Viscous Holographic $f(Q)$ Cosmology with Some Versions of Holographic Dark Energy with Generalized Cut-offs

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Abstract The work reported in this paper demonstrates the cosmology of $f(Q)$ gravity and the reconstruction of various associated parameters with different versions of holographic dark energy with generalized cut-offs, where $Q = 6H^2$. The universe is considered to be filled with viscous fluid characterised by a viscous pressure $\Pi = -3H\xi$, where $\xi = \xi_0 + \xi_1 H + \xi_2 (H + H^2)$ and $H$ is the Hubble parameter. Considering the power law form of expansion, we have derived the expression of $f(Q)$ under non-viscous holographic framework and it is then extended to viscous cosmological settings with extended generalized holographic Ricci dark energy. The forms of $f(Q)$ for both the cases are found to be monotone increasing functions of $Q$. In the viscous holographic framework, $f(Q)$ is reconstructed as a function of cosmic time $t$ and is found to stay at positive level with Nojiri-Odintsov cut-off. In these cosmological settings, the slow roll parameters are computed and a scope of exit from inflation and quasi exponential expansion are found to be available. Finally, it is observed that the warm inflationary expansion can be obtained from this model.

Key words: Holographic dark energy; Ricci dark energy; $f(Q)$ gravity; bulk-viscosity; equation of state parameter; slow roll parameter.

1 INTRODUCTION

Accelerated expansion of the universe in the late 90’s was observationally reported by Riess et al. [Riess & et al.]$^{1998}$ and Perlmutter et al. [Perlmutter & et al.$^{1999}$]. Immediately after the Big-Bang, there was another phase of accelerated expansion, which is known as the inflationary epoch. In between these two epochs, there were phases of radiation and matter domination. Prior to late 90’s, it was believed that the universe was expanding with deceleration. However in 1998, two independent research groups led by Riess [Riess & et al.$^{1998}$] and Perlmutter [Perlmutter & et al.$^{1999}$] respectively observed that the universe is expanding with acceleration rather than deceleration. This was breakthrough in modern cosmology and got support from subsequent observational studies with Supernovae Ia (SNeIa), the cosmic microwave background (CMB) radiation anisotropies, Large Scale Structure (LSS) and X-ray experiments [de Bernardis & et al.$^{2000}$, Seljak & et al.$^{2005}$, Abazajian & et al.$^{2005}$, Ade et al.$^{2013}$, Tegmark & et al.$^{2004}$, Allen & et al.$^{2004}$]. It is thought that this acceleration is driven by some exotic matter characterized by negative pressure. This exotic matter is coined as “Dark Energy” (DE) [Bamba
The early phase of evolution of the universe is named as "Inflationary Scenario" (Brevik et al. 2015). In this very late time acceleration is not only an accelerated phase of the universe. There was very early phase of evolution of universe named as "Inflationary Scenario" (Brevik et al. 2015). Hence, we can say that there are many challenges to make it viable (Nojiri et al. 2017). The main motive of modified gravity is to unify early inflation with the late time acceleration of the universe. It is to be noted that in $f(R)$ gravity, the unified $\Lambda$CDM with inflation was achieved (Nojiri & Odintsov 2011, 2007b; Appleby & Battye 2007; Nojiri & Odintsov 2008; Cognola et al. 2008; Artymowksi & Lalak 2014; Fay et al. 2007). The $R$ in $f(R)$ is the fundamental term. Apart from Einstein’s gravity, we can consider teleparallelism with the Weitzenbck connection. Here the fundamental term is the torsion $T$. Another type of modified gravity which is very useful in studying of inflationary era and transition of acceleration from deceleration regimes of the universe is $f(G)$ gravity. This is Gauss Bonnet gravity. $G$ is the topological invariant in 4 dimensions of spacetime. It is also observed that $f(G)$ gravity is less constrained than the $f(R)$ gravity (Bamba et al. 2017).

About 68.3% of the total density is due to DE, 26.8% is due to dark matter (DM). Baryonic matter contributes about 4.9% of the total energy density. Practically, the contribution of radiation is very negligible.

Modifications of Einstein’s gravity gave rise to a new degrees of freedom in the gravitational sector. In the models of DE, the infrared cutoff are modified, which is also the example of the reason of modified gravity. The concept modified gravity is of utmost importance in studying the late time acceleration of the universe. The models of modified gravity are represented by (i) Braneworld models (Nojiri & Odintsov 2000), (ii) $f(G)$ gravity, where $G = R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\lambda\sigma}R^{\mu\nu\lambda\sigma}$, $G$ represents the Gauss-Bonnet invariant, $R$ is the Ricci scalar curvature, $R_{\mu\nu}$ representing the Ricci curvature tensor and $R_{\mu\nu\lambda\sigma}$ is the Reimann curvature tensor (Garca & et al. 2011; Bamba et al. 2017), (iii) $f(T)$ gravity where $T$ represents the torsion (Bamba et al. 2012a), (iv) $f(R)$ gravity, $R$ indicates the Ricci scalar curvature (Nojiri & Odintsov 2011), (v) $f(Q)$ gravity, where $Q = 6H^2$ (Lazkot et al. 2019), (vi) DBI models (Chakraborty & Chattopadhyay 2020b), etc.

At this juncture we would like to mention to exhaustive review works by (Nojiri et al. 2017; Nojiri & Odintsov 2011), where an extensive study has been demonstrated on the various aspects of modified gravity theory and its consequences. To be accepted as a cosmological theory, the constraints on modified gravity are not only have to satisfy local astrophysical data but also from global constraints. Hence, we can say that there are many challenges to make it viable (Nojiri et al. 2017). There are many components in the universe. Each components have different cooling rates. Hence, the term bulk viscosity arises. In general, viscosity means resistance to flow. There are coupling among the different component of cosmic substratum and due to which the bulk viscous pressure in cosmic media emerges. The DE and DM models developed by Nojiri and Odintsov (Nojiri & Odintsov 2005, 2006a) were treated as imperfect fluids. A plethora of literatures have suggested us that the bulk viscous pressure plays a major role in the late time acceleration of the universe (Brevik et al. 2004, 2010, 2015; Chakraborty & Chattopadhyay 2020abc, 2021; Chattopadhyay & Chakraborty 2021; Chakraborty et al. 2021). The late time acceleration is not only accelerated phase of the universe. There was very early phase of evolution of universe named as "Inflationary Scenario" (Brevik et al. 2015). In this very
early phase both bulk and shear viscosity thought to play a very significant role (Pun & et al. 2008). The work of Chimento et al. (Chimento et al. 2000), reported that in the combination of a cosmic fluid characterized by bulk dissipative pressure and a quintessence matter - it is possible to have an accelerated expansion of the universe. The entropy for a coupled fluid and a relationship between the entropy of closed FRW universe to the energy contained in it established by Brevik et al. (Brevik et al. 2010). Brevik et al. (Brevik et al. 2015) by considering the bulk viscous pressure as a function of Hubble parameter, demonstrated Little Rip, Pseudo Rip and Bounce cosmology in bulk-viscosity framework.

The paper is organised as follows: In Section 2, we have discussed the holographic dark energy (HDE) and holographic principle (HP). In Section 3, we have discussed the mathematical background of \( f(Q) \) gravity. In Section 4, we have studied the non-viscous holographic \( f(Q) \) gravity. Here we have found the expression of \( f(Q) \) and studied its behaviour. In the Section 5, we have studied the viscous extended generalized holographic Ricci \( f(Q) \) gravity. Here, we have found the expression of \( f(Q) \) and studied its behaviour. We have reconstructed the pressure \( p_{rec,R} \), density \( \rho_{rec,R} \) and EoS parameter \( w_{rec,R} \). In Section 6, we have discussed the case namely viscous holographic \( f(Q) \) gravity with Nojiri-Odintsov cut-off. We have reconstructed the infrared cut-off \( L \), density \( \rho_{NO,rec} \). We have have studied the behaviour of \( f(Q) \) in this case. In Section 7, we have aimed to study the holographic Ricci \( f(Q) \) gravity as scalar field in bulk-viscosity framework. Here, we have reconstructed the Hubble slow roll parameters \( \epsilon_H \) and \( \eta_H \). We have reconstructed the dissipative coefficient \( \Gamma \) and \( 2V - \dot{\phi}^2 \). Lastly, we have concluded in the Section 8.

2 HOLOGRAPHIC DARK ENERGY

Holographic Dark Energy (HDE) model is one of the mostly studied model of DE. It is extensively discussed in the references (Li 2004; Myung & Seo 2009; Li et al. 2009; Nojiri et al. 2019b; Salako et al. 2015). The HDE models is based on the Holographic Principle (HP). HP is a tenet of string theories. It states that the description of a volume of space can be thought of as encoded on a lower dimensional boundary to the region. The energy density of HDE is given by \( \rho_{\Lambda} = 3c^2 M_p^2 L^{-2} \), where \( c \) is the numerical constant, \( L \) is the infrared cut-off and \( M_p \) is the reduced Planck mass. \( M_p = \frac{1}{\sqrt{8\pi G}} \approx 1 \).

Therefore, \( \rho_{\Lambda} = 3c^2 L^{-2} \) (Li et al. 2009; Pasqua et al. 2019; Nojiri et al. 2019a; 2021; Sarkar et al. 2021). If we assume \( L \) as the size of the current universe, for instance, the Hubble radius \( H^{-1} \), then the DE density will be very close to the observational value (Nojiri et al. 2019b). Therefore, density of HDE becomes

\[
\rho_{\Lambda} = 3c^2 H^2
\]

In this context we would like to mention that the HDE with Nojiri - Odintsov (NO) cut-off or in other words NO generalised HDE is the most general version of HDE. Ricci HDE, generalised Ricci HDE, etc are some particular cases of NO HDE with corresponding choice of NO cut-off. Such generalised NO cut-offs have been discussed in (Nojiri & Odintsov 2006; Chakraborty et al. 2021; Nojiri et al. 2021; Sarkar et al. 2021; Khurshudyan 2016a,b; Sarkar & Chattopadhyay 2021; Elizalde & Timoshkin 2019).

3 \( f(Q) \) GRAVITY-A BRIEF OVERVIEW

In this Section, we give overview of \( f(Q) \) gravity. Earlier \( f(Q) \) gravity has been discussed in (Mandal et al. 2020; 2021; Lazkoz et al. 2019). As a purpose of the current work is to present a holographic viscous \( f(Q) \) cosmology, we first present a mathematical background of \( f(Q) \) gravity. The action for symmetric teleparallel gravity is given by (Jimenez et al. 2018)

\[
S = \int \frac{1}{2} f(Q) \sqrt{-g} d^4 x + \int L_m \sqrt{-g} d^3 x
\]
where \( f \) is the function of nonmetricity \( Q \), \( g \) is the determinant of the metric \( g_{\mu\nu} \), \( \mathcal{L}_m \) is the matter Lagrangian density. The nonmetricity tensor is given by

\[
Q_{\lambda\mu\nu} = \nabla_\lambda g_{\mu\nu}
\]  
(3)

Its traces are

\[
Q_\alpha = Q_{\alpha\mu}{}^{\mu}
\]  
(4)

and

\[
\tilde{Q}_\alpha = Q_{\mu\alpha}{}^{\mu}
\]  
(5)

Therefore, superpotential will be

\[
P_{\alpha\mu\nu} = \frac{1}{4} \left[ -Q_{\alpha\mu\nu} + 2Q_{(\mu}{}^{\alpha}{}^{\nu)} + Q_{\alpha} g_{\mu\nu} - \tilde{Q}_{\alpha} g_{\mu\nu} - \delta_{\alpha}{}^{(\mu} Q_{\nu)} \right]
\]  
(6)

where, \( Q = -Q_{\alpha\mu\nu} P^{\alpha\mu\nu} \). Now, the energy momentum tensor will be

\[
T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \delta(\sqrt{-g} \mathcal{L}_m) g^{\mu\nu}
\]  
(7)

Varying Eq. (2) with respect to metric tensor \( g_{\mu\nu} \), the equations of motion will be

\[
2 \sqrt{-g} \nabla_\gamma (\sqrt{-g} f_Q P_{\gamma\mu\nu}) + \frac{1}{2} g_{\mu\nu} f + f_Q (P_{\nu}{}^{\gamma} Q_{\alpha}{}^{\nu} - 2\tilde{Q}_{\gamma\mu} P_{\gamma\nu}) = -T_{\mu\nu}
\]  
(8)

where \( f_Q = \frac{df}{dQ} \)

Also, varying Eq. (2) with respect to the connection, one obtains

\[
\nabla_\mu \nabla_\nu (\sqrt{-g} f_Q P_{\gamma\mu\nu}) = 0
\]  
(9)

We consider the FLRW universe represented by isotropic, homogeneous and spatially flat line element

\[
ds^2 = -dt^2 + a^2(t) \delta_{\mu\nu} dx^\mu dx^\nu
\]  
(10)

where \( a(t) \) is the scale factor. For the line element Eq. (10), the trace of nonmetricity tensor will be \( Q = 6H^2 \).

The energy momentum tensor of the cosmological fluid is given by

\[
T_{\mu\nu} = (\rho + p_{\text{eff}}) u_\mu u_\nu + p_{\text{eff}} g_{\mu\nu}
\]  
(11)

where \( p_{\text{eff}} \) is effective pressure and is given by \( p_{\text{eff}} = \text{thermodynamic pressure} + \Pi \), viscous pressure \( \Pi = -3H \xi, \xi = \xi_0 + \xi_1 H + \xi_2 (H + H^2) \), \( \xi_0, \xi_1 \) and \( \xi_2 \) are positive constants and \( \xi > 0 \). Using Eqs. (10) and (11) in Eq. (8), modified field equations will be (Jimenez et al. 2018; Mandal et al. 2021; Lazkoz et al. 2019):

\[
3H^2 = \frac{1}{2f_Q} \left( -\rho + \frac{f}{2} \right)
\]  
(12)

and

\[
\dot{H} + 3H^2 + \frac{\dot{f}}{f_Q} H = \frac{1}{2f_Q} (p_{\text{eff}} + \frac{f}{2})
\]  
(13)

Conservation equation in this case will be

\[
\dot{\rho} + 3H (\rho + p_{\text{eff}}) = 0
\]  
(14)

where \( \rho \) and \( p \) is the density and pressure respectively under \( f(Q) \) gravity.
4 NON-VISCOUS HOLOGRAPHIC $f(Q)$ GRAVITY

In this section, our aim is to study the behaviour of $"f"$ in non-viscous holographic $f(Q)$ gravity. To proceed, we consider power law form of scale factor $a(t)$.

$$a(t) = a_0 t^n$$

, $n > 0$. Then

$$\dot{a}(t) = a_0 n t^{-1+n}$$

We know, Hubble parameter $H = \frac{\dot{a}(t)}{a(t)}$. Using, the expression of $a(t)$ and $\dot{a}(t)$ from Eqns. 15 and 16 respectively, we get $H$ as

$$H = \frac{n}{t}$$

Differentiating Eq. 17 with respect to $t$, we get

$$\dot{H} = -\frac{n}{t^2}$$

Now, using expression of $H$ from Eq. 17 in Eq. 1, we get density of HDE.

$$\rho_\Lambda = \frac{3c^2n^2}{t^2}$$

Differentiating Eq. 19 with respect to $t$, we get $\dot{\rho}_\Lambda$.

$$\dot{\rho}_\Lambda = -\frac{6c^2n^2}{t^3}$$

Now, using $\dot{\rho}_\Lambda$ from Eq. 20, $\rho_\Lambda$ from Eq. 19, $H$ from Eq. 17 in the conservation equation 14, we get thermodynamic pressure of non viscous HDE $p_\Lambda$ as

$$p_\Lambda = \frac{n (-2c^2 + 3c^2n)}{t^2}$$

Now, we will find the solution of $f(Q)$. As $Q = 6H^2$. So, using $H$ from Eq. 17, we get $Q$ as

$$Q = \frac{6n^2}{t^2}$$

Differentiating Eq. 22 with respect to $t$, we get $\dot{Q}$.

$$\dot{Q} = -\frac{12n^2}{t^3}$$

Again, differentiating Eq. 23 with respect to $t$, we get $\ddot{Q}$.

$$\ddot{Q} = \frac{36n^2}{t^4}$$

Using the expression of $H$ from Eq. 17, $\dot{H}$ from Eq. 18, $p_\Lambda$ from Eq. 21 (in place of $p_{eff}$) in the second field equation 13, we get $f(t)$ as

$$f(t) = -\frac{6c^2n^2}{t^2} + t^{-3n}C_1 + \frac{C_2}{t}$$

From expression 22, we get $t$ in terms of $Q$ as

$$t = \sqrt{\frac{6n^2}{Q}}$$
We considered $C_1 = 0.008$, $C_2 = 0.005$, $c = 0.02$. The red, green and blue line corresponds to $n = 0.30, 0.34, 0.38$ respectively.

Using expression of $t$ from Eq. (26) in Eq. (25), we get $f(Q)$

$$\begin{align*}
f(Q) &= \frac{C_2}{\sqrt{6}} \sqrt{\frac{n^2}{Q}} + 6^{-3n/2}C_1 \left( \frac{n^2}{Q} \right)^{-3n/2} - c^2Q \\
&= 2(\dot{H} + 2H^2 + \frac{\kappa}{a^2})
\end{align*}$$

(27)

We have plotted the evolution of $f(Q)$ against $Q$ of non-viscous holographic $f(Q)$ gravity in the Fig. 1. The Fig. 1 suggests us that $f(Q)$ stays at positive level. The Fig. 1 also suggests us that with the increase of $Q$, $f(Q)$ increases. We can interpret that $f$ is a monotone increasing function of $Q$. As $f(Q) \to 0$, $Q \to 0$. This indicates that one of the sufficient conditions for a realistic reconstruction model is satisfied under the reconstruction in absence of bulk viscosity.

5 VISCIOUS EXTENDED GENERALIZED HOLOGRAPHIC RICCI $F(Q)$ GRAVITY

Holographic DE unify phantom inflation with late time acceleration of the universe. It is suggested by Nojiri and Odintsov (Nojiri & Odintsov 2006). Extended holographic Ricci Dark Energy (EHRDE) is the generalized form of HDE (Zhang 2009; George et al. 2019; Praseetha & Mathew 2014; Rao et al. 2018). The Ricci scalar is given by

$$R = -6(\dot{H} + 2H^2 + \frac{\kappa}{a^2})$$

(28)

where $\kappa$ is the curvature of the universe. The density of Ricci holographic dark energy (RHDE) is given by $\rho_R = 3c^2 \left( \dot{H} + 2H^2 + \frac{\kappa}{a^2} \right)$, where $c$ is a numerical parameter which tells us the characteristic of holographic Ricci DE and at $c^2 < 1/2$, the quintom behaviour of EoS parameter is observed. In our study, we will consider the flat universe. So, $\kappa = 0$. Therefore, $\rho_R = 3c^2 \left( \dot{H} + 2H^2 \right)$. The density of RHDE is generalized to EHRDE by Granda and Oliveros and is given by

$$\rho_R = 3c^2 \left( \alpha \dot{H} + \beta H^2 \right)$$

(29)
Here, also we have assumed the power law form of scale factor as in Eq. (15). Using the expression of $H$ and $\dot{H}$ from Eqs. (17) and (18) respectively in Eq. (29), we get the density of EHRDE as

$$\rho_R = 3c^2 \left( -\frac{n\alpha}{t^2} + \frac{n^2\beta}{t^2} \right)$$  

(30)

Differentiating equation (30) with respect to $t$, we get

$$\dot{\rho}_R = 3c^2 \left( \frac{2n\alpha}{t^3} - \frac{2n^2\beta}{t^3} \right)$$  

(31)

Now, using expression of $H$ and $\dot{H}$ from Eqs. (17) and (18) respectively in the viscous coefficient $\xi = \xi_0 + \xi_1H + \xi_2(\dot{H} + H^2)$, we get

$$\xi = \xi_0 + \frac{n\xi_1}{t} + \left( -\frac{n}{t^2} + \frac{n^2}{t^2} \right)\xi_2$$  

(32)

Using the expression of $\xi$ from Eq. (32), $H$ from Eq. (17) in the viscous pressure $\Pi = -3H\xi$, we get

$$\Pi = -\frac{3n}{t} \left( \xi_0 + \frac{n\xi_1}{t} + \left( -\frac{n}{t^2} + \frac{n^2}{t^2} \right)\xi_2 \right)$$  

(33)

Using $\dot{\rho}_R$ from Eq. (31), $\rho_R$ from Eq. (30), $H$ from Eq. (17), $\Pi$ from Eq. (33) in the conservation equation (14), we get thermodynamic pressure $\rho_R$ of viscous extended holographic Ricci dark energy (EHRDE).

$$\rho_R = -2c^2n\alpha - 3c^2nt\beta - c^2nt^3\beta + c^2n^2t\xi_0 + c^2n^2t\xi_1 - c^2n^2\xi_2 + 3n^3\xi_2$$  

(34)

Now, using $p_R$ from Eq. (34), $H$ from Eq. (17), $\dot{H}$ from Eq. (18) in the second field equation (13), we get $f(t)$ as

$$f(t) = \frac{1}{(1-3n)^2(-2+3n)^3} \left( -3(1+n) \left( C_4(1-3n)^2(-2+3n)t^3 + t^{3n} \left( (1-3n)^2 + (1-3n)^2n^2\xi_0 + (1-3n)^2n^2\xi_1 + (1-3n)^2n^3(-2+3n)\xi_2 \right) - \right. \right.$$  

$$\left. 18n^2(2+9(-1+n)n)t^{2+3n}\xi_0\log(t) \right)$$  

(35)

Using expression of $t$ from Eq. (26) in Eq. (35), we get $f(Q)$

$$f(Q) = -\frac{c_4}{\sqrt{6\sqrt{\pi}}} + 6^{-3n/2}C_3 \left( \frac{\alpha}{\sqrt{\pi}} \right)^{-3n/2} + \frac{c^2Q(\alpha-n\beta)}{n} + \frac{3\sqrt{2\sqrt{\pi}Q\xi_0}}{(1-3n)^3} + \frac{3nQ\xi_1}{2+3n} +$$  

$$\frac{nQ\xi_2}{2\sqrt{6\sqrt{\pi}}} - \frac{3\sqrt{2\sqrt{\pi}Q\xi_0\log \left[ \frac{6\pi\alpha}{\sqrt{\pi}} \right]}}{(1-3n)^3}$$  

(36)

We have plotted the evolution of $f(Q)$ against $Q$ of viscous extended generalized holographic Ricci $f(Q)$ gravity in the Fig.2. The Fig.2 indicates that $f(Q)$ stays at positive level. The Fig.2 suggests us that with the increase of $Q$, there is also a increase of $f(Q)$. We can interpret that $f$ is a monotone increasing function of $Q$. As $f(Q) \rightarrow 0$, $Q \rightarrow 0$. This indicates that one of the sufficient conditions for a realistic reconstruction model is satisfied under the reconstruction in presence of bulk viscosity.

Differentiating Eq. (36) with respect to $Q$, we get

$$f_Q = -\frac{1}{(1-3n)^n(-2+3n)^3}2^{1/2}(5-3n)3^{-1/2}(1+n) \left( \frac{\alpha}{\sqrt{\pi}} \right)^{1/2} -$$  

$$\frac{18\sqrt{6\sqrt{\pi}}C_3(1-3n)^2n(-2+3n)}{2^{1/2}} +$$  

$$6^{3n/2} \left( \frac{\alpha}{\sqrt{\pi}} \right)^{3n/2} \left( \sqrt{6(-2+3n)} \right)^{\alpha-n\beta} \left( 1-3n \right)^2 - \frac{12c^2n(-\alpha+n\beta)}{2^{3/2}} +$$  

$$54\sqrt{6} \left( \frac{\alpha}{\sqrt{\pi}} \right)^{3/2}Q\xi_0 - 36\sqrt{6}(1-3n)^2n \left( \frac{\alpha}{\sqrt{\pi}} \right)^{3/2}Q\xi_1 - 9(1-3n)^2n^3(-2+3n)\xi_2 +$$  

$$2^{1+3n/2}3^{3+2n}(2+9(-1+n)n) \left( \frac{\alpha}{\sqrt{\pi}} \right)^{2+3n/2}Q\xi_0\log \left[ \frac{6\pi\alpha}{\sqrt{\pi}} \right]$$  

(37)
Using this expressions of $t$, we get

$$\dot{f}_Q = \frac{1}{(1-3n)^2n} \left( 2 - \frac{3}{2}(2+n) \right) \left( 2 - \frac{3}{2}(2+n) \right) \left( \frac{n}{Q} \right)^{-3n/2} Q \left( \frac{-6\sqrt{6}C_n(1-3n)^2(-2+3n) \left( \frac{n}{Q} \right)^{3/2} - \frac{6n^{3/2}n}{2} \left( \frac{n}{Q} \right)^{3n/2} \left( -2C_n(1-3n)^2+3n(-12\xi_0(1-3n)^2Q\xi_2)+18n^2(-1+3n)\xi_0\log\left( \frac{n^2}{Q^2} \right) \right) }{f_Q} \right) \right)$$  \hspace{1cm} (38)

Using this expressions of $f(t)$, $f(Q)$, $\dot{f}_Q$ and $\ddot{f}_Q$ in the first and second field equations (12) and (13), we get reconstructed pressure $\rho_{rec,R}$ and density $\rho_{rec,R}$ of viscous extended generalized holographic Ricci $f(Q)$ gravity.

$$\rho_{rec,R} = \frac{c^2(-2+3n)\dot{t}(\alpha - n\beta) + 6n(t^2\xi_0 + nt\xi_1 + (-1 + n)n\xi_2)}{t^3}$$  \hspace{1cm} (39)

and

$$\rho_{rec,R} = \frac{1}{2(2+9(-1+n))} \left( t^{-3(1+n)} \left( -C_n(1-3n)^2(-2+3n)t^3 - 6nt^{3n} \left( c^2(2 + 9(-1 + n)n)t(\alpha - n\beta) + n(3t((-2+3n)t\xi_0 + n(-1 + 3n)\xi_1) + n(2 + 9(-1 + n)n)\xi_2)) \right) \right) \right)$$  \hspace{1cm} (40)

The reconstructed EoS parameter of viscous extended generalized holographic Ricci $f(Q)$ gravity is given by $\omega_{rec,R} = \frac{\rho_{rec,R}}{\rho_{rec,R}}$. Therefore, using the expression of $\rho_{rec,R}$, $\rho_{rec,R}$ from Eqs. (39), (40) respectively, we get $\omega_{rec,R}$ as

$$\omega_{rec,R} = - \left( \frac{\left( 2(2 + 9(-1 + n)n)t^{3n} \left( c^2(-2 + 3n)t(\alpha - n\beta) + 6n(t^2\xi_0 + nt\xi_1 + (-1 + n)n\xi_2) \right) \right) \left( C_n(1 - 3n)^2(-2 + 3n)t^3 + 6nt^{3n} \left( c^2(2 + 9(-1 + n)n)t(\alpha - n\beta) + n(3t((-2+3n)t\xi_0 + n(-1 + 3n)\xi_1) + n(2 + 9(-1 + n)n)\xi_2)) \right) \right)^{-1} }{c^2(-2 + 3n)t(\alpha - n\beta) + 6n(t^2\xi_0 + nt\xi_1 + (-1 + n)n\xi_2) \left( C_n(1 - 3n)^2(-2 + 3n)t^3 + 6nt^{3n} \left( c^2(2 + 9(-1 + n)n)t(\alpha - n\beta) + n(3t((-2+3n)t\xi_0 + n(-1 + 3n)\xi_1) + n(2 + 9(-1 + n)n)\xi_2)) \right) \right) \left( C_n(1 - 3n)^2(-2 + 3n)t^3 + 6nt^{3n} \left( c^2(2 + 9(-1 + n)n)t(\alpha - n\beta) + n(3t((-2+3n)t\xi_0 + n(-1 + 3n)\xi_1) + n(2 + 9(-1 + n)n)\xi_2)) \right) \right) \right)$$  \hspace{1cm} (41)

We have plotted Evolution of reconstructed EoS parameter $\omega_{rec,R}$ (Eq. (41)) against cosmic time $t$ of

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Fig. 2: Evolution of reconstructed $f(Q)$ (Eq. (36)) versus $Q$ of viscous extended generalized holographic Ricci $f(Q)$ gravity. We considered $C_1 = 0.008, C_4 = 0.005, c = 0.02, \xi_0 = 0.06, \xi_1 = 0.001, \xi_2 = 0.009, \alpha = 0.05, \beta = 0.04$. The red, green and blue line corresponds to $n = 0.30, 0.31, 0.32$ respectively.
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Fig. 3: Evolution of reconstructed EoS parameter $w_{\text{rec},R}$ (Eq. (41)) against cosmic time $t$ of viscous extended generalized holographic Ricci $f(Q)$ gravity. We considered $C_3 = 0.008$, $c = 0.02$, $\xi_0 = 0.06$, $\xi_1 = 0.001$, $\xi_2 = 0.009$, $\alpha = 0.05$, $\beta = 0.04$. The red, green and blue line corresponds to $n = 0.30, 0.31, 0.32$ respectively.

viscous extended generalized holographic Ricci $f(Q)$ gravity in the Fig. 3. We can see from the figure that with the evolution of universe i.e., with the increase in cosmic time $t$, $w_{\text{rec},R}$ is increasing going from phantom to quintessence. This indicates that there may be an avoidance of Big-Rip.

6 VISCOUS HOLOGRAPHIC $F(Q)$ GRAVITY WITH NOJIRI-ODINTSOV CUT-OFF

Nojiri and Odintsov demonstrated an approach to unify any inflationary cosmology with the DE cosmology of the late time universe (Nojiri & Odintsov 2006b; Nojiri et al. 2014; Nojiri & Odintsov 2017). They deduced that there is a possibility of transition of phantom to non-phantom in a such a manner that the universe could have the phantom EoS in the early as well as in the late time. Nojiri et al. 2014 and Nojiri and Odintsov 2017 demonstrated unification of inflation with $\Lambda$CDM under the purview of $f(R)$ gravity write this as generalised NO model and found the corresponding cut-off. The density for Nojiri-Odintsov (NO) HDE is defined as

$$\rho_{\text{NO}} = \frac{3c^2}{L^2} \quad (42)$$

where, $c$ is a numerical constant and $L$ is the infrared cut-off. Let, $R_h$ be the event horizon, $\alpha_0$, $\alpha_1$ and $\alpha_2$ are the numerical constants. Then, $\frac{c}{L}$ is given by

$$\frac{c}{L} = \frac{1}{R_h}(\alpha_0 + \alpha_1 R_h + \alpha_2 R_h^2) \quad (43)$$

The event horizon is given by

$$\dot{R}_h = H R_h - 1 \quad (44)$$

Solving the differential equation (44) using $H$ from Eq. (17), we get

$$R_h = -\frac{t}{1 - n} + t^n C_5 \quad (45)$$
Using the expression of $R_h$ from Eq. (45) in Eq. (43), we get the infrared cut off $L$ as

$$L = \frac{c(-1 + n) (t + C_5(-1 + n) t^n)}{(-1 + n)^2 \alpha_0 + (t + C_5(-1 + n) t^n) ((-1 + n) \alpha_1 + (t + C_5(-1 + n) t^n) \alpha_2)}$$  \hspace{1cm} (46)

Using expression of $L$ from Eq. (46) in the expression (42), we get reconstructed density $\rho_{NO,rec}$ of NO HDE as

$$\rho_{NO,rec} = 3 \left((1 - 1 + n)^2 \alpha_0 + (t + C_5(-1 + n) t^n) ((-1 + n) \alpha_1 + (t + C_5(-1 + n) t^n) \alpha_2)\right)^2 \frac{1}{(-1 + n)^2 (t + C_5(-1 + n) t^n)^2}$$  \hspace{1cm} (47)

Taking the correspondence between the density of NO HDE (Eq. (47)) and density of HDE (Eq. (19)), we get two expressions of $c$. Let it be, $c_1$ and $c_2$.

$$c_1 = \frac{t \left((-1 + n)^2 \alpha_0 + (t + C_5(-1 + n) t^n) ((-1 + n) \alpha_1 + (t + C_5(-1 + n) t^n) \alpha_2)\right)}{(-1 + n) n (t + C_5(-1 + n) t^n)}$$  \hspace{1cm} (48)

and

$$c_2 = \frac{t \left(\frac{(-1 + n) \alpha_0}{1 + C_5(-1 + n) t^n} + \alpha_1 + \frac{t \alpha_2}{1 + n} + C_5 t^n \alpha_2\right)}{n}$$  \hspace{1cm} (49)

Case I: Behaviour of $f(t)$ when $c = c_1$ Using $c = c_1$ from Eq. (48) in Eq. (19), we get the density $\rho_{\Lambda,NO,c_1}$ of HDE in terms of NO cut-off.

$$\rho_{\Lambda,NO,c_1} = \frac{3 \left((-1 + n)^2 \alpha_0 + (t + C_5(-1 + n) t^n) ((-1 + n) \alpha_1 + (t + C_5(-1 + n) t^n) \alpha_2)\right)^2}{(-1 + n)^2 (t + C_5(-1 + n) t^n)^2}$$  \hspace{1cm} (50)

From the first Field equation (Eq. (12)) of $f(Q)$ Gravity and using $H$ from Eq. (17), $\dot{Q}$ from Eq. (23), we get the differential equation

$$tf'[t] = 2\rho - f[t]$$  \hspace{1cm} (51)

Using this $\rho_{\Lambda,NO,c_1}$ from Eq. (50) in place of $\rho$ in Eq. (51), we solved it numerically by taking initial condition as $f(0.09) = 2$ and the numerical value of parameters as $n = 0.9$, $\alpha_0 = 0.6$, $\alpha_1 = 0.3$, $\alpha_2 = 0.2$ and $C_5 = 0.004$. Taking the cosmic time $t$ interval $[0.09, 3]$, we plotted the evolution $f_{reconstructed,NO,c_1}$ versus $t$ of HDE in terms of viscous NO cut-off for $c = c_1$ (Eq. (48)) in the Fig. 4. The Fig. 4 suggests us that with the evolution of universe i.e., with the increase in cosmic time $t$, $f_{reconstructed,NO,c_1}$ increases. The figure indicates that $f_{reconstructed,NO,c_1}$ is found to stay positive.

Case II: Behaviour of $f(t)$ when $c = c_2$ Using $c = c_2$ from Eq. (49) in Eq. (19), we get the density $\rho_{\Lambda,NO,c_2}$ of HDE in terms of NO cut-off.

$$\rho_{\Lambda,NO,c_2} = \frac{3 \left((-1 + n)^2 \alpha_0 + (t + C_5(-1 + n) t^n) ((-1 + n) \alpha_1 + (t + C_5(-1 + n) t^n) \alpha_2)\right)^2}{t^2 (1 + C_5(-1 + n) t^n)^2}$$  \hspace{1cm} (52)

Using this $\rho_{\Lambda,NO,c_2}$ from Eq. (52) in place of $\rho$ in Eq. (51), we solved it numerically by taking initial condition as $f(0.05) = 2$ and the numerical value of parameters as $n = 0.9$, $\alpha_0 = 0.6$, $\alpha_1 = 0.3$, $\alpha_2 = 0.2$ and $C_5 = 0.004$. Taking the cosmic time $t$ interval $[0.05, 3]$, we plotted the evolution of $f_{reconstructed,NO,c_2}$ versus $t$ of HDE in terms of viscous NO cut-off for $c = c_2$ (Eq. (49)) in the Fig. 5. The Fig. 5 suggests us that with the evolution of universe i.e., with the increase in cosmic time $t$, $f_{reconstructed,NO,c_2}$ increases.
The equation of motion for single-field inflation can be rewritten by Hamiltonian-Jacobi formulation. Here, scalar field is time variable and varies monotonically with time during any epoch. Hence, slow roll parameters can be written as:

\[ \epsilon_H = 3 \frac{\dot{\phi}^2}{V + \frac{\phi^2}{2}} \]  \hspace{1cm} (53)
and

\[ \eta_H = \epsilon_H - \frac{1}{2} \frac{\dot{\epsilon}_H}{H \epsilon_H} \quad (54) \]

During inflationary phase, warm inflation invokes a significant component of radiation. Let us assume, the inflation decay rate or dissipative coefficient by \( \Gamma \). It is responsible for decay of the scalar field into radiation during the inflationary expansion of the universe. It is given by (Lyth & Liddle 2009):

\[ \ddot{\phi} + (3H + \Gamma) \dot{\phi} + \frac{dV}{d\phi} = 0 \quad (55) \]

To study inflation, we assume the quintessence scalar field model of DE as it is very effective in the theory. The equation of energy density and pressure of quintessence scalar field are:

\[ \rho_\phi = \frac{1}{2} \dot{\phi}^2 + V(\phi) \quad (56) \]
\[ p_\phi = \frac{1}{2} \dot{\phi}^2 - V(\phi) \quad (57) \]

\( \phi \) is a scalar field and \( V(\phi) \) be its potential. From Eqns. (56) and (57), we have

\[ \dot{\phi} = \sqrt{\rho_\phi + p_\phi} \quad (58) \]

and

\[ V(\phi) = \frac{\rho_\phi - p_\phi}{2} \quad (59) \]

Let us consider the correspondence between viscous extended generalized holographic Ricci \( f(Q) \) gravity and quintessence scalar field model in bulk viscous framework. So, we have \( p_\phi = p_{rec,R} \quad (p_{rec,R} \text{ from Eq. (59)}) \) and \( \rho_\phi = \rho_{rec,R} \quad (p_{rec,R} \text{ from Eq. (60)}) \). Hence, from Eqns. (59), (60), (58) and (59), we have expressions of scalar field and potential as:

\[ \dot{\phi} = \frac{1}{\sqrt{2}} \left( -C_3(1 - 3n)^2(-2 + 3n)t^3 + 2t^{3n}(2c^2(2 + 9(-1 + n)n)t(-\alpha + n\beta) + (2 - 3n)^2t^2\xi_0 + n(4 + 3n(-5 + 3n))t\xi_1 + (-2 + n)n(-2 + 3n)(-1 + 3n)\xi_2) \right) \left( 2 + 9(-1 + n)n^{-1} \right)^2 \quad (60) \]

and

\[ V(t) = \frac{1}{6n} \left( -C_3(-1 + 3n)t^{3 - 3n} + (2(-2 + 3n)t (2c^2(1 - 3n)^2(-\alpha + n\beta) + 3(2 - 9n)t\xi_0) - 6n^2(-1 + 3n)(-4 + 9n)\xi_1 - 6(2 - 3n)^2n^2(-1 + 3n)\xi_2) \right) \left( 2 + 9(-1 + n)n^{-1} \right) \quad (61) \]

Using expression of \( \dot{\phi} \) from Eq. (60), \( V(t) \) from Eq. (61) in Eqns. (53) and (54), we get the Hubble slow roll parameters as:

\[ \epsilon_H = -3 \left( -C_3(1 - 3n)^2(-2 + 3n)t^4 + 2t^{3n}(2c^2(2 + 9(-1 + n)n)t(-\alpha + n\beta) + 3n((2 - 3n)^2t^2\xi_0 + n(4 + 3n(-5 + 3n))t\xi_1 + (-2 + n)n(-2 + 3n)(-1 + 3n)\xi_2)) \right) \left( 2 + 9(-1 + n)n^{-1} \right)^{-1} \]

\[ \eta_H = -3 \left( -C_3(1 - 3n)^2(-2 + 3n)t^4 + 2t^{3n}(2c^2(2 + 9(-1 + n)n)t(-\alpha + n\beta) + 3n((2 - 3n)^2t^2\xi_0 + n(4 + 3n(-5 + 3n))t\xi_1 + (-2 + n)n(-2 + 3n)(-1 + 3n)\xi_2)) \right) \left( 2 + 9(-1 + n)n^{-1} \right)^{-1} \quad (62) \]
and

\[ \eta_H = \left( -3 \left( C_3(1 - 3n)^2(-2 + 3n)t^3 - 2t^{3n} \left( 2c^2(2 + 9(-1 + n)n)\right(-\alpha + n\beta) + 3n \left( (2 - 3n)^2t^2\xi_0 + n(4 + 3n(-5 + 3n))\xi_1 + (-2 + n)n(-2 + 3n)(-1 + 3n)\xi_2) \right) \right) + \frac{1}{n}2t^{1+3n} \left( c^2(2 - 3n)^2(-1 + 3n)(-\alpha + n\beta) \left( C_3(2 + 9(-1 + n)n)^2t^3 - 6n^2t^{3n} \left( -9t^2\xi_0 + (4 + 3n(-5 + 3n))\xi_2 \right) \right) - 6n(2 + 9(-1 + n)n) \left( C_3(-2 + 3n)(t - nt)^2 \left( (-1 + 3n)^2\xi_0 + n(-2 + 3n)\xi_1 + 3(-1 + n)^2n\xi_2 \right) + 6n^3t^{3n} \left( 3t^2\xi_0\xi_1 + 4(-2 + 3n)\xi_0\xi_2 + n(-1 + 3n)\xi_1 \xi_2) \right) \right) \left( 2 \left( -C_3(1 - 3n)^2(-2 + 3n)t^3 + 2t^{3n} \left( 2c^2(2 + 9(-1 + n)n)\right(-\alpha + n\beta) + 3n \left( (2 - 3n)^2t^2\xi_0 + n(4 + 3n(-5 + 3n))\xi_1 + (-2 + n)n(-2 + 3n)(-1 + 3n)\xi_2) \right) \right) \right) \left( C_3(1 - 3n)^2(-2 + 3n)t^3 + 6n^3t^{3n} \left( c^2(2 + 9(-1 + n)n)\right(-\alpha + n\beta) + 3n(3t^2(-2 + 3n)\xi_0 + n(-1 + 3n)\xi_1 + n(2 + 9(-1 + n)n)\xi_2) \right) \right)^{-1} \]

(63)

We have plotted the evolution of Hubble slow roll parameters \( \epsilon_H \) and \( \eta_H \) against the cosmic time \( t \) in the Fig. 6 and Fig. 7 respectively. We observed from the figures that the Hubble slow roll parameters are less than 1 and are increasing with the cosmic time \( t \). This indicates that the model has a scope of exit from inflation. In the very early stages of inflation, the scalar field \( \phi \) is overdamped. Therefore, the condition of inflationary period is constrained by \( \frac{1}{2}\dot{\phi}^2 = V \) \( \text{[Kumar \\& Xu 2014]} \). In this section, viscous extended generalized holographic Ricci \( f(Q) \) gravity are assumed to provide the inflationary scenario. Hence, for the inflationary expansion to happen we should have \( 2V - \dot{\phi}^2 > 0 \) \( \text{[Brevik et al. 2017a, Odintsov et al. 2018]} \). Using \( \phi \) and \( V(t) \) from Eqs. (60) and (61) respectively, we have

\[ 2V - \dot{\phi}^2 = -2 \left( c^2(-2 + 3n)t(-\alpha + n\beta) + 6n \left( t^2\xi_0 + nt\xi_1 + (-1 + n)n\xi_2) \right) \right) t^{-3} \]

(64)

We plotted \( 2V - \dot{\phi}^2 \) against cosmic time \( t \) in the Fig. 8. From the figure we see that \( 2V - \dot{\phi}^2 > 0 \) for the range \( 0.00001 < t < 0.1 \). Hence, quasi exponential expansion is available for \( 0.00001 < t < 0.1 \) of viscous extended generalized holographic Ricci \( f(Q) \) gravity. Now we will study the behaviour of
Fig. 7: Evolution of Hubble slow roll parameter $\eta_H$ (Eq. (63)) against the cosmic time $t$ of viscous extended generalized holographic Ricci $f(Q)$ gravity. We consider $c = 0.009, \xi_0 = 0.001, \xi_2 = 0.0023, \beta = 0.0004, n = 0.006, C_3 = -0.4$. The red green and blue line corresponds to $\alpha = 0.999997, 0.999998$ and $0.999999$ respectively.

Fig. 8: Evolution of $2V - \dot{\phi}^2$ (Eq. (64)) against the cosmic time $t$ of viscous extended generalized holographic Ricci $f(Q)$ gravity. We consider $c = 0.02, \xi_0 = 0.06, \xi_1 = 0.009, \alpha = 0.05, \beta = 0.04$. The red, green and blue line corresponds to $n = 0.40, 0.41$ and $0.42$ respectively.

dissipative coefficient $\Gamma$. Differentiating Eqns. (60) and (61) with respect to $t$, we get

$$\ddot{\phi} = t^{-4-3n} \left( 3C_3(1-3n)^2n(-2+3n)t^{3} - 2t^{3n} \left( 4c^2(2+9(-1+n))n((-\alpha+n\beta)+3n((2-3n)^2t^2\xi_0+2n(4+3n(-5+3n))t\xi_1+3(-2+n)n(-2+3n)(-1+3n)\xi_2) )\right) \right)$$

$$\left( t^{-3(1+n)} \left( -C_3(1-3n)^2(-2+3n)t^{3} + 2t^{3n} \left( 2c^2(2+9(-1+n))n((-\alpha+n\beta)+3n((2-3n)^2t^2\xi_0+n(4+3n(-5+3n))t\xi_1+3(-2+n)n(-2+3n)(-1+3n)\xi_2) )\right) \right) \right)$$

$$\left( 2+9(-1+n)n^{-1}\right)^{-1}$$

(Eq. (65))
Fig. 9: Evolution of dissipative coefficient $\Gamma$ (Eq. (68)) against the cosmic time $t$ of viscous extended generalized holographic Ricci $f(Q)$ gravity. We consider $C_3 = 0.8, c = 0.02, \xi_0 = 0.06, \xi_1 = 0.001, \xi_2 = 0.009, \alpha = 0.4, \beta = 0.04$. The red, green and blue line corresponds to $n = 0.31, 0.32$ and $0.33$ respectively.

and

$$\dot{V}(t) = t^{-4-3n}(3C_3(1-3n)^2n(-2+3n)t^3 + 2t^{3n}((-2+3n)t (-4c^2(1-3n)^2(-\alpha + n\beta) + 3n(-2+9n)\xi_0) + 6n^2(-1+3n)(-4+9n)t\xi_1 + 9(2-3n)^2n^2(-1+3n)\xi_2)) \left(4(2 + 9(-1 + n)n)^{-1}\right)$$

Hence,

$$\frac{dV}{d\phi} = t^{-4-3n}(3C_3(1-3n)^2n(-2+3n)t^3 + 2t^{3n}((-2+3n)t (-4c^2(1-3n)^2(-\alpha + n\beta) + 3n(-2+9n)\xi_0) + 6n^2(-1+3n)(-4+9n)t\xi_1 + 9(2-3n)^2n^2(-1+3n)\xi_2)) \left(2\sqrt{2}(2 + 9(-1 + n)n)\right)\left((t^{-3(1+n)}(-3C_3(1-3n)^2(-2+3n)t^3 + 2t^{3n}(2c^2(2 + 9(-1 + n)n)(-\alpha + n\beta) + 3n((2-3n)^2t^2\xi_0 + n(4 + 3n(-5 + 3n))\xi_1 + (-2 + n)n(-2 + 3n)(-1 + 3n)\xi_2))\right)^{-1}$$

Using expression of $\ddot{\phi}$ from Eq. (65), $\frac{dV}{d\phi}$ from Eq. (67), $H$ from Eq. (17), $\dot{\phi}$ from Eq. (60) in Eq. (55) we get $\Gamma$ as

$$\Gamma = 18n^2(2 + 9(-1 + n)n)t^{-1+3n}(t^2\xi_0 + nt\xi_1 + (-1 + n)n\xi_2)$$

$$\frac{C_3(1-3n)^2}{(2 + 3n)t^3 - 2t^{3n}(2c^2(2 + 9(-1 + n)n)(-\alpha + n\beta) + 3n((2-3n)^2t^2\xi_0 + n(4 + 3n(-5 + 3n))\xi_1 + (-2 + n)n(-2 + 3n)(-1 + 3n)\xi_2))^{-1}$$

We have plotted $|\Gamma|$ against cosmic time $t$ in the Fig 8. From the figure, we observed that $|\Gamma| > 1$. This implies warm inflation which means the decay of scalar field into radiation during the inflationary phase.
8 CONCLUSIONS

The study demonstrated in this paper has been aimed at studying the behaviour of $f(Q)$ gravity with different cut-offs. We have reconstructed thermodynamic pressure and density of viscous extended generalized holographic Ricci $f(Q)$ gravity. We then reconstructed its EoS parameter and studied it accordingly. For studying the viscous case, we have adopted an Eckart approach. This approach has been adopted because it is consistent with the observational data. The bulk viscous pressure, $\Pi = -3H\xi$, where $\xi = \xi_0 + \xi_1 H + \xi_2(H + H^2)$ (Jimenez et al. 2018; Ren & Meng 2006). We have studied the $f(Q)$ gravity with different cut-offs also. From the history of the expansion of the universe, the reconstruction of modified gravity was demonstrated (Nojiri & Odintsov 2007a).

In the Section 2, we have assumed that the size of the universe to be Hubble radius $H^{-1}$. It is to be noted that, by considering this, the density of HDE becomes $\rho_\Lambda = 3c^2H^2$. This density is very close to the observational value.

In the Section 3, we have demonstrated the Field Equations (12 and 13) of $f(Q)$ gravity. For this we have written action of symmetric teleparallel gravity in equation (2).

In Section 4, we have studied the case non-viscous holographic $f(Q)$ gravity. To study this, we have considered power law form of scale factor $a(t) = a_0t^n, n > 0$. From the conservation equation (14), we deduced thermodynamic pressure of non-viscous HDE as $p_\Lambda$ in Eq. (21). Using $p_\Lambda$ in the second field equation (13), we derived $f(t)$ in Eq. (25). Hence, we got $f(Q)$ in Eq. (27). We then plotted $f(Q)$ against $Q$ of non-viscous holographic $f(Q)$ gravity in Fig. 1. We can see that $f(Q)$ stays at positive level and with the increase of $Q$, $f(Q)$ increases. We also noted that as $f(Q) \to 0$, $Q \to 0$.

In Section 5, we have demonstrated the case ”viscous extended generalized holographic Ricci $f(Q)$ gravity”. Here, we have reconstructed $f(Q)$ gravity in Eq. (58) and plotted it in Fig. 2 against $Q$ of $f(Q)$. The figure also indicates same as that of Fig. 1. It explains us that $f(Q)$ is positive and as $Q$ increases, $f(Q)$ increases. As $f(Q) \to 0$, $Q \to 0$. In Eqs. (49) and (50), we reconstructed the pressure $p_{rec,R}$ and density $\rho_{rec,R}$ of viscous extended generalized holographic Ricci $f(Q)$ gravity respectively. Hence, reconstructed EoS parameter $w_{rec,R}$ of Eq. (41) of viscous extended generalized holographic Ricci $f(Q)$ gravity is plotted against cosmic time $t$ in Fig. 3. The figure suggests us that this model indicates the avoidance of Big-Rip singularity.

In Section 6, the case ”viscous holographic $f(Q)$ gravity with Nojiri-Odintsov cut-off”. In this case, we have reconstructed infrared cut-off $L$ in Eq. (66) and reconstructed density in Eq. (47) of NO HDE. Two expressions of $c$ we deduced and named it as $c_1$ and $c_2$ in Eqs. (48) and (49). We then proceeded in two cases i.e., case I for expression $c_1$ (Eq. (48)) and case II for expression $c_2$ (Eq. (49)). In case I, we deduced the density $\rho_{A,NO,c_1}$ (Eq. (50)) of HDE in terms of No cut-off. Hence, plotted $f_{\text{reconstructed,NO,c_1}}$ versus cosmic time $t$ in Fig. 4 of this case. In case II, we deduced the density $\rho_{A,NO,c_2}$ (Eq. (52)) of HDE in terms of No cut-off. Hence, plotted $f_{\text{reconstructed,NO,c_2}}$ versus cosmic time $t$ in Fig. 5 of this case. Both the figures i.e., Figs. 4 and 5 suggests us that with the cosmic time $t$, $f_{\text{reconstructed,NO,c_1}}$ and $f_{\text{reconstructed,NO,c_2}}$ increases and is found to stay at positive level.

In Section 7, we have studied the contribution of slow roll parameters in $f(Q)$ gravity. Here we have assumed the background evolution to be viscous extended generalized holographic Ricci $f(Q)$ gravity. The Hubble slow roll parameters are less than 1 and $\epsilon_H$ (Eq. (62)) and $\eta_H$ (Eq. (63)) are increasing (Figs. 6 and 7). This indicates that there is a scope of exit from inflation. It is observed from the Fig. 8 that $2V - \Omega' > 0$. This indicates that quasi exponential expansion is available for this model. In the Fig. 8 we can see that $|\Gamma'| > 1$ i.e., the warm inflationary expansion can be obtained from this model.
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