + mx^2) - x(10g + m + gm)^2)] \\
\times [128\pi(x(h - gx) - 1)]^{-1}.
\] (27a)

Integrating (27a) yields

\[
y = \exp \left[\frac{100bgx - 2m(3bh + 4\pi + 4\pi d)x^2 + \frac{1}{3}bmn(20g + m)x^3 - \frac{1}{4}bhm^2x^4 + \frac{1}{5}bgm^2x^5}{128\pi \sqrt{4g - h^2}}\right] - \frac{1}{4}\pi bh
\]

Then the matter variables for the envelope layer become

\[
e^{2\lambda_e} = [1 - hx + gx^2]^{-1},
\]

\[
e^{2\nu_e} = A^2y^2(x) = A^2[\exp(G_0(x))]^2,
\]

\[
\rho_e = [h(6 + mx^2 - x(m + g(10 + mx^2)))]^{-1},
\]

\[
p_{\nu_e} = b \left([h(6 + mx^2 - x(m + g(10 + mx^2)))]^{-1} \right)^2 + d \left([h(6 + mx^2 - x(m + g(10 + mx^2)))]^{-1} \right)^2
\]

\[
\Delta_e = p_{\nu_e} - p_{\nu_e},
\]

\[
\sigma_e = 2(mx(1 - hx + gx^2) + \frac{1}{2} (3 + [hx - 2(1 - hx + gx^2)]^{-1}) \sqrt{1 - hx + gx^2}x^{-1},
\]

\[
E_e^2 = mx(1 - hx + gx^2),
\]

\]
where for simplicit we have set

\[ G_0(x) = \begin{align*}
&= [100bgx - 2m(3bh + 4π + 4πd)x^2 + \frac{1}{3}bm(20g + m)x^3 - \frac{1}{4}bgm^2x^4 + \frac{1}{5}bgm^2x^5 \\
&+ 2(5bh + 8π + 40rd)log(1 - hx + gx^2)]128π^{-1} + 4(13bh^2 - 50bg + 8hπ + 8dhπ \\
&- 128fπ^2)arctan([2gx - h] \sqrt{4g - h^2}^{-1}) \end{align*} \]

\[ G_1(x) = [-4096π^2(h - 2gx) - 2048mx^2(1 - hx + gx^2)]3276π^{-3}, \]

\[ G_2(x) = [32π(4 - 6hx + 8gx^2)(16π(16fπ + (2g + 10dg + m + dm)x + (1 + d)gm^2))] \\
\times 3276π^{-3}(x(h - gx) - 1)]^{-1}, \]

\[ G_3(x) = [-h(2 + 6d + mx^2 + dm^2) - b(h(6 + mx^2) - x(10g + m + gm^2))] \\
\times 3276π^{-3}(x(h - gx) - 1)]^{-1}, \]

\[ G_4(x) = [x(1 - hx + gx^2)(16π(2(10 + 3dh - 8fπ)(2g + 10dg + m + dm)x)))] \\
\times 3276π^{-3}(x(h - gx) - 1)]^{-1}, \]

\[ G_5(x) = [x(1 - hx + gx^2)(1 + d)hm^2 - (1 + d)gm^2 + b(h(6 + mx^2) - x(10g + m + gm^2))] \\
\times 3276π^{-3}(x(h - gx) - 1)]^{-1}, \]

\[ G_6(x) = [128πx(1 - hx + gx^2)(2gx - h)(16π(16fπ + 2g + 10dg + m + dm)x)))] \\
\times 3276π^{-3}(x(h - gx) - 1)]^{-1}, \]

\[ G_7(x) = [128πx(1 - hx + gx^2)(1 + d)gm^2 - h(2 + 6d + mx^2 + dm^2) - b(h(6 + mx^2) \\
- x(10g + m + gm^2)))]3276π^{-3}(x(h - gx) - 1)]^{-1}, \]

\[ G_8(x) = [128πx(1 - hx + gx^2)(16π(2(5 + 5dg + 3(1 + d)gm^2 + (1 + d)m(1 - 2hx)))) \\
\times 3276π^{-3}(x(h - gx) - 1)]^{-1}, \]

\[ G_9(x) = [128πx(1 - hx + gx^2)(2b(m - 2hx - g)(10 + 3mx^2))(h(6 + mx^2) + x(m \\
+ g(10 + mx^2)))))]3276π^{-3}(x(h - gx) - 1)]^{-1}. \]

The total mass of the stellar sphere for the envelope layer is given in the form

\[ M_e(r) = 4π \int r^2 ρ_e dr, \quad r^2 = x \]

\[ M_e = [x^3(h(6 + mx^2) - x(m + g(10 + mx^2)))96π^{-1}. \]

4 BOUNDARY CONDITIONS

Here we match the radial pressure, gravitational potentials, and electric field at the interfaces as follows:

**Junction conditions at the core-intermediate boundary.**

This is expressed by

\[ e^{-2λn}(R_0) = e^{-2λc}(R_0), \quad (38a) \]

\[ e^{2νn}(R_0) = e^{2νc}(R_0), \quad (38b) \]

\[ p_{en}(R_0) = p_{ec}(R_0), \quad (38c) \]

\[ E_{en}^2(R_0) = E_{ec}^2(R_0). \quad (38d) \]

**Junction conditions at the intermediate-envelope interface.**

This is written as

\[ e^{-2λn}(R_e) = e^{-2λc}(R_e), \quad (39a) \]

\[ e^{2νn}(R_e) = e^{2νc}(R_e), \quad (39b) \]

\[ p_{en}(R_e) = p_{ec}(R_e), \quad (39c) \]

\[ E_{en}^2(R_e) = E_{ec}^2(R_e). \quad (39d) \]
Junction conditions at the envelope-surface boundary.

The interior and exterior line elements (1) and (2) should match smoothly at the surface \( r = R_e \). Therefore

\[
e^{2\lambda}(R_e) = \left(1 - \frac{2M}{R_e} + \frac{Q^2}{R_e^2}\right)^{-1},
\]
\[
e^{2\nu}(R_e) = \left(1 - \frac{2M}{R_e} + \frac{Q^2}{R_e^2}\right),
\]
\[
p_{r e}(R_e) = 0,
\]
\[
E^2_e(R_e) = \frac{Q^2}{R_e^4}.
\]

They yield

\[
(1 - hR_e^2 + gR_e^2) = \left(1 + \frac{2P_o(x)}{R_e} + \frac{m(1 - hR_e + gR_e^2)}{R_e^2}\right)^{-1},
\]
\[
A^2[\exp(G_0(x))]^2 = \left(1 + \frac{2P_o(x)}{R_e} + \frac{m(1 - hR_e + gR_e^2)}{R_e^2}\right),
\]
\[
0 = b(P_1(x))^2 + d\left([h(6 + mx^2) - x(m + g(10 + mx^2))]\right] \\
\times[16\pi]^{-1}) - f,
\]
\[
E^2 = \frac{Q^2}{R_e^2} = mR_e(1 - hR_e + gR_e^2).
\]

where for simplicit we have set

\[
P_o(x) = x^3(h(6 + mx^2) - x(m + g(10 + mx^2)))[96\pi]^{-1} + k_1,
\]
\[
P_1(x) = \left([h(6 + mx^2) - x(m + g(10 + mx^2))]\right][16\pi]^{-1}.
\]

We observe from the system (41) that there are sufficient number of free parameters in four equations. This indicates that the matching conditions are easily satisfied. In system (41) above, it is complicated to solve the value of \( R_e \) analytically. However the value of \( R_e \) can be obtained geometrically from the plots especially at the vanishing point of radial pressure which is clearly shown to be \( R_e = 9.96 \) km.

The same choice of electric field \( E^2 \) is applied for all three layers. From the systems (38) and (39) we observe that at the core-intermediate junction at \( R_\eta \) and at the intermediate-envelope junction at \( R_\iota \), the electric field exhibit continuity throughout the interior of the stellar sphere.

5 PHYSICAL CONDITIONS

For a well behaved physical model, the matter variables and other physical quantities have to satisfy the following conditions.

(i) The energy density, radial and tangential pressures should be regular, finite and continuous, greater or equal to zero.

(ii) The metric potentials \( e^{2\lambda} \) and \( e^{2\nu} \) should be greater or equal to zero.

(iii) The radial sound speed should be less than the speed of light so as to obey the causality condition.

These conditions in each layer are written as

\[
\nu_\eta = A,
\]
\[
\nu_\iota = \left([h(6 + mx^2) - x(m + g(10 + mx^2))]\mu[8\pi]^{-1},
\]
\[
\nu_\epsilon = \left([8\pi d + bh(6 + mx^2) - bx(m + g(10 + mx^2))]\mu[8\pi]^{-1}.
\]
(iv) The model should satisfy the energy conditions: strong energy condition (SEC), weak energy condition (WEC) or the null energy condition (NEC). i.e SEC: $\rho - p_r - 2p_t \geq 0$, WEC: $\rho - 3p_r \geq 0$, $\rho - 3p_t \geq 0$, NEC: $\rho - p_r \geq 0$, $\rho - p_t \geq 0$.

(v) In general relativity, stability of the stellar sphere should satisfy the condition $\Gamma = \frac{\rho + p_r}{p_r} \frac{dp_r}{d\rho} \geq \frac{4}{3}$.

This condition for each region becomes

\[ \Gamma_\eta = \left[ \frac{(\alpha - h(6 + mx^2) + x(m + g(10 + mx^2))(1 + \alpha) + 16\pi\beta)) - h((6 + mx^2)\alpha + x(m + g(10 + mx^2))\alpha + 16\pi\beta)]}{(1 + \alpha + x(m + g(10 + mx^2))\alpha + 16\pi\beta)} \right]^{-1}, \]

\[ \Gamma_\chi = \left[ (B_1(x)(-a + (B_2(x)(16\pi + h(6 + mx^2))f - x(m + g(10 + mx^2))) + 8\pi(-a + ((h(6 + mx^2) - x(10g + m + gm^2)^2)\mu)))[(256\pi)^2]^{-1}, \]

\[ \Gamma_\epsilon = \left[ (B_3(x)(b(h(6 + mx^2) + x(10g + m + gm^2)^2) + 16\pi(1 - B_5(x)))[(2048\pi)^2(-\pi f + \frac{1}{16}(h(6 + mx^2) - x(m + g(10 + mx^2))) + B_4(x)))]^{-1}, \]

where

\[ B_1(x) = (h(6 + mx^2) - x(m + g(10 + mx^2)))\mu, \]

\[ B_2(x) = (h(6 + mx^2) - x(10g + m + gm^2)), \]

\[ B_3(x) = (8\pi d + bh(6 + mx^2) - bx(m + g(10 + mx^2))), \]

\[ B_4(x) = (h(6 + mx^2) - x(m + g(10 + mx^2)))^2)[256\pi]^{-1}, \]

\[ B_5(x) = 16\pi f + (1 + d)h(6 + mx^2) - (1 + d)x(10g + m + gm^2)), \]

\[ B_6(x) = -2hx + 8gx^2)(-16\pi(16\pi a + h(2 - mx^2) + x(-2g + m + gm^2) + h(6 + mx^2) - x(m + g(10 + mx^2)))\mu. \]

(vi) The surface redshift for the relativistic stellar objects have been discussed by many authors in the past [Baraco & Hamity (2002)]. In the paper by Bohmer & Harko (2006) showed that for imperfect fluid the value of the redshift should not exceed 5. The value of the surface redshift was later extended in the treatment by Ivanov (2002) who suggested the maximum value be $z_s \leq 5.211$ for anisotropic relativistic stellar object. The surface redshift is calculated using the form

\[ z_s = \frac{1}{\sqrt{1 - \frac{2M(r)}{r}}} - 1. \]

where $M$ stands for the total mass of the stellar object. In our study we obtain

\[ z_s = \frac{1}{\sqrt{1 + \frac{x^5/2(-h(6 + mx^2) + x(m + g(10 + mx^2))}{48\pi}}} - 1. \]

(vii) A thorough analysis of the mass radius ratio for a relativistic stellar object is described in Jasim et al. (2021). It has been noted that the compactification factor for the relativistic object is categorised into different classes: (i) normal star: \[ \frac{M(r)}{r} \sim 10^{-5}, \] (ii) white dwarf: \[ \frac{M(r)}{r} \sim 10^{-3}, \] (iii) neutron star: \[ 10^{-1} < \frac{M(r)}{r} < \frac{1}{4}, \] (iv) ultra dense compact star: \[ \frac{1}{4} < \frac{M(r)}{r} < \frac{1}{2}, \] (v) black hole: \[ \frac{M(r)}{r} = \frac{1}{2}. \]

The compactification factor is obtained in the form

\[ \mu(r) = \frac{2M(r)}{r}. \]

In our treatment we get

\[ \mu(x) = \frac{x^5/2(-h(6 + mx^2) + x(m + g(10 + mx^2))}{48\pi}. \]
6 RESULTS

We discuss the physical features of the matter variables and other quantities which were generated by using the Python programming language. The plots generated include energy density (Fig. 1), the radial pressure (Fig. 2), the tangential pressure (Fig. 3), the gravitational potentials (Figs. 4 and 5), mass (Fig. 6), the energy conditions (Figs. 7, 8, 9 and 10), the measure of anisotropy (Fig. 11), adiabatic index (Fig. 12), radial sound speed (Fig. 13), charge density (Fig. 14), electric field (Fig. 15), mass ratio (Fig. 16) and surface redshift (Fig. 17). We generate graphs using the following specified values of the constants: \(a = \pm 1.187 \times 10^{-5}\), \(b = 319.2\), \(d = 0.7283\), \(f = \pm 1.547 \times 10^{-5}\), \(g = \pm 1.5 \times 10^{-5}\), \(h = \pm 0.004587\), \(k_0 = 0.5\), \(k_1 = 0.0342\), \(k_2 = 0.2\), \(k_3 = 0.2\), \(m = \pm 7.15 \times 10^{-5}\), \(A = 0.1\), \(\alpha = 0.19897\), \(\beta = 4.9166 \times 10^{-5}\) and \(\mu = \pm 481.124\). All graphs are plotted against the radial distance in the specified domain of radius as per Pant et al. (2020).

In our model, Figs. 1 and 2 shows that the energy density and the radial pressure are continuous functions, maximum at the centre, and monotonically decreasing towards the surface. We also observe that these trends are similar to the findings in the papers by Pant et al. (2020); Bisht et al. (2021); Maharaj et al. (2014); Pant et al. (2019); Gedela et al. (2019) and Sunzu & Danford (2017). It is clearly shown that the radial pressure vanishes at the surface \((R_c = 9.96 \text{ km})\). Fig. 3 indicates that tangential pressure is an increasing function towards the surface. This physical behaviour is also found in the models by Thirukkanes & Ragel (2014); Maharaj & Mafa Takisa (2013) and Ngubelanga & Maharaj (2015). Figs. 4 and 5 show that metric potentials are continuously increasing functions. Fig. 6 indicates that the mass is monotonically increasing function with the increase of the radial distance. The energy conditions in Figs. 7, 8, 9 and 10 are decreasing functions that are positive. These physical features are also found in the works by Pant et al. (2020); Maurya et al. (2018) and Maurya & Govender (2017). The measure of anisotropy is an increasing function with uncontinuity point between intermediate layer and the envelope layer (Fig. 11). This shows that tangential pressure is greater than the radial pressure. The similar profile is also obtained by Pant et al. (2020); Thirukkanes & Ragel (2014); Bhar et al. (2017); Ngubelanga et al. (2015) and Murad (2016). In Fig. 12, the stability condition is found to satisfy the criteria within the stellar interior \(\Gamma \geq \frac{4}{3}\). This shows a gradual increase in \(\Gamma\) towards the surface.

In our treatment, we obtained the minimum adiabatic index \(\Gamma = 1.9769\). This shows that the object is stable from gravitational collapse. We can also see a similar physical feature in the treatments by Pant et al. (2020); Bisht et al. (2021); Jasim et al. (2018) and Bhar et al. (2017). Fig. 13 shows that the speed of sound is less than the speed of light. It is found to be in the range \((0.25523 \leq \nu \leq 0.69613)\). Similar physical feature are also found by Pant et al. (2020); Pant et al. (2019); Gedela et al. (2019); Maharaj & Mafa Takisa (2013); Bhar et al. (2017); Murad (2016) and Maharaj & Mafa Takisa (2012). We observe in Figs. 14 and 15 that charge density and electric field are monotonically increasing functions. From Fig. 16, we observe that the trend of the mass radius ratio is increasing function, reached the maximum value \(\mu(x) = 0.3933\) which satisfying the condition for ultra dense compact star. The same physical profile is also observed in the treatments of Pant et al. (2020); Bisht et al. (2021), and Jasim et al. (2018). In Fig. 17 we observe that the surface redshift is continuous, monotonically increasing in nature with radial coordinate \((r)\), attained maximum value at \(z_s = 1.05188\) which is physical and in acceptable range. The physical analysis show that the presence of charge continues to yield a physically reasonable model of a three-layered stellar object.

7 CONCLUSION

In this paper we have formulated a new class of exact solutions for superdense stellar spheres comprising three interior layers. In our model we used the Einstein-Maxwell field equations incorporated with different equations of state. The core region obeys a linear equation of state, the intermediate layer has a quadratic equation of state (with no linear term), and the envelope layer is outfitted with a general quadratic equation of state. In our model we have made a specific choice for the electric field and one of the gravitational potentials. Our model contains the electric field which is absent in Pant et al.
(2020) and Bisht et al. (2021). Setting electric field zero \( (E = 0) \), we regain the anisotropic uncharged model developed by Pant et al. (2020), and other earlier models in the core-envelope setting. The plots generated indicates that the gravitational potentials, matter variables, and other physical quantities are well behaved and compatible with astrophysical studies. In this case, they are regular, continuous, and free from singularity throughout the interior of the stellar objects. The physical analysis shows that our model is well behaved. The results obtained in this paper are significant and allow us to describe the physical structures, features, and properties of charged anisotropic superdense stellar objects with three layers. In future work, other results can be generated by considering different forms of gravitational potential, equations of state, and electric field.

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Fig. 1: Energy density against radial distance

Fig. 2: Radial pressure against radial distance

Fig. 3: Tangential pressure against radial distance
Charged stellar model with three layers

**Fig. 4:** Gravitational potential against radial distance

**Fig. 5:** Gravitational potential against radial distance

**Fig. 6:** Mass against radial distance
Fig. 7: Energy conditions against radial distance

Fig. 8: Energy conditions against radial distance

Fig. 9: Energy conditions against radial distance
Fig. 10: Energy conditions against radial distance

Fig. 11: Measure of anisotropy against radial distance

Fig. 12: Adiabatic index against radial distance
Fig. 13: Radial speed against radial distance

Fig. 14: Charge density against radial distance

Fig. 15: Electric field against radial distance
Charged stellar model with three layers

Fig. 16: Mass radius ratio against radial distance

Fig. 17: Surface redshift ($z_s$) against radial distance

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