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A Linear Operator Method to Compute the Normal Modes with Rotation of any Asymmetric 3D Planet with Pure Vector Spherical Harmonics

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Abstract

In order to compute the free core nutation of the terrestrial planets, such as Earth and Mars, the Moon and lower degree normal modes of the Jovian planets, we propose a linear operator method (LOM). Generalized surface spherical harmonics (GSSHs) are usually applied to the elliptical models with a stress tensor, which cannot be expressed in vector spherical harmonics explicitly. However, GSSHs involve complicated math. LOM is an alternative to GSSHs, whereas it only deals with the coupling fields of the same azimuthal order m, as is the case when a planet model is axially symmetric and rotates about that symmetry axis. We extend LOM to any asymmetric 3D model. The lower degree spheroidal modes of the Earth are computed to validate our method, and the results agree very well with what is observed. We also compute the normal modes of a two-layer Saturn model as a simple application.

Key words: planets and satellites: interiors – Earth – planets and satellites: gaseous planets

1. Introduction

Free core nutation (FCN) is a normal or eigenmode of the rotating Earth, if only there is a fluid core which is elliptical rather than spherical, and it is related to slight misalignment of the rotation axes of the Earth's fluid core and mantle. When studying rotational normal modes, nutation or solid tides of Earth, researchers usually start from an infinite set of coupled ordinary differential equations (ODEs) along a radius that governs the infinitesimal free elastic-gravitational oscillations of a rotating, slightly elliptical Earth, as well as a set of boundary conditions on displacement vector, stress tensor and gravity potential, as described in Smith (1974). Earth is usually treated as a slightly elliptical symmetrical oblate body, i.e., a rotating symmetric model. The classical way to expand these equations with such an oblate model is by means of generalized surface spherical harmonics (GSSHs), which were proposed for quantum mechanics originally (Edmonds 2016) and introduced into the geophysics area by Phinney & Burridge (1973) (see also Huang & Liao 2003 for corrections and comments), who presented the expansions of any order tensors in GSSH representation and computed the excitation of a point force. Smith (1974) adopted this GSSH representation, and projected all parameters from an elliptical domain onto an equivalent spherical domain, and obtained notable results of Earth nutation and wobble (Smith 1977). Along the way, a series of theoretical studies on nutation and rotational modes of the non-rigid Earth have been performed, e.g., Wahr (1981), Dehant & Defraigne (1997), Schastok (1997), Huang et al. (2001, 2011), Rogister (2001), etc. All these studies treated Earth as a rotating symmetric oblate object and its flattening was small. However, it is not easy to understand GSSH representation which involves the knowledge of group theory, representation theory and Lie algebra. Rochester et al. (2014) developed a linear operator method (LOM) as an alternative to the combination of J-squares and GSSHs, and used it in their subsequent studies of wobble/nutation. In their view, the new combination has at least two advantages: (1) using the familiar conventional Y_m^n removes any need for the more complicated GSSH notation and (2) using the operators facilitates writing the coefficients of the ODEs in a much more compact form. However, their work only deals with the coupling fields of the same azimuthal order m, as is the case when the reference model is axially symmetric and rotates about that symmetry axis.

We follow their way and develop LOM so that LOM can deal with any asymmetric three-dimensional (3D) Earth (or other planet) model rather than a slightly elliptical model. This method does not rely on GSSHs or Wigner 3-j symbols, which makes it easier for physicists and engineers.

To demonstrate and validate our method and code, we first compute the lower degree normal modes of Earth, which agree very well with what is observed. Then we compute the normal modes of simple two-layer polytropic Saturn models which are the non-rotating sphere, rotating sphere and rotating oblate planet.

2. Linear Operator Method

2.1. Background

Let P be a particle in reference domain V_E in hydrostatic equilibrium, and $\boldsymbol{r}$ its position vector. The equilibrium density field and gravitational potential field are denoted by $\rho(\boldsymbol{r})$ and $\phi(\boldsymbol{r})$

respectively, while $\boldsymbol{u}(\boldsymbol{r}, t)$ signifies the infinitesimal Lagrangian displacement vector of particle P at time t.

The dynamic equation for infinitesimal elastic-gravitational motion $\boldsymbol{u}$ for a rotating Earth model, in a steadily rotating reference frame with constant speed Ω_0, is given in Smith (1974) (or Dahlen & Tromp 1998 for more information),

$$\begin{aligned}&\rho D_t^2\boldsymbol{u} + 2\rho\boldsymbol{\Omega}_0 \times D_t\boldsymbol{u}\\ &= -\rho\boldsymbol{\Omega}_0 \times (\boldsymbol{\Omega}_0 \times \boldsymbol{u}) + \nabla \cdot \overleftrightarrow{S^e} - \nabla(\gamma\nabla \cdot \boldsymbol{u})\\ &\quad -\rho\nabla\phi_1 - \rho\boldsymbol{u} \cdot \nabla\nabla\phi + \nabla \cdot [\gamma(\nabla\boldsymbol{u})^T],\end{aligned} \quad (1)$$

where γ is the equilibrium pressure, and ϕ_1 is the incremental gravitational potential induced by the mass redistribution due to deformation which satisfies the Poisson equation,

$$\nabla^2\phi_1 = 4\pi G\nabla \cdot (\rho\boldsymbol{u}). \quad (2)$$

The stress tensor $\overleftrightarrow{S^e}$ is the incremental elastic stress with respect to the reference stress, $\overleftrightarrow{S^{\text{ref.}}} = -\gamma\overleftrightarrow{I}$, where $\overleftrightarrow{I}$ is the identity tensor, and is related to the displacement field by two Lamé parameters (λ, μ). For an isotropic medium,

$$\overleftrightarrow{S^e} = \lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I} + \mu[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T], \quad (3)$$

where the rigidity $\mu = 0$ in a liquid part.

Usually, all the physical field functions, like $\boldsymbol{u}(\boldsymbol{r}, t)$, are combinations of oscillations $\boldsymbol{u}(\boldsymbol{r}, t) = \sum_\omega \boldsymbol{u}(\boldsymbol{r}, \omega)e^{j\omega t}$. We will omit $e^{j\omega t}$ parts in the following discussion. Then, these field functions are expanded by spherical harmonics (SHs), vector spherical harmonics (VSHs) or GSSHs. Here is an example to expand a complex square-integrable tensor field of second order $\overleftrightarrow{S^e}(\theta, \phi)$ on the surface of a unit sphere in terms of GSSHs,

$$\begin{aligned}\overleftrightarrow{S^e}(\boldsymbol{r}) &= \overleftrightarrow{S^e}(r, \theta, \phi)\\ &= \sum_{n=0}^{\infty}\sum_{m=-n}^{n} S_n^{m\ \alpha\beta}(r)Y_n^{m\ (\alpha+\beta)}(\theta, \phi)\hat{e}_\alpha\hat{e}_\beta,\end{aligned} \quad (4)$$

where α and β take one of $(-, 0, +)$ (Dahlen & Tromp 1998 for more information).

Scalar and vector fields can be expanded in a similar way in GSSHs. However, LOM does not have a unified form like Equation (4), and the tensor is represented in $\hat{r}\nabla_1 Y_n^m(\theta, \phi)$, $\nabla_1\nabla_1 Y_n^m(\theta, \phi)$, $\nabla_1[\hat{r} \times \nabla_1 Y_n^m(\theta, \phi)]^T$, etc. There is no need to represent stress tensor $\overleftrightarrow{S}$ in a stand-alone form, and it only comes with the form $\hat{n} \cdot \overleftrightarrow{S}$ or $\nabla \cdot \overleftrightarrow{S}$ in the dynamical equations and the boundary conditions, which can be represented by LOM.

2.2. Overview of Three Basic Linear Operators

SHs are indeed a linear combination of $\cos\theta$ and $\sin\theta e^{\pm i\phi}$. The basic idea of LOM is to decompose $Y_n^m(\theta, \phi)$ to $(\cos\theta)^k$ and $(\sin\theta e^{\pm i\phi})^l$. $\cos\theta$ and $\sin\theta e^{\pm i\phi}$ are repeatedly used as three basic operators. If the actions of these three basic operators are known, then any actions of SHs are also known.

In Appendix A.1, $\cos\theta$ is shown how to act on VSHs ($\boldsymbol{R}_n^m(\theta, \phi)$, $\boldsymbol{S}_n^m(\theta, \phi)$ and $\boldsymbol{T}_n^m(\theta, \phi)$), for instance

$$\begin{aligned}\cos\theta * \boldsymbol{S}_n^m(\theta, \phi) &= d_7(n, m) * \boldsymbol{S}_{n+1}^m(\theta, \phi)\\ &\quad + d_8(n, m) * \boldsymbol{S}_{n-1}^m(\theta, \phi) + d_9(n, m) * \boldsymbol{T}_n^m(\theta, \phi).\end{aligned} \quad (5)$$

This $\cos\theta*$ operator is enough to deal with rotating symmetric (ellipsoid) models. In such kind of Earth models, all the dynamical variables and parameters are independent of longitude ϕ; thus we do not need to deal with $(\sin\theta e^{\pm i\phi})^l$. Appendix A.2 presents the action of $\sin\theta e^{\pm i\phi}$. For instance,

$$\begin{aligned}\sin\theta e^{i\phi} * \boldsymbol{S}_n^m(\theta, \phi) &= d_{14}(n, m) * \boldsymbol{S}_{n+1}^{m+1}(\theta, \phi)\\ &\quad + d_{15}(n, m) * \boldsymbol{S}_{n-1}^{m+1}(\theta, \phi) + d_{16}(n, m) * \boldsymbol{T}_n^{m+1}(\theta, \phi).\end{aligned} \quad (6)$$

After the three basic operators' actions are specified, $Y_n^m(\theta, \phi)$'s action can be built up by them. In Appendix B.1, we will discuss how to express the dot product of two VSHs. The basic idea is to transform vectors into the form of three basic operators. For instance, the dot product of $\boldsymbol{S}_n^m(\theta, \phi) \cdot \boldsymbol{\Psi}_a^b(\theta, \phi)$ (where $\boldsymbol{\Psi}_a^b(\theta, \phi)$ denotes any of $\boldsymbol{R}_a^b(\theta, \phi)$, $\boldsymbol{S}_a^b(\theta, \phi)$ or $\boldsymbol{T}_a^b(\theta, \phi)$) is complicated, so we can decompose $\boldsymbol{S}_n^m(\theta, \phi)$ into

$$\begin{aligned}\boldsymbol{S}_n^m(\theta, \phi) &= \cdots(\cos\theta)^k\,(\sin\theta e^{\pm i\phi})^m\,\nabla_1\cos\theta\\ &\quad + \cdots(\cos\theta)^l\,(\sin\theta e^{\pm i\phi})^{m-1}\,\nabla_1(\sin\theta e^{\pm i\phi}).\end{aligned} \quad (7)$$

$\nabla_1\cos\theta \cdot \boldsymbol{\Psi}_a^b(\theta, \phi)$ and $\nabla_1(\sin\theta e^{\pm i\phi}) \cdot \boldsymbol{\Psi}_a^b(\theta, \phi)$ are easily transformed to the combinations of $Y_a^b(\theta, \phi)$, while $\cos\theta * Y_a^b(\theta, \phi)$ and $\sin\theta e^{\pm i\phi} * Y_a^b(\theta, \phi)$ are already known, so $\boldsymbol{S}_n^m(\theta, \phi) \cdot \boldsymbol{\Psi}_a^b(\theta, \phi)$ can also be easily represented as a combination of the SHs.

In Appendix C we will discuss the divergence of a stress tensor. A tensor $\overleftrightarrow{S}$ is usually represented in a combination in dyadic form

$$\overleftrightarrow{S} = \boldsymbol{v}_1 \otimes \boldsymbol{v}_2 + \boldsymbol{v}_3 \otimes \boldsymbol{v}_4 + \cdots, \quad (8)$$

where the symbol $\otimes$ means linking two vectors to make a dyad, and it can be omitted for short. The divergence of a dyadic and the dot product of a vector with a dyadic are easily obtained. However, not all parts of a stress tensor can be written easily in dyadic form by VSHs. Fortunately, only $\hat{n} \cdot \overleftrightarrow{S}$ and $\nabla \cdot \overleftrightarrow{S}$ are needed, which could be written in VSHs. When a part cannot be written in dyadic form explicitly, we can follow this principle: transform it to three basic operators' form. For instance, to compute $\boldsymbol{\Psi}_a^b(\theta, \phi) \cdot \nabla_1\nabla_1 Y_n^m(\theta, \phi)$, as $Y_n^m(\theta, \phi)$ is a combination of $(\cos\theta)^k$ and $(\sin\theta e^{\pm i\phi})^l$, the tensor $\nabla_1\nabla_1 Y_n^m(\theta, \phi)$ can be represented as below

$$\begin{aligned}
&\nabla_1\nabla_1[(\cos\theta)^k(\sin\theta e^{\pm i\phi})^l]\\
&= \nabla_1[(\sin\theta e^{\pm i\phi})^l * k * (\cos\theta)^{k-1}\nabla_1\cos\theta]\\
&+ \nabla_1[(\cos\theta)^k * l * (\sin\theta e^{\pm i\phi})^{l-1}\nabla_1(\sin\theta e^{\pm i\phi})]\\
&= \cdots\nabla_1\nabla_1\cos\theta + \cdots\nabla_1(\sin\theta e^{\pm i\phi})\otimes\nabla_1\cos\theta\\
&+\cdots\nabla_1\cos\theta\otimes\nabla_1\cos\theta + \cdots\nabla_1\nabla_1(\sin\theta e^{\pm i\phi})\\
&+\cdots\nabla_1\cos\theta\otimes\nabla_1(\sin\theta e^{\pm i\phi}) + \cdots\nabla_1(\sin\theta e^{\pm i\phi})\\
&\otimes\nabla_1(\sin\theta e^{\pm i\phi}).
\end{aligned} \tag{9}$$

The second, third, fifth and sixth terms in the above equation are dyadics. The dot product of a vector and a dyad is

$$\boldsymbol{\Psi}_a^b(\theta,\phi)\cdot\boldsymbol{v}_1\otimes\boldsymbol{v}_2 = [\boldsymbol{\Psi}_a^b(\theta,\phi)\cdot\boldsymbol{v}_1]\boldsymbol{v}_2. \tag{10}$$

$\nabla_1\nabla_1\cos\theta$ and $\nabla_1\nabla_1(\sin\theta e^{\pm i\phi})$ cannot be written in VSH form explicitly, so they are transformed into a spherical coordinate basis, for instance,

$$\nabla_1\nabla_1\cos\theta = \sin\theta * \hat{\theta}\hat{r} - \cos\theta(\hat{\theta}\hat{\theta} + \hat{\phi}\hat{\phi}). \tag{11}$$

Moreover $\boldsymbol{\Psi}_a^b(\theta,\phi)\cdot\nabla_1\nabla_1\cos\theta$ is easy to be transformed to VSHs. So, $\boldsymbol{\Psi}_a^b(\theta,\phi)\cdot\nabla_1\nabla_1 Y_n^m(\theta,\phi)$ can be written in VSHs.

The cross product of two VSHs is discussed in Appendix B.4 using a method similar to dot product. The abstract forms of the final dynamic equations are listed in Appendix D. All the useful and fundamental formulae are attached in appendices. We hope this paper can be a useful manual for those who are challenged by complex maths and want a more straightforward approach.

The LOM uses three kinds of relations: the recursion relations of Legendre functions, the relations transforming $\boldsymbol{u} = u_r\hat{r} + u_\theta\hat{\theta} + u_\phi\hat{\phi}$ to VSHs and the Leibniz rule reducing complex composition to the three basic operators.

In this paper, $c_n(n=0, 1, 2\cdots)$ are coefficients. c_n^{-1} is the reciprocal of c_n. $M_n(n=0, 1, 2, \cdots)$ are linear maps which satisfy

$$\begin{aligned}
M_n[\psi + \varphi] &= M_n[\psi] + M_n[\varphi]\\
M_n[x_1\psi] &= x_1 M_n[\psi],
\end{aligned} \tag{12}$$

where ψ and φ are SHs or VSHs, and x_1 is a constant number. The symbol $*$ denotes the algebraic product of a scalar field (or an algebraic constant) with a tensor, a vector or a scalar. The symbol $\times$ signifies a cross product of two vectors. The symbol $\cdot$ means a dot (or inner) product of two vectors.

2.3. Equation Expansion

The displacement field $\boldsymbol{u}$ is represented in spherical coordinates by VSHs

$$\boldsymbol{u} = \sum_{n=0}^{\infty}\sum_{m=-n}^{n}\mathbb{S}_n^m + \boldsymbol{T}_n^m, \tag{13}$$

with

$$\mathbb{S}_n^m = [u_n^m\hat{r} + v_n^m\nabla_1]Y_n^m = u_n^m\boldsymbol{R}_n^m + v_n^m\boldsymbol{S}_n^m, \tag{14}$$

$$\boldsymbol{T}_n^m = -w_n^m\boldsymbol{r}\times\nabla_1 Y_n^m, \tag{15}$$

where u_n^m, v_n^m and w_n^m are functions of r only. $\boldsymbol{S}_n^m$ is a spheroidal oscillation, and $\boldsymbol{T}_n^m$ is a toroidal oscillation. Suppose that a normal mode has the eigen angular velocity ω, and its displacement field is

$$\boldsymbol{u}(t) = \boldsymbol{u}(\omega)e^{i\omega t}. \tag{16}$$

We will omit ω in $\boldsymbol{u}(\omega)$, and just use $\boldsymbol{u}$ for short. Dropping terms higher than the first order in $\boldsymbol{u}$, Equation (1) becomes

$$\begin{aligned}
&\omega^2\boldsymbol{u} - 2i\omega\boldsymbol{\Omega}_0\times\boldsymbol{u} + \nabla\phi_1 + \nabla(\boldsymbol{u}\cdot\boldsymbol{g}_0) - \boldsymbol{g}_0\nabla\cdot\boldsymbol{u}\\
&+ \frac{1}{\rho_0}\nabla\cdot\overleftrightarrow{S^e} = 0.
\end{aligned} \tag{17}$$

To compute the normal modes, we should know how vector $\boldsymbol{u}$ is operated on. As $\boldsymbol{u}$ is expanded in the VSHs, we just need to know how VSHs are operated on. There are six terms in the above equation. The first term $\omega^2\boldsymbol{u}$ is trivial because it is just the $\boldsymbol{u}$ with a coefficient. The second term $2i\omega\boldsymbol{\Omega}_0\times\boldsymbol{u}$ shows that we should deal with the cross product of $\boldsymbol{\Omega}_0$ (which is $\hat{z}$) and $\boldsymbol{u}$, which is studied in Appendix B.4. The third term $\nabla\phi_1$ is a gradient of SHs. The gradient in spherical coordinates is

$$\nabla = \hat{r}\partial_r + \frac{1}{r}\nabla_1. \tag{18}$$

∇_1 is the surface gradient on a unit sphere

$$\nabla_1 = \hat{\theta}\partial_\theta + \frac{\hat{\phi}}{\sin\theta}\partial_\phi. \tag{19}$$

So, the gradient of the SH $Y_n^m(\theta,\phi)$ is $r^{-1}\boldsymbol{S}_n^m(\theta,\phi)$.

The fourth term of Equation (17) is $\nabla(\boldsymbol{u}\cdot\boldsymbol{g}_0)$, which is the gradient of a dot product. The dot product of two VSHs is discussed in Appendix B.2, and the gradient of an SH is just discussed above.

The fifth term is $\boldsymbol{g}_0\nabla\cdot\boldsymbol{u}$, which is the product of $\boldsymbol{g}_0$ and the divergence of $\boldsymbol{u}$. The divergence of a VSH is discussed at the beginning of Appendix C.

The sixth term is $\frac{1}{\rho_0}\nabla\cdot\overleftrightarrow{S^e}$, which is the divergence of the stress tensor $\overleftrightarrow{S^e}$,

$$\nabla\cdot\overleftrightarrow{S^e} = \nabla\cdot\{\lambda(\nabla\cdot\boldsymbol{u})\boldsymbol{I} + \mu[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T]\}. \tag{20}$$

This derivation is pretty complex, and we discuss this term in Appendix C.

There are so many symbolic computations in this work that it exceeds a human's ability. Mathematica, Maxima and Maple are good software for symbolic computation, however they cannot meet our requirements. So, we write software to do these special symbolic computations in Common Lisp.

2.4. Simple Instances

We will give two instances to demonstrate this method. The first is to expand the governing Equation (17) with a very simple $\boldsymbol{u}$. The second instance is to expand $\nabla \cdot \overleftrightarrow{S^e}$.

2.4.1. The First Instance

Suppose the displacement vector has a very simple form

$$\boldsymbol{u} = u_0 \boldsymbol{S}_2^0, \tag{21}$$

for which u_0 is a constant. The first term in Equation (17) is

$$\omega^2 \boldsymbol{u} = \omega^2 u_0 \boldsymbol{S}_2^0. \tag{22}$$

The second term in Equation (17) is $-2i\omega\boldsymbol{\Omega}_0 \times \boldsymbol{u}$, and assuming $\boldsymbol{\Omega}_0 = \hat{z}$, it then becomes

$$-2i\omega\boldsymbol{\Omega}_0 \times \boldsymbol{u} = -2i\omega\hat{z} \times \boldsymbol{u} = -2i\omega u_0 \hat{z} \times \boldsymbol{S}_2^0. \tag{23}$$

By Equation (G6), d_5 and d_6 in Equation (25) are

$$\begin{aligned} d_5(2, 0) &= -2d_1(2, 0) - d_3(2, 0)\\ d_6(2, 0) &= -2d_2(2, 0) - d_4(2, 0). \end{aligned} \tag{26}$$

By Equation (G4), d_3 and d_4 in Equation (26) are

$$\begin{aligned} d_3(2, 0) &= c_3(2, 0) * 2 * c_4(2, 0)c_3^{-1}(3, 0)\\ d_4(2, 0) &= c_3(2, 0) * 2 * c_5(2, 0)c_3^{-1}(1, 0)\\ &\quad - c_3(2, 0)2c_3^{-1}(1, 0). \end{aligned} \tag{27}$$

By Equation (G2), d_1 and d_2 in Equations (25) and (26) are

$$\begin{aligned} d_1(2, 0) &= c_3(2, 0)c_4(2, 0)c_3^{-1}(3, 0)\\ d_2(2, 0) &= c_3(2, 0)c_5(2, 0)c_3^{-1}(1, 0). \end{aligned} \tag{28}$$

By Equations (E3), (E4) and (E5), we can get

$$\begin{aligned} c_3(1, 0) &= (-1)^0\left(\frac{2+1}{4\pi}\right)^{1/2}\left[\frac{(1-0)!}{(1+0)!}\right]^{1/2} = 0.4886025119029199\\ c_3(2, 0) &= (-1)^0\left(\frac{2*2+1}{4\pi}\right)^{1/2}\left[\frac{(2-0)!}{(2+0)!}\right]^{1/2} = 0.6307831305050401\\ c_3(3, 0) &= (-1)^0\left(\frac{2*3+1}{4\pi}\right)^{1/2}\left[\frac{(3-0)!}{(3+0)!}\right]^{1/2} = 0.7463526651802308\\ c_4(2, 0) &= \frac{2-0+1}{2*2+1} = 0.6\\ c_5(2, 0) &= \frac{2+0}{2*2+1} = 0.4. \end{aligned} \tag{29}$$

By Equation (H5), $\hat{z} \times \boldsymbol{S}_2^0(\theta, \phi)$ becomes

$$\begin{aligned} \hat{z} \times \boldsymbol{S}_2^0(\theta, \phi) &= -d_7(2, 0) * \boldsymbol{T}_{2+1}^0(\theta, \phi) - d_8(2, 0) * \boldsymbol{T}_{2-1}^0(\theta, \phi)\\ &\quad + d_9(2, 0) * \boldsymbol{S}_2^0(\theta, \phi) - i * 0 * c_1(2)\boldsymbol{R}_2^0(\theta, \phi),\\ &= -d_7(2, 0) * \boldsymbol{T}_3^0(\theta, \phi) - d_8(2, 0) * \boldsymbol{T}_1^0(\theta, \phi)\\ &\quad + d_9(2, 0) * \boldsymbol{S}_2^0(\theta, \phi). \end{aligned} \tag{24}$$

By Equation (G8), the coefficients d_7, d_8 and d_9 are

$$\begin{aligned} d_7(2, 0) &= c_1(2)[d_1(2, 0)c_1^{-1}(2+1)\\ &\quad + d_5(2, 0)] = c_1(2)[d_1(2, 0)c_1^{-1}(3) + d_5(2, 0)]\\ d_8(2, 0) &= c_1(2)[d_2(2, 0)c_1^{-1}(2-1)\\ &\quad + d_6(2, 0)] = c_1(2)[d_2(2, 0)c_1^{-1}(1) + d_6(2, 0)]\\ d_9(2, 0) &= -i * 0 * c_1(2) = 0. \end{aligned} \tag{25}$$

Then d_1, d_2, d_3, d_4, d_5 and d_6 in Equations (28), (27) and (26) become

$$\begin{aligned} d_1(2, 0) &= 0.5070925528371101\\ d_2(2, 0) &= 0.5163977794943224\\ d_3(2, 0) &= 1.014\,185\,105\,674\,220\,2\\ d_4(2, 0) &= -1.5491933384829668\\ d_5(2, 0) &= -2.0283702113484403\\ d_6(2, 0) &= 0.516\,397\,779\,494\,322. \end{aligned} \tag{30}$$

By Equation (E1), $c_1(n)$ becomes

$$\begin{aligned} c_1(2) &= \frac{1}{2(2+1)} = \frac{1}{6}\\ c_1^{-1}(1) &= \frac{1}{c_1(1)} = \frac{1(1+1)}{1} = 2\\ c_1^{-1}(3) &= \frac{1}{c_1(3)} = \frac{3(3+1)}{1} = 12. \end{aligned} \tag{31}$$

By Equations (30) and (31), d_7 and d_8 in Equation (25) are

$$d_7(2,\,0) = 0.676\,123\,403\,782\,813\,5$$
$$d_8(2,\,0) = 0.258\,198\,889\,747\,161\,1. \tag{32}$$

Then $\hat{z} \times \boldsymbol{S}_2^0(\theta,\,\phi)$ in Equation (24) becomes

$$\hat{z} \times \boldsymbol{S}_2^0(\theta,\,\phi) = -0.6761234037828135 * \boldsymbol{T}_3^0(\theta,\,\phi) - 0.2581988897471611 * \boldsymbol{T}_1^0(\theta,\,\phi). \tag{33}$$

Finally the second term in Equation (17) becomes

$$-2i\omega\boldsymbol{\Omega}_0 \times \boldsymbol{u} = 1.3522468075656267 i\omega u_0 \boldsymbol{T}_3^0 + 0.5163977794943223 i\omega u_0 \boldsymbol{T}_1^0. \tag{34}$$

Suppose

$$\phi_1 = \phi_{10} Y_2^0(\theta,\,\phi), \tag{35}$$

and ϕ_{10} is a constant, then the third term $\nabla\phi_1$ becomes

$$\nabla\phi_1 = \frac{\partial}{\partial r}\{\phi_{10}Y_2^0(\theta,\,\phi)\}\hat{r} + \frac{1}{r}\nabla_1\{\phi_{10}Y_2^0(\theta,\,\phi)\} = 0 + \frac{1}{r}\nabla_1 Y_2^0(\theta,\,\phi) = \frac{6}{r}\frac{\nabla_1 Y_2^0(\theta,\,\phi)}{6} = \frac{6\phi_{10}}{r}\boldsymbol{S}_2^0. \tag{36}$$

In the above equation, $\frac{\nabla_1 Y_2^0(\theta,\,\phi)}{6} = \boldsymbol{S}_2^0$ is the definition of $\boldsymbol{S}_2^0$ from Equation (A3).

If we suppose $\boldsymbol{g}_0 = g_0\hat{r}$, then

$$\boldsymbol{u}\cdot\boldsymbol{g}_0 = u_0\boldsymbol{S}_2^0\cdot\{g_0\hat{r}\} = u_0 g_0\frac{1}{6}\{\nabla_1 Y_2^0\}\cdot\hat{r} = 0. \tag{37}$$

The fourth term becomes

$$\nabla(\boldsymbol{u}\cdot\boldsymbol{g}_0) = 0. \tag{38}$$

The fifth term is

$$-\boldsymbol{g}_0\nabla\cdot\boldsymbol{u} = -\boldsymbol{g}_0\nabla\cdot\{u_0\boldsymbol{S}_2^0\}. \tag{39}$$

Also by Equation (C3)

$$\nabla\cdot\{u_0\boldsymbol{S}_2^0\} = -\frac{u_0}{r}Y_2^0, \tag{40}$$

and the fifth term becomes

$$-\boldsymbol{g}_0\nabla\cdot\boldsymbol{u} = \frac{g_0 u_0}{r}\boldsymbol{R}_2^0. \tag{41}$$

The sixth term is pretty complex, if we suppose $\rho_0 = 1$, $\mu = 1$ and $\lambda = 1$ for simplicity, then the divergence of the tensor becomes

$$\begin{aligned}\nabla\cdot\overleftrightarrow{S^e} &= \nabla\cdot[\lambda(\nabla\cdot\boldsymbol{u})\overleftrightarrow{I}] + \nabla\cdot\{\mu[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T]\}\\ &= \nabla\cdot[1*(\nabla\cdot\boldsymbol{u})\overleftrightarrow{I}] + 1*\nabla\cdot(\nabla\boldsymbol{u})\\ &\quad + 1*\nabla\cdot(\nabla\boldsymbol{u})^T + 1*\mu\cdot(\nabla\boldsymbol{u}) + \nabla 1\cdot(\nabla\boldsymbol{u})^T\\ &= \nabla\cdot[(\nabla\cdot\boldsymbol{u})\overleftrightarrow{I}] + \nabla\cdot(\nabla\boldsymbol{u}) + \nabla\cdot(\nabla\boldsymbol{u})^T.\end{aligned} \tag{42}$$

By Equation (40), the first term in the above Equation (42) becomes

$$\nabla\cdot[\lambda(\nabla\cdot\boldsymbol{u})\overleftrightarrow{I}] = \nabla\cdot\left[-\frac{u_0}{r}Y_2^0\overleftrightarrow{I}\right]. \tag{43}$$

By Equation (C6), (43) becomes

$$\begin{aligned}\nabla\cdot[\lambda(\nabla\cdot\boldsymbol{u})\overleftrightarrow{I}] &= \nabla\cdot\left[-\frac{u_0}{r}Y_2^0\overleftrightarrow{I}\right]\\ &= -u_0\partial_r\left(\frac{1}{r}\right)\boldsymbol{R}_2^0 - \frac{1}{r}\frac{u_0}{r}c_1^{-1}(2)\boldsymbol{S}_2^0\\ &= \frac{u_0}{r^2}\boldsymbol{R}_2^0 - \frac{6u_0}{r^2}\boldsymbol{S}_2^0.\end{aligned} \tag{44}$$

By Equation (C8), the second term in Equation (42) becomes

$$\begin{aligned}\mu\nabla\cdot(\nabla\boldsymbol{u}) &= \nabla(\nabla\cdot\boldsymbol{u}) - \nabla\times\nabla\times\boldsymbol{u}\\ &= \nabla\left(-\frac{u_0}{r}Y_2^0\right) - \nabla\times\nabla\times(u_0\boldsymbol{S}_2^0)\\ &= \frac{\partial}{\partial r}\left\{-\frac{u_0}{r}Y_2^0\right\}\hat{r} + \frac{1}{r}\nabla_1\left\{-\frac{u_0}{r}Y_2^0\right\}\\ &\quad - u_0\nabla\times\nabla\times\boldsymbol{S}_2^0\\ &= \frac{u_0}{r^2}\boldsymbol{R}_2^0 - \frac{6u_0}{r^2}\boldsymbol{S}_2^0 - u_0\nabla\times\nabla\times\boldsymbol{S}_2^0.\end{aligned} \tag{45}$$

By Equation (C10), $\nabla\times\boldsymbol{S}_2^0$ becomes

$$\nabla\times\boldsymbol{S}_2^0 = -\frac{1}{r}\boldsymbol{T}_2^0, \tag{46}$$

$$\begin{aligned}\nabla\times\nabla\times\boldsymbol{S}_2^0 &= \nabla\times\left\{-\frac{1}{r}\boldsymbol{T}_2^0\right\}\\ &= \frac{-1}{r^2}\boldsymbol{R}_2^0 + \left[\frac{\partial\left(\frac{-1}{r}\right)}{\partial r} + \frac{-1}{r^2}\right]\boldsymbol{S}_2^0(\theta,\,\phi)\\ &= -\frac{1}{r^2}\boldsymbol{R}_2^0.\end{aligned} \tag{47}$$

So, Equation (45) becomes

$$\begin{aligned}\mu\nabla\cdot(\nabla\boldsymbol{u}) &= \frac{u_0}{r^2}\boldsymbol{R}_2^0 - \frac{6u_0}{r^2}\boldsymbol{S}_2^0 - u_0\left(-\frac{1}{r^2}\boldsymbol{R}_n^m\right)\\ &= \frac{2u_0}{r^2}\boldsymbol{R}_2^0 - \frac{6u_0}{r^2}\boldsymbol{S}_2^0.\end{aligned} \tag{48}$$

By Equation (C12), the third term in Equation (42) becomes

$$\nabla\cdot\{\nabla[u_0\boldsymbol{S}_2^0]\}^T = \partial_{rr}^2(u_0)*[\hat{r}\cdot\boldsymbol{S}_2^0\otimes\hat{r}] + \partial_r\left(\frac{1}{r}u_0\right)*[\hat{r}\cdot(\nabla_1\boldsymbol{S}_2^0)^T] + \frac{1}{r}\frac{\partial u_0}{\partial r}\nabla_1\cdot[\boldsymbol{S}_2^2\otimes\hat{r}] + \frac{1}{r^2}u_0\nabla_1\cdot[\nabla_1\boldsymbol{S}_2^0]^T$$
$$= 0 - \frac{u_0}{r^2}[\hat{r}\cdot(\nabla_1\boldsymbol{S}_2^0)^T] + 0 + \frac{1}{r^2}u_0\nabla_1\cdot[\nabla_1\boldsymbol{S}_2^0]^T = \frac{-u_0}{r^2}[\hat{r}\cdot(\nabla_1\boldsymbol{S}_2^0)^T] + \frac{u_0}{r^2}\nabla_1\cdot[\nabla_1\boldsymbol{S}_2^0]^T. \quad (49)$$

By Equations (C18) and (C28), we can get

$$\hat{r}\cdot[\nabla_1\boldsymbol{S}_2^0]^T = -\boldsymbol{S}_2^0, \quad (50)$$
$$\nabla_1\cdot[\nabla_1\boldsymbol{S}_2^0]^T = \boldsymbol{R}_2^0 - 7\boldsymbol{S}_2^0. \quad (51)$$

So, Equation (49) becomes

$$\nabla\cdot\{\nabla[u_0\boldsymbol{S}_2^0]\}^T = \frac{-u_0}{r^2}*(-\boldsymbol{S}_2^0) + \frac{u_0}{r^2}\{\boldsymbol{R}_2^0 - 7\boldsymbol{S}_2^0\}$$
$$= \frac{-6u_0}{r^2}\boldsymbol{S}_2^0 + \frac{u_0}{r^2}\boldsymbol{R}_2^0. \quad (52)$$

With Equations (44), (48) and (52), the divergence of the tensor in Equation (42) becomes

$$\nabla\cdot\overleftrightarrow{S^e} = \frac{4u_0}{r^2}\boldsymbol{R}_2^0 - \frac{18u_0}{r^2}\boldsymbol{S}_2^0. \quad (53)$$

With Equations (22), (34), (36), (38), (41) and (53), the final expansion of Equation (17) becomes

$$\omega^2u_0\boldsymbol{S}_2^0 + 1.352\,246\,807\,565\,626\,7i\omega u_0\boldsymbol{T}_3^0$$
$$+ 0.5163977794943223i\omega u_0\boldsymbol{T}_1^0$$
$$+ \frac{6\phi_{10}}{r}\boldsymbol{S}_2^0 + \frac{g_0u_0}{r}\boldsymbol{R}_2^0 + \frac{4u_0}{r^2}\boldsymbol{R}_2^0 - \frac{18u_0}{r^2}\boldsymbol{S}_2^0 = 0. \quad (54)$$

Equation (53) gives a very simple example of the derivation of the divergence of a tensor. The most difficult part of LOM is to derive the operations of the tensor, and we will provide a little more complex instance in the following Section 2.4.2 to show how it works.

2.4.2. The Second Instance

Suppose

$$\boldsymbol{u} = \boldsymbol{R}_2^1,$$
$$\mu = Y_2^0,$$
$$\lambda = Y_2^0. \quad (55)$$

$\nabla\cdot\overleftrightarrow{S^e}$ has five parts,

$$\nabla\cdot\overleftrightarrow{S^e} = \nabla\cdot[\lambda(\nabla\cdot\boldsymbol{u})\overleftrightarrow{I}] + \mu\nabla\cdot(\nabla\boldsymbol{u})$$
$$+ \mu\nabla\cdot(\nabla\boldsymbol{u})^T + \nabla\mu\cdot(\nabla\boldsymbol{u}) + \nabla\mu\cdot(\nabla\boldsymbol{u})^T. \quad (56)$$

The expansions of the first three terms of Equation (56), which are $\nabla\cdot\{\lambda(\nabla\cdot\boldsymbol{u})\boldsymbol{I}\}$, $\mu\nabla\cdot(\nabla\boldsymbol{u})$ and $\mu\nabla\cdot(\nabla\boldsymbol{u})^T$, are shown in the first instance. We list their results, and do not give the details.

By Equation (C6), the first term in Equation (56) becomes

$$\nabla\cdot\{\lambda(\nabla\cdot\boldsymbol{u})\boldsymbol{I}\} = -0.4414562522349272\frac{1}{r^2}\boldsymbol{R}_4^1$$
$$+ 8.829125044698543\frac{1}{r^2}\boldsymbol{S}_4^1 - 0.18022375157286863\frac{1}{r^2}\boldsymbol{R}_2^1$$
$$+ 1.0813425094372118\frac{1}{r^2}\boldsymbol{S}_2^1. \quad (57)$$

By Equations (C8)–(C11), the second term in Equation (56) becomes

$$\mu\nabla\cdot(\nabla\boldsymbol{u}) = -1.7658250089397087\frac{1}{r^2}\boldsymbol{R}_4^1$$
$$- 0.7208950062914745\frac{1}{r^2}\boldsymbol{R}_2^1 + 4.41456252234927\frac{1}{r^2}\boldsymbol{S}_4^1$$
$$+ 0.5406712547186066\frac{1}{r^2}\boldsymbol{S}_2^1 - 1.8094322036008625i\frac{1}{r^2}\boldsymbol{T}_3^1$$
$$- 1.6925688412059197i\frac{1}{r^2}\boldsymbol{T}_1^1. \quad (58)$$

By Equations (C12)–(C30), the third term in Equation (56) becomes

$$\mu\nabla\cdot(\nabla\boldsymbol{u})^T = 4.41456252234927\frac{1}{r^2}\boldsymbol{S}_4^1$$
$$+ 0.5406712547186066\frac{1}{r^2}\boldsymbol{S}_2^1 - 1.8094322036008625i\frac{1}{r^2}\boldsymbol{T}_3^1$$
$$-1.6925688412059197i\frac{1}{r^2}\boldsymbol{T}_1^1 - 0.4414562522349272\frac{1}{r^2}\boldsymbol{R}_4^1$$
$$-0.18022375157286863D0\frac{1}{r^2}\boldsymbol{R}_2^1. \quad (59)$$

By Equation (C32), the fourth term in Equation (56) becomes

$$\nabla\mu\cdot\nabla[\chi_n^m(r)\boldsymbol{\Psi}_n^m(\theta,\phi)] = \frac{\partial Y_2^0}{\partial r}\frac{\partial 1}{\partial r}\boldsymbol{R}_2^1(\theta,\phi)$$
$$+ \frac{1}{r^2}*1*[\nabla_1Y_2^0\cdot\nabla_1\boldsymbol{R}_2^1]$$
$$= \frac{1}{r^2}[\nabla_1Y_2^0\cdot\nabla_1\boldsymbol{R}_2^1]. \quad (60)$$

By Equation (C34), we can get

$$\nabla_1Y_2^0\cdot\nabla_1\boldsymbol{R}_2^1(\theta,\phi) = M_9[\nabla_1Y_2^0\cdot\nabla_1Y_2^1]$$
$$+ c_1^{-1}(2)Y_2^1(\theta,\phi)*\boldsymbol{S}_2^0(\theta,\phi). \quad (61)$$

By Equation (A52), the second term on the right hand of Equation (61) becomes

$$\begin{aligned} Y_2^1(\theta,\,\phi)\,*\,\boldsymbol{S}_2^0 &= \sum_{s=0}^{[(2-|1|)/2]} c_{10}(2,\,1,\,s)(M_6)^{2-2s-|1|} \\ &\quad\times\,[(M_{21}^{(\zeta(1))})^{|1|}[\boldsymbol{S}_2^0]] \\ &= c_{10}(2,\,1,\,0)(M_6)^{1-0}[(M_{21}^{(1)})[\boldsymbol{S}_2^0]] \\ &= -0.772548432160781\,*\,M_6[M_{19}[\boldsymbol{S}_2^0]]. \end{aligned} \quad (62)$$

By Equation (A47), we can get

$$\begin{aligned} M_{27}[\boldsymbol{S}_2^0(\theta,\,\phi)] &= +iM_1[M_{25}[M_2[M_{12}[\boldsymbol{S}_2^0(\theta,\,\phi)]]]] \\ &= iM_1[M_{25}[M_2[c_1(2)Y_2^0]]] = \frac{i}{6}M_1[M_{25}[c_3(2,\,0)P_2^0]] \\ &= 0.10513052175084iM1[P_2^1]{=}0.10513052175084ic_3^{-1}(2,\,1)Y_2^1 \\ &= -0.4082482821789322iY_2^1. \end{aligned} \quad (65)$$

$$\begin{aligned} M_6[\boldsymbol{S}_3^1] &= M_{10}[M_6[M_{12}[\boldsymbol{S}_3^1]]] + M_{16}[M_{12}[\boldsymbol{S}_3^1]] = M_{10}[M_6[c_1(3)Y_3^1]] + M_{16}[c_1(3)Y_3^1] \\ &= c_1(3)M_{10}[M_1[M_0[M_2[Y_3^1]]]] + c_1(3)M_{14}\{-2M_6[Y_3^1] + M_7[Y_3^1]\} - c_1(3)M_{15}[iY_3^1] \\ &= c_1(3)M_{10}[M_1[M_0[c_3(3,\,1)P_3^1]]] + c_1(3)M_{14}\{-2M_0[c_3(3,\,1)P_3^1] + M_1[M_5[M_2[Y_3^1]]]\} - c_1(3)i\boldsymbol{T}_3^1 \\ &= c_1(3)c_3(3,\,1)M_{10}[M_1[c_4(3,\,1)P_4^1 + c_5(3,\,1)P_2^1]] \\ &\quad + c_1(3)M_{14}\{-2c_3(3,\,1)\,*\,[c_4(3,\,1)P_4^1 + c_5(3,\,1)P_2^1] + M_1[M_5[c_3(3,\,1)P_3^1]]\} - c_1(3)i\boldsymbol{T}_3^1 \\ &= c_1(3)c_3(3,\,1)c_4(3,\,1)c_3^{-1}(4,\,1)c_1^{-1}(4)\boldsymbol{S}_4^1 + c_1(3)c_3(3,\,1)c_5(3,\,1)c_1^{-1}(1)c_3^{-1}(2,\,1)\boldsymbol{S}_2^1 \\ &\quad - 2c_1(3)c_3(3,\,1)c_4(3,\,1)\boldsymbol{S}_4^1 - 2c_1(3)c_3(3,\,1)c_5(3,\,1)\boldsymbol{S}_2^1 \\ &\quad + c_1(3)c_3(3,\,1)M_{14}[M_1[(3+1)P_2^1 - 3M_0[P_3^1]]] - c_1(3)i\boldsymbol{T}_3^1 \\ &= c_1(3)c_3(3,\,1)c_4(3,\,1)c_3^{-1}(4,\,1)c_1^{-1}(4)\boldsymbol{S}_4^1 + c_1(3)c_3(3,\,1)c_5(3,\,1)c_1^{-1}(1)c_3^{-1}(2,\,1)\boldsymbol{S}_2^1 \\ &\quad - 2c_1(3)c_3(3,\,1)c_4(3,\,1)\boldsymbol{S}_4^1 - 2c_1(3)c_3(3,\,1)c_5(3,\,1)\boldsymbol{S}_2^1 \\ &\quad + 4c_1(3)c_3(3,\,1)c_3^{-1}(2,\,1)\boldsymbol{S}_2^1 - 3c_1(3)c_3(3,\,1)[c_4(3,\,1)c_3^{-1}(4,\,1)\boldsymbol{S}_4^1 + c_5(3,\,1)c_3^{-1}(2,\,1)\boldsymbol{S}_2^1] - c_1(3)i\boldsymbol{T}_3^1 \\ &= 0.6099375419488038\boldsymbol{S}_4^1 + 0.3187276118361799\boldsymbol{S}_2^1 - \frac{1}{12}i\boldsymbol{T}_3^1, \end{aligned} \quad (67)$$

In fact $M_6[M_{19}[\boldsymbol{S}_2^0]]$ is $\cos\theta\,*\,\sin\theta e^{i\phi}\,*\,\boldsymbol{S}_2^0$. By Equation (A48),

$$\begin{aligned} M_{19}[\boldsymbol{S}_2^0(\theta,\,\phi)] &= \sin\theta e^{i\phi}\,*\,\boldsymbol{S}_2^0(\theta,\,\phi) \\ &= M_{14}[M_{24}[\boldsymbol{S}_2^0(\theta,\,\phi)]] + M_{15}[M_{27}[\boldsymbol{S}_2^0(\theta,\,\phi)]]. \end{aligned} \quad (63)$$

By Equations (A45), (A20) and other relations, we can get

$$\begin{aligned} M_{24}[\boldsymbol{S}_2^0] &= 6M_{19}[M_{12}[\boldsymbol{S}_2^0]] + M_1[M_0[M_{25}[M_2[M_{12}[\boldsymbol{S}_2^0]]]]] \\ &= 6M_{19}[c_1(2)Y_2^0] + M_1[M_0[M_{25}[M_2[c_1(2)Y_2^0]]]] \\ &= M_1[M_3[M_2[Y_2^0]]] + \frac{1}{6}M_1[M_0[M_{25}[c_3(2,\,0)P_2^0]]]] \\ &= M_1[M_3[c_3(2,\,0)P_2^0]] + 0.10513052175084002M_1[M_0[P_2^1]] \\ &= 0.6307831305050401M_1[c_6(2,\,0)(P_3^1 - M_0[P_2^1])] \\ &\quad + 0.10513052175084002M_1[c_4(2,\,1)P_3^1 + c_5(2,\,1)P_1^1] \\ &= 0.21026104350168M_1[(P_3^1 - c_4(2,\,1)P_3^1 - c_5(2,\,1)P_1^1] \\ &\quad + 0.10513052175084002M_1[0.4P_3^1 + 0.6P_1^1] \\ &= 0.21026104350168M_1[0.6P_3^1 - 0.6P_1^1] \\ &\quad + 0.10513052175084002M_1[0.4P_3^1 + 0.6P_1^1] \\ &= -0.5855400775457997Y_3^1 + 0.36514839242905195Y_1^1 \\ &\quad - 0.19518002100125173Y_3^1 - 0.182574196214526Y_1^1 \\ &= -0.7807200723713712Y_3^1 + 0.18257418895967958Y_1^1. \end{aligned} \quad (64)$$

By Equations (A24) and (64), Equations (65) and (63) become

$$\begin{aligned} M_{19}[\boldsymbol{S}_2^0(\theta,\,\phi)] &= -0.7807200723713712\boldsymbol{S}_3^1 \\ &\quad + 0.18257418895967958\boldsymbol{S}_1^1 - 0.4082482756069868i\boldsymbol{T}_2^1. \end{aligned} \quad (66)$$

By Appendices E and F, we can expand $M_6[\boldsymbol{S}_3^1]$ in detail, and list the expansions of $M_6[\boldsymbol{S}_1^1]$ and $M_6[\boldsymbol{T}_2^1]$ here

$$M_6[\boldsymbol{S}_1^1] = 0.6708203573569965\boldsymbol{S}_2^1 - 0.5i\boldsymbol{T}_1^1, \quad (68)$$

$$\begin{aligned} M_6[\boldsymbol{T}_2^1] &= 0.6374552929509951\boldsymbol{T}_3^1 \\ &\quad + 0.22360680971429314\boldsymbol{T}_1^1 + \frac{1}{6}i\boldsymbol{S}_2^1. \end{aligned} \quad (69)$$

With Equations (66), (67), (68) and (69), $Y_2^1 * \boldsymbol{S}_2^0$ in Equation (62) becomes

$$\begin{aligned} Y_2^1\,*\,\boldsymbol{S}_2^0 &= 0.36788021019577277\boldsymbol{S}_4^1 \\ &\quad + 0.045055937893217185\boldsymbol{S}_2^1 + 0.15078601696673857i\boldsymbol{T}_3^1 \\ &\quad + 0.14104740343382657i\boldsymbol{T}_1^1. \end{aligned} \quad (70)$$

Let us discuss the first term on the right hand of Equation (61). With Equation (B9), $\nabla_1 Y_2^0 \cdot \nabla_1 Y_2^1$ becomes

$$
\begin{aligned}
B_1[\nabla_1 Y_2^0, \nabla_1 Y_2^1] =& H_0[Y_2^0] B_1[\nabla_1 \cos\theta, \nabla_1 Y_2^1] \\
&+ H_1[Y_2^0] B_1[\nabla_1(\sin\theta e^{\zeta(1)i\phi}), \nabla_1 Y_2^1] \\
=& H_0[Y_2^0]\nabla_1 \cos\theta \cdot \nabla_1 Y_2^1 + H_1[Y_2^0]\nabla_1(\sin\theta e^{i\phi}) \cdot \nabla_1 Y_2^1. \quad (71)
\end{aligned}
$$

By Equation (B7), we can get

$$
\begin{aligned}
H_0[Y_2^0] =& \sum_{s=0}^{[(2-|0|)/2]} (2 - 2s - |0|) * c_{10}(2, 0, s)(\sin\theta e^{\zeta(0)i\phi})^{|0|} (\cos\theta)^{2-2s-|0|-1} \\
=& (2 - 0) * c_{10}(2, 0, 0)(\cos\theta)^{2-0-|0|-1} + (2 - 2 * 1 - 0) c_{10}(2, 0, 1)(\cos\theta)^{2-2*1-|0|-1} \\
=& 2c_{10}(2, 0, 0)\cos\theta, \\
H_1[Y_2^0] =& \sum_{s=0}^{[(2-|0|)/2]} |0| * c_{10}(2, 0, s)(\cos\theta)^{2-2s-|0|} (\sin\theta e^{\zeta(0)i\phi})^{|0|-1} \\
=& 0. \quad (72)
\end{aligned}
$$

So Equation (71) becomes

$$
B_1[\nabla_1 Y_2^0, \nabla_1 Y_2^1] = 2c_{10}(2, 0, 0)\cos\theta \nabla_1 \cos\theta \cdot \nabla_1 Y_2^1. \quad (73)
$$

With Equation (B10),

$$
\begin{aligned}
\nabla_1 \cos\theta \cdot \nabla_1 Y_2^1 =& M_7[Y_2^1] = M_1[M_5[M_2[Y_2^1]]] \\
=& M_1[M_5[c_3(2, 1)P_2^1]] \\
=& c_3(2, 1) M_1[(2 + 1)P_1^1 - 2M_0[P_2^1]] \\
=& 3c_3(2, 1)c_3^{-1}(1, 1)Y_1^1 \\
&-2c_3(2, 1)M_1[c_4(2, 1)P_3^1 + c_5(2, 1)P_1^1] \\
=& 3c_3(2, 1)c_3^{-1}(1, 1)Y_1^1 \\
&-2c_3(2, 1)c_4(2, 1)c_3^{-1}(3, 1)Y_3^1 \\
&-2c_3(2, 1)c_5(2, 1)c_3^{-1}(1, 1)Y_1^1 \\
=& 1.3416408582857589 Y_1^1 \\
&-0.9561829394264927 Y_3^1. \quad (74)
\end{aligned}
$$

By Equations (60), (61), (70) and (74), the fourth term of Equation (56) finally becomes

$$
\begin{aligned}
\nabla\mu \cdot (\nabla\boldsymbol{u}) =& 13.243687567047811 \frac{1}{r^2}\boldsymbol{S}_4^1 \\
&+1.6220137641558197 \frac{1}{r^2}\boldsymbol{S}_2^1 \\
&- 5.428296610802588 i \frac{1}{r^2}\boldsymbol{T}_3^1 \\
&-5.077706523617759 i \frac{1}{r^2}\boldsymbol{T}_1^1 \\
&+ 0.27033562735930283 \frac{1}{r^2}\boldsymbol{R}_2^1 \\
&-0.8829125044698543 \frac{1}{r^2}\boldsymbol{R}_4^1. \quad (75)
\end{aligned}
$$

By Equation (C43), the last term in Equation (56) becomes

$$
\nabla Y_2^0 \cdot \{\nabla[1 * \boldsymbol{R}_2^1]\}^T = +\frac{1}{r^2}\nabla_1 Y_2^0 \cdot [\nabla_1 \boldsymbol{R}_2^1]^T. \quad (76)
$$

With Equation (C44) and Equation (70),

$$
\begin{aligned}
\nabla_1 Y_2^0 \cdot [\nabla_1 \boldsymbol{R}_2^1]^T =& Y_2^1 * \nabla_1 Y_2^0 \\
=& c_1^{-1}(2) Y_2^1 * \nabla_1 Y_2^0 \\
=& 2.2072812611746366 \boldsymbol{S}_4^1 \\
&+ 0.2703356273593031 \boldsymbol{S}_2^1 \\
&+ 0.9047161018004315 i\boldsymbol{T}_3^1 \\
&+ 0.8462844206029594 i\boldsymbol{T}_1^1. \quad (77)
\end{aligned}
$$

So, the last term in Equation (56) becomes

$$
\begin{aligned}
\nabla\mu \cdot (\nabla\boldsymbol{u})^T =& 2.2072812611746366 \frac{1}{r^2}\boldsymbol{S}_4^1 \\
&+ 0.2703356273593031 \frac{1}{r^2}\boldsymbol{S}_2^1 \\
&+ 0.9047161018004315 i \frac{1}{r^2}\boldsymbol{T}_3^1 \\
&+ 0.8462844206029594 i \frac{1}{r^2}\boldsymbol{T}_1^1. \quad (78)
\end{aligned}
$$

By summing up Equations (57), (58), (59), (75) and (78), we can obtain

$$
\begin{aligned}
\nabla \cdot \overleftrightarrow{S^e} =& \nabla \cdot \{\lambda(\nabla \cdot \boldsymbol{u})\boldsymbol{I} + \mu[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T]\} \\
&-3.531650017879417 \frac{1}{r^2}\boldsymbol{R}_4^1 \\
&+33.109218917619536 \frac{1}{r^2}\boldsymbol{S}_4^1 \\
&- 0.8110068820779089 \frac{1}{r^2}\boldsymbol{R}_2^1 \\
&+ 4.055034410389547 \frac{1}{r^2}\boldsymbol{S}_2^1 \\
&-8.142444916203882 i \frac{1}{r^2}\boldsymbol{T}_3^1 \\
&- 7.616559785426638 i \frac{1}{r^2}\boldsymbol{T}_1^1. \quad (79)
\end{aligned}
$$

This example of Equation (79) from Equation (20) shows that the most tedious part ($\nabla \cdot \overleftrightarrow{S^e}$) can be derived and expressed by their components ($\boldsymbol{R}$, $\boldsymbol{S}$, $\boldsymbol{T}$).

The above instances are simple, and the real computation is much more complex. The parameters are like Equation (A30):

$$\begin{aligned}\rho &= \rho_0(r) + \sum_{n,m} \rho_n^m(r) Y_n^m(\theta, \phi)\\ \lambda &= \lambda_0(r) + \sum_{n,m} \lambda_n^m(r) Y_n^m(\theta, \phi)\\ \mu &= \mu_0(r) + \sum_{n,m} \mu_n^m(r) Y_n^m(\theta, \phi), \end{aligned} \quad (80)$$

as is the displacement field $\boldsymbol{u}$. All the symbolic computations exceed human capacities, but can be done by a computer.

3. Numerical Validation and a Simple Application

In this section, we will validate our method in Section 3.1 and give a simple application to the lower degree normal modes of Saturn in Section 3.2.

3.1. Normal Modes of Earth and Numerical Validation

To validate our method, we compute the spheroidal modes of a spherical, rotating, elastic and isotropic Earth model. We take the Preliminary Reference Earth Model (Dziewonski & Anderson 1981, PREM) without an ocean. ${}_0S_n$ means the fundamental mode of harmonic degree n.

We list our results in Table 1, and it shows that our computed periods of the normal modes agree very well with those in Dziewonski & Anderson (1981).

3.2. A Two-layer Saturn Model

The normal modes of the planet Saturn are computed to demonstrate our method. The rotational modes are free oscillations which involve the redistribution of angular momentum, in the absence of external torques produced by the gravitational forces. Such modes require planet models to rotate, which usually have nonzero ellipticities.

Vorontsov et al. (1976) followed Alterman's approach (Alterman et al. 1959) to compute the free oscillations of the giant planets. Vorontsov & Zharkov (1981) studied the free oscillations of the giant planets with rotation and ellipticity by perturbation method which took rotation and ellipticity as small perturbations to their initially spherically symmetric, non-rotating Jovian models (Le Bihan & Burrows 2013). If a planet rotates fast and has a large ellipticity, then the perturbation method becomes inappropriate. Instead of the perturbation method we use a direct integration method to compute the normal modes in this work.

For the planet Saturn, we take $R_p = 58{,}242$ km, $M_p = 5.68 * 10^{26}$ kg, $\Omega_0 = 10.55$ hr and $N_p = 1.5$, where R_p, M_p, ω and N_p are mean radius, total mass, rotation speed and polytropic index respectively. By solving (Zhang & Huang 2018)

$$P = K\rho^{\frac{n+1}{n}}, \quad (81)$$

and

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2 \frac{d\theta}{d\xi}\right) + \theta^n = 0, \quad (82)$$

we can get the density profile of Saturn. Based on this profile, we make a two-layer model which consists of a solid core and a liquid shell, and the radius of the core is 1/4 of the whole radius. We modify this profile by making the density of the core four times the density of the liquid shell at the boundary. The density profile is

$$\rho_0^{\text{core}} = 9.03 * 10^{12} - 1.14 * 10^6 * r - 4.13 * 10^3 * r^2 \quad \text{kg km}^{-3}, \quad (83)$$

$$\rho_0^{\text{shell}} = 2.22 * 10^{12} + 7.24 * 10^6 * r - 1.56 * 10^3 * r^2 \quad \text{kg km}^{-3}. \quad (84)$$

For a simple polytropic fluid model, Equation (1) can be simplified as

$$\rho_0\omega^2\boldsymbol{u} - 2i\rho_0\omega\boldsymbol{\Omega}_0 \times \boldsymbol{u} - \nabla p_1 + \rho_0\nabla V_1 + \rho_1\boldsymbol{g}_0 = 0, \quad (85)$$

which governs the small isentropic oscillations of an inviscid liquid core given by the conservation laws for mass, momentum, gravitational flux and entropy (Rochester 1989), where

$$\rho_1 = -\nabla \cdot (\rho_0\boldsymbol{u}), \quad (86)$$

$$p_1 = -\boldsymbol{u} \cdot \nabla p_0 + \alpha^2\rho_1 + \alpha^2\boldsymbol{u} \cdot \nabla\rho_0. \quad (87)$$

In the above Equations (85)–(87), $\boldsymbol{u}$, ρ_0, p_1 and V_1 stand for displacement, Eulerian perturbation in density, Eulerian pressure disturbance and Eulerian perturbation in the gravitational potential respectively (all regarded as first-order departures from the equilibrium reference state). The local compressional wave speed α has the form

$$\alpha^2 = (1 - \beta)\rho_0\frac{\boldsymbol{g}_0}{\nabla\rho_0}. \quad (88)$$

We take $\beta = 0$ for a purely adiabatic model to solve α^2. The bulk modulus profile is

$$K^{\text{core}} = 2.43 * 10^{16} - 3.23 * 10^{10} * r - 1.80 * 10^7 * r^2 - 847 * r^3 + 0.0328 * r^4 \quad \text{Pa}, \quad (89)$$

$$K^{\text{shell}} = 2.84 * 10^{16} - 3.09 * 10^{12} * r + 1.42 * 10^8 * r^2 - 3341 * r^3 + 0.0393 * r^4 - 1.84 * 10^{-7} * r^5 \text{ Pa}. \quad (90)$$

We also suppose the core to be a Poisson solid, so we can get the profile of Lamé parameters.

The ellipticity profile $\epsilon(r)$ is solved from the classical Clairaut equation of the Wavre integro-differential equation

Table 1
The Periods (minutes) of the Spheroidal Modes of PREM

Author	${}_0S_2$	${}_0S_3$	${}_0S_4$	${}_0S_5$
Dziewonski & Anderson (1981)	53.89	35.57	25.76	19.84
This work	53.65	35.45	25.65	19.70
Percentage difference	0.45	0.34	0.43	0.70

(Moritz 1990), or Huang et al. (2019) for more information,

$$\left(\frac{1}{r_0^2}\frac{d\epsilon_2}{dr_0} + \frac{2}{r_0^3}\epsilon_2\right)\int_0^{r_0} \rho q^2 dq - \int_{r_0}^{R} \rho\frac{d\epsilon_2}{dq}dq + \frac{5\omega^2}{12\pi G} = 0. \tag{91}$$

If a parameter in a non-rotating spherical model is $X_0(r)$, it then becomes

$$X(r, \theta) = X_0(r) - \frac{1}{r}\frac{\partial X_0(r)}{\partial r}P_2(\theta, \phi), \tag{92}$$

in a rotating oblate case. In this work we drop the equivalent spherical domain, and just use the original figure, which will resolve lots of obscure problems. By solving Equation (91), the flattening profile is

$$f^{\text{core}} = 0.0241 + 6.60 * 10^{-19} * r + 2.02 * 10^{-12} * r^2 - 9.88 * 10^{-16} * r^3 + 6.79 * 10^{-22} * r^4, \tag{93}$$

$$f^{\text{shell}} = 0.0338 - 2.71 * 10^{-6} * r + 1.92 * 10^{-10} * r^2 - 3.85 * 10^{-15} * r^3 + 2.78 * 10^{-20} * r^4. \tag{94}$$

The surface flattening of this Saturn model is 0.0867, which is very large and almost 26 times the Earth's flattening. So, the effect of flattening (ellipticity) of Saturn cannot be treated as a small perturbation anymore, as is usually done in Earth models.

3.3. Variable Expansion and Boundary Conditions

The variables are expressed in spherical coordinates by SHs in Equations (13), (14) and (15). The displacement field (for any given SH order m) in a rotating planet model takes one of two possible forms (Smith 1974), either

$$\boldsymbol{u} = \boldsymbol{T}_{|m|}^m + \boldsymbol{S}_{|m|+1}^m + \boldsymbol{T}_{|m|+2}^m + \boldsymbol{S}_{|m|+3}^m + \cdots, \tag{95}$$

or

$$\boldsymbol{u} = \boldsymbol{S}_{|m|}^m + \boldsymbol{T}_{|m|+1}^m + \boldsymbol{S}_{|m|+2}^m + \boldsymbol{T}_{|m|+3}^m + \cdots. \tag{96}$$

Ellipticity ($P_2(\cos\theta)$) couples $\boldsymbol{S}_n^m$ to $\boldsymbol{S}_{n+2}^m + \boldsymbol{T}_{n+1}^m + \boldsymbol{S}_n^m + \boldsymbol{T}_{n-1}^m + \boldsymbol{S}_{n-2}^m$ and couples $\boldsymbol{T}_n^m$ to $\boldsymbol{T}_{n+2}^m + \boldsymbol{S}_{n+1}^m + \boldsymbol{T}_n^m + \boldsymbol{S}_{n-1}^m + \boldsymbol{T}_{n-2}^m$. Both rotation and ellipticity can couple a single SH into a chain, which increases difficulty of the computation.

At a planet's free surface the boundary conditions require

$$\hat{n} \cdot \overleftrightarrow{S}|_-^+ = 0, \tag{97}$$

$$\hat{n} \cdot [\nabla V_1 - 4\pi G\rho_0 \boldsymbol{u}]|_-^+ = 0, \tag{98}$$

where $\boldsymbol{n}$ is the normal vector at the boundary surface.

Also, the stress tensor $\overleftrightarrow{S}$ in fluid shell takes the form (Huang et al. 2004),

$$\overleftrightarrow{S} = -(p_1 + \boldsymbol{u} \cdot \nabla p_0)\overleftrightarrow{I}. \tag{99}$$

At the center of a planet, variables are required to be regular.

We adopt the Galerkin method here to solve Equation (85) instead of Alterman's approach. Seyed-Mahmoud & Rochester (2006) applied the Galerkin method on the oscillatory dynamics of a rotating compressible self-gravitating inviscid fluid in terms of three potential scalar dependent variables (Seyed-Mahmoud 1994). We express the variables in the form of Equation (95) or (96) instead of the 3-scalar potential description. The variables ψ_n^m, u_n^m, v_n^m and w_n^m are expanded in power polynomials of r as $\sum_{k=0}^{k_{\max}} c_k r^k$, where c_k are the coefficients. Multiplying trial functions χ_j on Equation (85) and integrating them in a sphere, we get

$$\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\pi}\int_{r=0}^{r=R} \chi_j(\text{Governing Equation})r^2 \sin\theta dr d\phi d\theta = 0, \quad j = 0 \ldots j_{\max}, \tag{100}$$

or for an oblate situation, we get

$$\int_{\theta=0}^{\theta=\pi}\int_{\phi=0}^{\phi=2\phi}\int_{r=0}^{r=R[1+\epsilon(R)P_2(\cos\theta)]} \chi_j(\text{Governing Equation})r^2 \sin\theta dr d\phi d\theta = 0, \quad j = 0 \ldots j_{\max}. \tag{101}$$

3.4. Result

The final results are listed in Table 2. We started from a non-rotating spherical model and obtained the fundamental periods

Table 2
Lower Degree Normal Modes of Saturn

Model	Mode	Truncated Chain	Result (minutes)
Non-rotating Sphere	$\boldsymbol{S}_2$	$\boldsymbol{S}_2$	168.4
	$\boldsymbol{S}_3$	$\boldsymbol{S}_3$	127.7
	$\boldsymbol{S}_4$	$\boldsymbol{S}_4$	107.4
Rotating Sphere	$\boldsymbol{S}_2^0$	$\boldsymbol{T}_1^0 + \boldsymbol{S}_2^0 + \boldsymbol{T}_3^0$	167.8
	$\boldsymbol{S}_2^1$	$\boldsymbol{T}_1^1 + \boldsymbol{S}_2^1 + \boldsymbol{T}_3^1$	158.3
	$\boldsymbol{S}_2^{-1}$	$\boldsymbol{T}_1^{-1} + \boldsymbol{S}_2^{-1} + \boldsymbol{T}_3^{-1}$	170.3
	$\boldsymbol{S}_2^2$	$\boldsymbol{S}_2^2 + \boldsymbol{T}_3^2$	150.4
	$\boldsymbol{S}_2^{-2}$	$\boldsymbol{S}_2^{-2} + \boldsymbol{T}_3^{-2}$	174.5
Rotating Oblate Body	$\boldsymbol{S}_2^0$	$\boldsymbol{T}_1^0 + \boldsymbol{S}_2^0 + \boldsymbol{T}_3^0$	147.1
	$\boldsymbol{S}_2^1$	$\boldsymbol{T}_1^1 + \boldsymbol{S}_2^1 + \boldsymbol{T}_3^1$	134.1
	$\boldsymbol{S}_2^{-1}$	$\boldsymbol{T}_1^{-1} + \boldsymbol{S}_2^{-1} + \boldsymbol{T}_3^{-1}$	157.5
	$\boldsymbol{S}_2^2$	$\boldsymbol{S}_2^2 + \boldsymbol{T}_3^2$	130.9
	$\boldsymbol{S}_2^{-2}$	$\boldsymbol{S}_2^{-2} + \boldsymbol{T}_3^{-2}$	183.1
Differentially Rotating Sphere	$\boldsymbol{S}_2^0$	$\boldsymbol{T}_1^0 + \boldsymbol{S}_2^0 + \boldsymbol{T}_3^0$	160.4
	$\boldsymbol{S}_2^1$	$\boldsymbol{T}_1^1 + \boldsymbol{S}_2^1 + \boldsymbol{T}_3^1$	140.4
	$\boldsymbol{S}_2^{-1}$	$\boldsymbol{T}_1^{-1} + \boldsymbol{S}_2^{-1} + \boldsymbol{T}_3^{-1}$	171.1
	$\boldsymbol{S}_2^2$	$\boldsymbol{S}_2^2 + \boldsymbol{T}_3^2$	127.7
	$\boldsymbol{S}_2^{-2}$	$\boldsymbol{S}_2^{-2} + \boldsymbol{T}_3^{-2}$	193.9

168.4 minutes for $\boldsymbol{S}_2$, 127.7 minutes for $\boldsymbol{S}_3$ and 107.4 minutes for $\boldsymbol{S}_4$.

The rotation cannot be treated as a small perturbation, because 168.2 minutes is almost $1/4$ of the rotation period compared with Earth: the fundamental S_2^0 period of Earth is about 54 minutes and is about $1/24$ of Earth's rotation period.

The rotation splits $\boldsymbol{S}_2$ into $\boldsymbol{S}_2^{-2}$, $\boldsymbol{S}_2^{-1}$, $\boldsymbol{S}_2^0$, $\boldsymbol{S}_2^1$ and $\boldsymbol{S}_2^2$. $\boldsymbol{S}_2^1$ has a period of 158.3 minutes with truncated series $\boldsymbol{T}_1^1 + \boldsymbol{S}_2^1 + \boldsymbol{T}_3^1$, and others are listed in the column of the rotating sphere in Table 2. The results show that the fundamental periods of $\boldsymbol{S}_2^m$ vary from 150.4 to 174.5 minutes, and the difference is 24.1 minutes which is about 14% of the base number. The rotation should not be regarded as a small negligible number.

The oblate model increases the difference. The results are listed in the column of the rotating oblate case in Table 2. The results show that the fundamental periods of $\boldsymbol{S}_2^m$ vary from 130.9 to 183.1 minutes, and the difference is 52.2 minutes which is about 35% of the base number. The flattening also should not be treated as a small perturbation factor.

We also compute a simple differentially rotating model. It has a differential rotation speed $\Omega(r)$ in the liquid shell, which is

$$\Omega(r) = \Omega_0 * \left(1 - \frac{r}{30} * \frac{4}{58232}\right). \tag{102}$$

The results are listed in the column of the differentially rotating sphere in Table 2. It shows that the differential rotation also causes a large difference and is a non-negligible factor.

The main theme of this paper is about the LOM, so we will not discuss the normal modes of Saturn in detail due to limited space. We will discuss effects of flattening and rotation on kronoseismology (Hedman & Nicholson 2013) in a future article.

4. Conclusion

In this paper, we show how to represent the dynamic equations and boundary conditions for 3D planet models by VSHs and the linear operations of these VSHs. These equations can be finally expressed in integrable form. The GSSH method relies on abstruse mathematics, which is difficult for many researchers. By contrast, the LOM uses simple math and it is easy to program for numerical computation. We compute the normal modes of Earth to validate our method, and compute the normal modes with rotation of Saturn to demonstrate this method. In future articles we will report the FCN of Earth computed by this method, which is very close to what is observed, and the effect of the topography of core–mantle boundary on FCN.

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Appendix A
Vector Spherical Harmonics

Any vector field, like displacement field $\boldsymbol{u}$, can be represented in a spherical coordinate basis ($\hat{r}$, $\hat{\theta}$, $\hat{\phi}$) where θ is co-latitude and ϕ is longitude

$$\boldsymbol{u} = u_r\hat{r} + u_\theta\hat{\theta} + u_\phi\hat{\phi}. \tag{A1}$$

It can also be represented by three VSHs

$$\boldsymbol{u} = \sum_{n,m}[U_n^m(r)\boldsymbol{R}_n^m(\theta, \phi) + V_n^m(r)\boldsymbol{S}_n^m(\theta, \phi) + W_n^m(r)\boldsymbol{T}_n^m(\theta, \phi)]. \tag{A2}$$

These VSHs are defined as

$$\begin{aligned}\boldsymbol{R}_n^m(\theta, \phi) &= \hat{r}Y_n^m(\theta, \phi)\\ \boldsymbol{S}_n^m(\theta, \phi) &= \frac{\nabla_1 Y_n^m(\theta, \phi)}{n(n+1)} = c_1(n)\nabla_1 Y_n^m(\theta, \phi)\\ \boldsymbol{T}_n^m(\theta, \phi) &= \frac{-\hat{r}\times\nabla_1 Y_n^m(\theta, \phi)}{n(n+1)} = c_2(n)\hat{r}\times\nabla_1 Y_n^m(\theta, \phi).\end{aligned} \tag{A3}$$

$\nabla_1 = \hat{\theta}\partial_\theta + \frac{\hat{\phi}}{\sin\theta}\partial_\phi$ is the surface gradient on a unit sphere of radius = 1.

The scalar SHs are defined as

$$\begin{aligned}Y_n^m(\theta, \phi) &= (-1)^m\left(\frac{2n+1}{4\pi}\right)^{1/2}\left[\frac{(n-m)!}{(n+m)!}\right]^{1/2}P_n^m(\cos\theta)e^{im\phi}\\ &= c_3(n, m)P_n^m(\cos\theta)e^{im\phi}.\end{aligned} \tag{A4}$$

Also, the associated Legendre functions are defined as

$$P_n^m(\cos\theta) = \frac{1}{2^n n!}(1-\cos^2\theta)^{m/2}\left(\frac{d}{d\cos\theta}\right)^{n+m}(\cos^2\theta - 1)^n. \tag{A5}$$

With these definitions, for the case where $\boldsymbol{u}$ is assumed to be real, $U_n^m(r)$ should satisfy

$$\begin{cases}\mathrm{Re}[U_n^m(r)] - (-1)^m\mathrm{Re}[U_n^{-m}(r)] = 0\\ \mathrm{Im}[U_n^m(r)] + (-1)^m\mathrm{Im}[U_n^{-m}(r)] = 0\\ \mathrm{Im}(U_n^0(r)] = 0.\end{cases} \tag{A6}$$

The same is true for $V_n^m(r)$ and $W_n^m(r)$.

The vector field in the form of Equation (A1) (coordinate bases are $\hat{r}$, $\hat{\theta}$ and $\hat{\phi}$) can be transformed to VSHs (coordinate bases are $\boldsymbol{R}_n^m(\theta, \phi)$, $\boldsymbol{S}_n^m(\theta, \phi)$ and $\boldsymbol{T}_n^m(\theta, \phi)$) by the relations below

$$\sum_{n,m}V_n^m(r)Y_n^m(\theta, \phi) = -\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta u_\theta) + \frac{\partial u_\phi}{\partial\phi}\right], \tag{A7}$$

$$\sum_{n,m}W_n^m(r)Y_n^m(\theta, \phi) = +\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta u_\phi) - \frac{\partial u_\theta}{\partial\phi}\right]. \tag{A8}$$

One of the associated Legendre function recursion relations is

$$\begin{aligned}\cos\theta * P_n^m(\cos\theta) &= \frac{n-m+1}{2n+1}P_{n+1}^m(\cos\theta) + \frac{n+m}{2n+1}P_{n-1}^m(\cos\theta)\\ &= c_4(n, m)P_{n+1}^m(\cos\theta) + c_5(n, m)P_{n-1}^m(\cos\theta)\\ &\equiv M_0[P_n^m(\cos\theta)].\end{aligned} \tag{A9}$$

$\cos\theta *$, i.e., linear map M_0, turns $P_n^m(\cos\theta)$ into $P_{n+1}^m(\cos\theta)$ and $P_{n-1}^m(\cos\theta)$.

$$M_1[P_n^m(\cos\theta)] \equiv P_n^m(\cos\theta) * e^{im\phi} = c_3^{-1}(n, m) Y_n^m(\theta, \phi)$$
$$M_2[Y_n^m(\theta, \phi)] \equiv Y_n^m(\theta, \phi) \div e^{im\phi} = c_3(n, m) P_n^m(\cos\theta), \tag{A10}$$

while $\sin\theta *$ turns m into $m+1$ or $m-1$

$$\begin{aligned} \sin\theta * P_n^m(\cos\theta) &= \frac{1}{n+m+1}[P_{n+1}^{m+1}(\cos\theta) - \cos\theta P_n^{m+1}(\cos\theta)] \\ &= c_6(n, m)\{P_{n+1}^{m+1}(\cos\theta) - M_0[P_n^{m+1}(\cos\theta)]\} \\ &\equiv M_3[P_n^m(\cos\theta)], \end{aligned} \tag{A11}$$

or

$$\begin{aligned} \sin\theta * P_n^m(\cos\theta) &= (n+m)\cos\theta P_n^{m-1}(\cos\theta) - (n-m+2)P_{n+1}^{m-1}(\cos\theta)] \\ &= (n+m)M_0[P_n^{m-1}(\cos\theta)] - (n-m+2)P_{n+1}^{m-1}(\cos\theta) \\ &\equiv M_4[P_n^m(\cos\theta)]. \end{aligned} \tag{A12}$$

M_3 and M_4 are two linear maps on P_n^m with different operational formats.

$\sin\theta\partial_\theta$ has many relations, and the following one leaves m unchanged,

$$\begin{aligned} -\sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta} &= (1-\cos^2\theta)\frac{\partial P_n^m(\cos\theta)}{\partial(\cos\theta)} \\ &= (n+m)P_{n-1}^m(\cos\theta) - n\cos\theta * P_n^m(\cos\theta) \\ &= (n+m)P_{n-1}^m(\cos\theta) - nM_0[P_n^m(\cos\theta)] \\ &\equiv M_5[P_n^m(\cos\theta)]. \end{aligned} \tag{A13}$$

In other words, $M_5[P_n^m(\cos\theta)]$ is

$$M_5[P_n^m(\cos\theta)] = -\sin\theta\frac{\partial P_n^m(\cos\theta)}{\partial\theta} = (n+m)P_{n-1}^m(\cos\theta) - nM_0[P_n^m(\cos\theta)]. \tag{A14}$$

The same relationship holds true for SHs,

$$\begin{aligned} \cos\theta * Y_n^m(\theta, \phi) &= c_3(n, m)e^{im\phi}\cos\theta * P_n^m(\cos\theta) \\ &= M_1[M_0[M_2[Y_n^m(\theta, \phi)]]] \equiv M_6[Y_n^m(\theta, \phi)], \end{aligned} \tag{A15}$$

$$\begin{aligned} -\sin\theta\frac{\partial Y_n^m(\theta, \phi)}{\partial\theta} &= (1-\cos^2\theta)\frac{\partial Y_n^m(\theta, \phi)}{\partial(\cos\theta)} \\ &= M_1[M_5[M_2[Y_n^m(\theta, \phi)]]] \equiv M_7[Y_n^m(\theta, \phi)]. \end{aligned} \tag{A16}$$

The above operator M_7 does not change m, because on the left side there is no $e^{\pm i\phi}$. The principle is: multiplying $e^{i\phi}$ on the left side, m must increase by 1 on the right side; and multiplying $e^{-i\phi}$ on the left side, m must decrease by 1 on the right side.

A.1. For a Rotating Oblate Earth Model

A steadily rotating Earth can be approximated using a slightly rotating symmetric model with a small polar ellipticity, while its equator is still a circle. In this model, each parameter, like density, Lamé parameters, etc., can be expressed by the following formula with one order small correction of ellipticity ϵ,

$$x(r, \theta) = x_0(r)[1 + \epsilon(r)P_2(\cos\theta)]. \tag{A17}$$

None of them depend on longitude ϕ, so ϕ and $e^{\pm i\phi}$ never appear. This correction generates many terms of $\cos\theta$ in the dynamic equation, so we need only care about the operator $\cos\theta$ and ignore the operators $\sin\theta e^{\pm i\phi}$ in this subsection.

The action of $\cos\theta$ on $\boldsymbol{R}_n^m(\theta, \phi)$ is

$$\cos\theta * \boldsymbol{R}_n^m(\theta, \phi) = M_9[M_6[M_8[\boldsymbol{R}_n^m(\theta, \phi)]]] \equiv M_6[\boldsymbol{R}_n^m(\theta, \phi)]. \tag{A18}$$

M_8 and M_9 are

$$\begin{aligned} M_8[\boldsymbol{R}_n^m(\theta, \phi)] &\equiv \hat{r} \cdot \boldsymbol{R}_n^m(\theta, \phi) = Y_n^m(\theta, \phi) \\ M_9[Y_n^m(\theta, \phi)] &\equiv Y_n^m(\theta, \phi)\hat{r} = \boldsymbol{R}_n^m(\theta, \phi). \end{aligned} \tag{A19}$$

Let

$$\begin{aligned} M_{10}[Y_n^m(\theta, \phi)] &\equiv \nabla_1 Y_n^m(\theta, \phi) = c_1^{-1}(n)\boldsymbol{S}_n^m(\theta, \phi) \\ M_{11}[Y_n^m(\theta, \phi)] &\equiv \hat{r} \times \nabla_1 Y_n^m(\theta, \phi) = c_2^{-1}(n)\boldsymbol{T}_n^m(\theta, \phi) \\ M_{12}[\boldsymbol{S}_n^m(\theta, \phi)] &\equiv c_1(n) Y_n^m(\theta, \phi) \\ M_{13}[\boldsymbol{T}_n^m(\theta, \phi)] &\equiv c_2(n) Y_n^m(\theta, \phi), \end{aligned} \tag{A20}$$

and applying integration by parts, $\cos\theta * \boldsymbol{S}_n^m(\theta, \phi)$ becomes

$$\begin{aligned} \cos\theta * \boldsymbol{S}_n^m(\theta, \phi) &= \cos\theta * c_1(n)\nabla_1 Y_n^m(\theta, \phi) \\ &= c_1(n)[\nabla_1(\cos\theta * Y_n^m(\theta, \phi)) - Y_n^m(\theta, \phi) * \nabla_1 \cos\theta] \\ &= M_{10}[M_6[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]]] + c_1(n)\sin\theta * Y_n^m(\theta, \phi)\hat{\theta}. \end{aligned} \tag{A21}$$

So, it just requires the VSH form of $\sin\theta * Y_n^m(\theta, \phi)\hat{\theta}$. Using the relation (A7), the equation below is obtained,

$$\begin{aligned} &-\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta u_\theta) + \frac{\partial u_\phi}{\partial\phi}\right] \\ &= -\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin^2\theta Y_n^m(\theta, \phi))\right] = -2\cos\theta Y_n^m(\theta, \phi) - \sin\theta\frac{\partial}{\partial\theta}Y_n^m(\theta, \phi) \\ &= -2M_6[Y_n^m(\theta, \phi)] + M_7[Y_n^m(\theta, \phi)]. \end{aligned} \tag{A22}$$

By relation (A8), it is easy to get

$$+\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta u_\phi) - \frac{\partial u_\theta}{\partial\phi}\right] = -imY_n^m(\theta, \phi). \tag{A23}$$

Utilizing the auxiliary operator

$$\begin{aligned} M_{14}[Y_n^m(\theta, \phi)] &\equiv \boldsymbol{S}_n^m(\theta, \phi) \\ M_{15}[Y_n^m(\theta, \phi)] &\equiv \boldsymbol{T}_n^m(\theta, \phi), \end{aligned} \tag{A24}$$

and (A7), (A8), (A22) and (A23), the VSH representation of $\sin\theta * Y_n^m(\theta, \phi)\hat{\theta}$ is

$$\begin{aligned} &\sin\theta * Y_n^m(\theta, \phi)\hat{\theta} \\ &\quad = M_{14}\{-2M_6[Y_n^m(\theta, \phi)] + M_7[Y_n^m(\theta, \phi)]\} - M_{15}[imY_n^m(\theta, \phi)] \\ &\quad \equiv M_{16}[Y_n^m(\theta, \phi)]. \end{aligned} \tag{A25}$$

Then the final VSH representation of $\cos\theta * \boldsymbol{S}_n^m(\theta, \phi)$ is

$$\begin{aligned} &\cos\theta * \boldsymbol{S}_n^m(\theta, \phi) \\ &\quad = M_{10}[M_6[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]]] + M_{16}[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]] \\ &\quad \equiv M_6[\boldsymbol{S}_n^m(\theta, \phi)]. \end{aligned} \tag{A26}$$

By defining the $\hat{r}$-cross-product operator M_{17}

$$M_{17}[\boldsymbol{u}] \equiv -\hat{r} \times \boldsymbol{u} = \begin{cases} \boldsymbol{T}_n^m, & \text{if} \boldsymbol{u} = \boldsymbol{S}_n^m \\ -\boldsymbol{S}_n^m, & \text{if} \boldsymbol{u} = \boldsymbol{T}_n^m, \end{cases} \tag{A27}$$

the VSH representation of $\cos\theta * \boldsymbol{T}_n^m(\theta, \phi)$ is

$$\begin{aligned} \cos\theta * \boldsymbol{T}_n^m(\theta, \phi) &= -\hat{r} \times [\cos\theta * \boldsymbol{S}_n^m(\theta, \phi)] = M_{17}[M_6[\boldsymbol{S}_n^m(\theta, \phi)]] \\ &= M_{17}[M_6[M_{18}[\boldsymbol{T}_n^m(\theta, \phi)]]] \equiv M_6[\boldsymbol{T}_n^m(\theta, \phi)], \end{aligned} \tag{A28}$$

and M_{18} is

$$M_{18}[\boldsymbol{T}_n^m] \equiv \boldsymbol{S}_n^m. \tag{A29}$$

A.2. For an Asymmetric Model

For an asymmetric Earth model rather than a rotating oblate model described in the above subsection, the related parameters should be expressed in the following general forms

$$\begin{aligned}\rho &= \rho_0(r) + \sum_{n,m} \rho_n^m(r) Y_n^m(\theta, \phi)\\ \lambda &= \lambda_0(r) + \sum_{n,m} \lambda_n^m(r) Y_n^m(\theta, \phi)\\ \mu &= \mu_0(r) + \sum_{n,m} \mu_n^m(r) Y_n^m(\theta, \phi).\end{aligned} \tag{A30}$$

The product of two SHs is deduced first for convenience' sake, as $\sin\theta e^{\pm i\phi} * Y_n^m(\theta, \phi)$,

$$\begin{aligned}\sin\theta e^{i\phi} * Y_n^m(\theta, \phi) &= e^{i\phi} * e^{im\phi} c_3(n, m) \sin\theta * P_n^m(\cos\theta)\\ &= M_1[M_3[M_2[Y_n^m(\theta, \phi)]]] \equiv M_{19}[Y_n^m(\theta, \phi)],\end{aligned} \tag{A31}$$

$$\begin{aligned}\sin\theta e^{-i\phi} * Y_n^m(\theta, \phi) &= e^{-i\phi} * e^{im\phi} c_3(n, m) \sin\theta * P_n^m(\cos\theta)\\ &= M_1[M_4[M_2[Y_n^m(\theta, \phi)]]] \equiv M_{20}[Y_n^m(\theta, \phi)].\end{aligned} \tag{A32}$$

Let

$$M_{21}^{(s)} = \begin{cases} M_{19}, & \text{if } s = 1\\ M_{20}, & \text{if } s = -1.\end{cases} \tag{A33}$$

$P_n^m(\cos\theta)$ is a combination of $\cos\theta$ and $\sin\theta$. For $m > 0$,

$$\begin{aligned}P_n^m(\cos\theta) &= \sin^m\theta \sum_{s=0}^{[(n-m)/2]} \frac{(-1)^s(2n-2s)!}{2^n s!(n-s)!(n-2s-m)!} (\cos\theta)^{n-2s-m}\\ &= \sin^m\theta \sum_{s=0}^{[(n-m)/2]} c_7(n, m, s)(\cos\theta)^{n-2s-m},\end{aligned} \tag{A34}$$

$$\begin{aligned}Y_n^m(\theta, \phi) &= c_3(n, m) \sin^m\theta e^{im\phi} \sum_{s=0}^{[(n-m)/2]} c_7(n, m, s)(\cos\theta)^{n-2s-m}\\ &= \sum_{s=0}^{[(n-m)/2]} c_8(n, m, s)(\cos\theta)^{n-2s-m} (\sin\theta e^{i\phi})^m.\end{aligned} \tag{A35}$$

For $m < 0$,

$$P_n^m(\cos\theta) = (-1)^m \frac{(n+m)!}{(n-m)!} P_n^{-m}(\cos\theta) = c_9(n, m) P_n^{-m}(\cos\theta), \tag{A36}$$

$$\begin{aligned}Y_n^m(\theta, \phi) &= c_3(n, m) c_9(n, m) P_n^{|m|}(\cos\theta) e^{-i|m|\phi}\\ &= \sum_{s=0}^{[(n-|m|)/2]} c_3(n, m) c_9(n, m) c_7(n, |m|, s)(\cos\theta)^{n-2s-|m|} (\sin\theta e^{-i\phi})^{|m|}\end{aligned} \tag{A37}$$

$$c_{10}(n, m, s) = \begin{cases} c_8(n, m, s), & m > 0\\ c_3(n, m) c_9(n, m) c_7(n, |m|, s), & m < 0.\end{cases} \tag{A38}$$

Let

$$\zeta(m) = \frac{m}{|m|}, \tag{A39}$$

which is a sign function. For any m, $Y_n^m(\theta, \phi)$ has the unified form

$$Y_n^m(\theta, \phi) = \sum_{s=0}^{[(n-|m|)/2]} c_{10}(n, m, s)(\cos\theta)^{n-2s-|m|}(\sin\theta e^{\zeta(m)i\phi})^{|m|}. \tag{A40}$$

The product of two SHs is now regarded as actions of the $\cos\theta$ and $\sin\theta e^{\pm i\phi}$ operating on the second SH,

$$\begin{aligned} Y_n^m(\theta, \phi) * Y_a^b(\theta, \phi) &= \sum_{s=0}^{[(n-|m|)/2]} c_{10}(n, m, s)\sigma(M_6)^{n-2s-|m|}[(M_{21}^{(\zeta(m))})^{|m|}[Y_a^b(\theta, \phi)]] \\ &\equiv M_{23}^{(n,m)}[Y_a^b(\theta, \phi)]. \end{aligned} \tag{A41}$$

By the above equation,

$$\begin{aligned} Y_n^m(\theta, \phi) * \boldsymbol{R}_a^b(\theta, \phi) &= M_9[M_{23}^{(n,m)}[M_8[\boldsymbol{R}_a^b(\theta, \phi)]]] \\ &\equiv M_{23}^{(n,m)}[\boldsymbol{R}_a^b(\theta, \phi)]. \end{aligned} \tag{A42}$$

Before deducing $\sin\theta e^{\pm i\phi} * \boldsymbol{S}_n^m(\theta, \phi)$, two important relations of $\sin\theta\frac{\partial Y_n^m(\theta,\phi)}{\partial\theta}$ are needed:

$$\begin{aligned} \sin\theta\frac{\partial Y_n^m(\theta,\phi)}{\partial\theta} &= -\sin^2\theta\frac{\partial Y_n^m(\theta,\phi)}{\partial\cos\theta} = -(1-\cos^2\theta)\frac{\partial Y_n^m(\theta,\phi)}{\partial\cos\theta} \\ &= m\cos\theta Y_n^m(\theta, \phi) - \sin\theta c_3(n, m)P_n^{m+1}(\cos\theta)e^{im\phi}, \end{aligned} \tag{A43}$$

$$\begin{aligned} &\sin\theta\frac{\partial Y_n^m(\theta, \phi)}{\partial\theta} = -(1-\cos^2\theta)\frac{\partial Y_n^m(\theta, \phi)}{\partial\cos\theta} \\ &= -m\cos\theta Y_n^m(\theta, \phi) + \sin\theta c_3(n, m)(n+m)(n-m+1)P_n^{m-1}(\cos\theta)e^{im\phi} \\ &= -m\cos\theta Y_n^m(\theta, \phi) + c_3(n, m)c_{11}(n, m)\sin\theta P_n^{m-1}(\cos\theta)e^{im\phi}. \end{aligned} \tag{A44}$$

The procedure of representing $\sin\theta * Y_n^m(\theta, \phi)\hat{\theta}$ in VSHs is applied to $\sin\theta e^{i\phi} * \boldsymbol{S}_n^m(\theta, \phi)$. Let $\boldsymbol{u} = \sin\theta e^{i\phi} * \boldsymbol{S}_n^m(\theta, \phi)$, then the right hand side of Equation (A7) is

$$\begin{aligned} &-\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta u_\theta) + \frac{\partial u_\phi}{\partial\phi}\right] \\ &= -\frac{c_1(n)e^{i\phi}}{\sin\theta}[-n(n+1)\sin^2\theta Y_n^m(\theta, \phi) + \cos\theta\sin\theta\frac{\partial Y_n^m(\theta, \phi)}{\partial\theta} - mY_n^m(\theta, \phi)] \\ &= c_1(n)(n^2+n+m)\sin\theta e^{i\phi}Y_n^m(\theta, \phi) + c_1(n)\cos\theta c_3(n, m)P_n^{m+1}(\cos\theta)e^{i(m+1)\phi} \\ &= (n^2+n+m)M_{19}[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]] + M_1[M_0[M_{25}[M_2[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]]]]] \\ &\equiv M_{24}[\boldsymbol{S}_n^m(\theta, \phi)]. \end{aligned} \tag{A45}$$

$$\begin{aligned} M_{25}[P_n^m(\cos\theta)] &\equiv P_n^{m+1}(\cos\theta) \\ M_{26}[P_n^m(\cos\theta)] &\equiv P_n^{m-1}(\cos\theta). \end{aligned} \tag{A46}$$

The right hand side of Equation (A8) is

$$\begin{aligned} &+\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta u_\phi) - \frac{\partial u_\theta}{\partial\phi}\right] \\ &=+\frac{c_1(n)e^{i\phi}}{\sin\theta}\left[im\cos\theta Y_n^m(\theta, \phi)e^{i\phi} - i\sin\theta\frac{\partial Y_n^m(\theta, \phi)}{\partial\theta}\right] \\ &=+ic_1(n)c_3(n, m)P_n^{m+1}(\cos\theta)e^{i(m+1)\phi} \\ &=+iM_1[M_{25}[M_2[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]]]] \equiv M_{27}[\boldsymbol{S}_n^m(\theta, \phi)]. \end{aligned} \tag{A47}$$

So $\sin\theta e^{i\phi} * \boldsymbol{S}_n^m(\theta, \phi)$ in VSHs is

$$\begin{aligned} \sin\theta e^{i\phi} * \boldsymbol{S}_n^m(\theta, \phi) &= M_{14}[M_{24}[\boldsymbol{S}_n^m(\theta, \phi)]] + M_{15}[M_{27}[\boldsymbol{S}_n^m(\theta, \phi)]] \\ &\equiv M_{19}[\boldsymbol{S}_n^m(\theta, \phi)]]. \end{aligned} \tag{A48}$$

Also, $\sin\theta e^{i\phi} * \boldsymbol{T}_n^m(\theta, \phi)$ is

$$\begin{aligned}\sin\theta e^{i\phi} * \boldsymbol{T}_n^m(\theta, \phi) &= -\hat{r} \times [\sin\theta e^{i\phi} * \boldsymbol{S}_n^m(\theta, \phi)] \\ &= M_{17}[M_{19}[M_{18}[\boldsymbol{T}_n^m(\theta, \phi)]]] \equiv M_{19}[\boldsymbol{T}_n^m(\theta, \phi)]. \end{aligned} \tag{A49}$$

$\sin\theta e^{-i\phi} * \boldsymbol{S}_n^m(\theta, \phi)$ and $\sin\theta e^{-i\phi} * \boldsymbol{T}_n^m(\theta, \phi)$ are obtained by a similar procedure

$$\begin{aligned}&\sin\theta e^{-i\phi} * \boldsymbol{S}_n^m(\theta, \phi)\\ &= c_1(n) * M_{14}[\sin\theta e^{-i\phi} * (n^2 + n)Y_n^m(\theta, \phi) - m\sin\theta e^{-i\phi} Y_n^m(\theta, \phi)\\ &\quad -c_{11}(n, m)c_3(n, m)\cos\theta P_n^{m-1}(\cos\theta)e^{i(m-1)\phi}]\\ &\quad -c_1(n)c_3(n, m)M_{15}[ic_{11}(n, m)P_n^{m-1}e^{i(m-1)\phi}]\\ &= (n^2 + n - m)M_{14}[M_{20}[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]]]\\ &\quad -c_{11}(n, m)M_{14}[M_1[M_0[M_{26}[M_2[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]]]]]]\\ &\quad -ic_{11}(n, m)M_{15}[M_1[M_{26}[M_2[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]]]]]\\ &\equiv M_{20}[\boldsymbol{S}_n^m(\theta, \phi)],\end{aligned} \tag{A50}$$

and

$$\begin{aligned}\sin\theta e^{-i\phi} * \boldsymbol{T}_n^m(\theta, \phi) &= -\hat{r} \times [\sin\theta e^{-i\phi}\boldsymbol{S}_n^m(\theta, \phi)]\\ &= M_{17}[M_{20}[M_{18}[\boldsymbol{T}_n^m(\theta, \phi)]]] \equiv M_{20}[\boldsymbol{T}_n^m(\theta, \phi)].\end{aligned} \tag{A51}$$

Like the product of two SHs, $Y_n^m(\theta, \phi) * \boldsymbol{S}_a^b(\theta, \phi)$ can be regarded as the action of a combination of $\cos\theta$ and $\sin\theta e^{\pm i\phi}$ operating on $\boldsymbol{S}_a^b(\theta, \phi)$,

$$\begin{aligned}Y_n^m(\theta, \phi) * \boldsymbol{S}_a^b(\theta, \phi) &= \sum_{s=0}^{[(n-|m|)/2]} c_{10}(n, m, s)(M_6)^{n-2s-|m|}[(M_{21}^{(\zeta(m))})^{|m|}[\boldsymbol{S}_a^b(\theta, \phi)]]\\ &\equiv M_{23}^{(n,m)}[\boldsymbol{S}_a^b(\theta, \phi)].\end{aligned} \tag{A52}$$

Similarly,

$$\begin{aligned}Y_n^m(\theta, \phi) * \boldsymbol{T}_a^b(\theta, \phi) &= -\hat{r} \times [Y_n^m(\theta, \phi) * \boldsymbol{S}_a^b(\theta, \phi)]\\ &= M_{17}[M_{23}^{(n,m)}[M_{18}[\boldsymbol{T}_a^b(\theta, \phi)]]]\\ &\equiv M_{23}^{(n,m)}[\boldsymbol{T}_a^b(\theta, \phi)].\end{aligned} \tag{A53}$$

Appendix B
Operations of Two VSHs

B.1. Boundary Conditions

The boundary conditions on the displacement field $\boldsymbol{u}$, tensor field $\overleftrightarrow{S}$ and incremental Eulerian gravitational potential ϕ_1 are also very important. They are: $\boldsymbol{u} \cdot \hat{n}$, $\hat{n} \cdot \overleftrightarrow{S}$ and ϕ_1; $\hat{n} \cdot (\nabla\phi_1 + 4\pi G\rho\boldsymbol{u})$ should be continuous across any kind of boundary, and $\boldsymbol{u}$ should be continuous across any welded boundary between two solid regions.

In an asymmetric 3D Earth model a boundary surface is described as

$$r = r_0 + \sum_{n,m} \kappa_n^m Y_n^m(\theta, \phi). \tag{B1}$$

For a rotating elliptical Earth model, the above equation degenerates to a simple format $r = r_0 + \epsilon(r_0)P_2(\cos\theta)$.

The normal vector of this surface is

$$\begin{aligned} \boldsymbol{n} &= \nabla\left[r - r_0 - \sum_{n,m} \kappa_n^m Y_n^m(\theta, \phi)\right] \\ &= \hat{r} - \sum_{n,m}\left[\frac{\kappa_n^m}{r}\nabla_1 Y_n^m(\theta, \phi)\right] \\ &= \hat{r} - \sum_{n,m}\left[\frac{\kappa_n^m}{r}c_1^{-1}(n)\boldsymbol{S}_n^m(\theta, \phi)\right]. \end{aligned} \tag{B2}$$

For an isotropic medium, its tensor is

$$\overleftrightarrow{S} = \lambda(\nabla\cdot\boldsymbol{u})\overleftrightarrow{I} + \mu[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T]. \tag{B3}$$

The boundary condition of the tensor assumes that $\boldsymbol{n}\cdot\overleftrightarrow{S}$ should be continuous.

$$\boldsymbol{n}\cdot\overleftrightarrow{S} = \lambda(\nabla\cdot\boldsymbol{u})\boldsymbol{n}\cdot\overleftrightarrow{I} + \mu * \boldsymbol{n}\cdot[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T]. \tag{B4}$$

The first term on the right hand is just $\lambda(\nabla\cdot\boldsymbol{u})\boldsymbol{n}$. The second term $\boldsymbol{n}\cdot[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T]$ will be discussed in the next section.

B.2. Dot-product of Two VSHs

$\boldsymbol{R}_a^b(\theta, \phi)\cdot\boldsymbol{R}_n^m(\theta, \phi)$ is quite easy.

$$\boldsymbol{R}_a^b(\theta, \phi)\cdot\boldsymbol{R}_n^m(\theta, \phi) = M_{23}^{(a,b)}[M_8[\boldsymbol{R}_n^m(\theta, \phi)]] \equiv M_{28}^{(a,b)}[\boldsymbol{R}_n^m(\theta, \phi)]. \tag{B5}$$

Before discussing $\boldsymbol{S}_n^m(\theta, \phi)\cdot\boldsymbol{v}_1$, the Leibniz rule is applied on $\nabla_1 Y_n^m(\theta, \phi)$,

$$\begin{aligned} \nabla_1 Y_n^m(\theta, \phi) &= \sum_{s=0}^{[(n-|m|)/2]} c_{10}(n, m, s)\nabla_1[(\cos\theta)^{n-2s-|m|}(\sin\theta e^{i\zeta(m)\phi})^{|m|}] \\ &= \sum_{s=0}^{[(n-|m|)/2]}[|m| * c_{10}(n, m, s)(\cos\theta)^{n-2s-|m|} * (\sin\theta e^{\zeta(m)i\phi})^{|m|-1}\nabla_1(\sin\theta e^{\zeta(m)i\phi}) \\ &\quad +(n - 2s - |m|) * c_{10}(n, m, s)(\sin\theta e^{\zeta(m)i\phi})^{|m|}(\cos\theta)^{n-2s-|m|-1}\nabla_1\cos\theta]. \end{aligned} \tag{B6}$$

Let

$$\begin{aligned} H_0[Y_n^m(\theta, \phi)] = &\sum_{s=0}^{[(n-|m|)/2]}(n - 2s - |m|) * c_{10}(n, m, s)(\sin\theta e^{\zeta(m)i\phi})^{|m|} \\ &*(\cos\theta)^{n-2s-|m|-1} \\ H_1[Y_n^m(\theta, \phi)] = &\sum_{s=0}^{[(n-|m|)/2]}|m| * c_{10}(n, m, s)(\cos\theta)^{n-2s-|m|} \\ &*(\sin\theta e^{\zeta(m)i\phi})^{|m|-1}, \end{aligned} \tag{B7}$$

then $\nabla_1 Y_n^m(\theta, \phi)$ is represented by two basic vectors, $\nabla_1\cos\theta$ and $\nabla_1(\sin\theta e^{\zeta(m)i\phi})$.

$$\nabla_1 Y_n^m(\theta, \phi) = H_0[Y_n^m(\theta, \phi)] * \nabla_1\cos\theta + H_1[Y_n^m(\theta, \phi)] * \nabla_1(\sin\theta e^{\zeta(m)i\phi}). \tag{B8}$$

Let $B_1[\boldsymbol{v}_1, \boldsymbol{v}_2]$ denote the dot product of any two vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, then

$$\begin{aligned} &B_1[\nabla_1 Y_n^m(\theta, \phi), \boldsymbol{v}_2] = \nabla_1 Y_n^m(\theta, \phi)\cdot\boldsymbol{v}_2 \\ &\quad = H_0[Y_n^m(\theta, \phi)]B_1[\nabla_1\cos\theta, \boldsymbol{v}_2] + H_1[Y_n^m(\theta, \phi)]B_1[\nabla_1(\sin\theta e^{\zeta(m)i\phi}), \boldsymbol{v}_2]. \end{aligned} \tag{B9}$$

So, $B_1[\nabla_1 \cos\theta, \boldsymbol{v}_2]$ and $B_1[\nabla_1(\sin\theta e^{\zeta(m)i\phi}), \boldsymbol{v}_2]$ just need to be solved. For $\boldsymbol{v}_2 = \boldsymbol{S}_a^b(\theta, \phi) = c_1(a)\nabla_1 Y_a^b(\theta, \phi)$, we have

$$\nabla_1 \cos\theta \cdot \nabla_1 Y_a^b(\theta, \phi) = -\sin\theta \frac{\partial Y_a^b(\theta, \phi)}{\partial\theta} = M_7[Y_a^b(\theta, \phi)], \tag{B10}$$

$$\begin{aligned}\nabla_1(\sin\theta e^{i\phi}) \cdot \nabla_1 Y_a^b(\theta, \phi) &= \tfrac{e^{i\phi}}{\sin\theta}[\cos\theta \sin\theta \tfrac{\partial Y_a^b(\theta,\phi)}{\partial\theta} - bY_a^b(\theta, \phi)]\\ &= -b\sin\theta e^{i\phi} Y_a^b(\theta, \phi) - \cos\theta c_3(a, b) P_a^{b+1}(\cos\theta) e^{i(b+1)\phi}\\ &= -bM_{19}[Y_a^b(\theta, \phi)] - M_6[M_1[M_{25}[M_2[Y_a^b(\theta, \phi)]]]] \equiv M_{29}[Y_a^b(\theta, \phi)],\end{aligned} \tag{B11}$$

$$\begin{aligned}\nabla_1(\sin\theta e^{-i\phi}) \cdot \nabla_1 Y_a^b(\theta, \phi) &= \frac{e^{-i\phi}}{\sin\theta}\left[\cos\theta \sin\theta \frac{\partial Y_a^b(\theta, \phi)}{\partial\theta} + bY_a^b(\theta, \phi)\right]\\ &= b\sin\theta e^{-i\phi} Y_a^b(\theta, \phi) + \cos\theta c_3(a, b) c_{11}(a, b) P_a^{b-1}(\cos\theta) e^{i(b-1)\phi}\\ &= bM_{20}[Y_a^b(\theta, \phi)] + c_{11}(a, b) M_6[M_1[M_{26}[M_2[Y_a^b(\theta, \phi)]]]] \equiv M_{30}[Y_a^b(\theta, \phi)].\end{aligned} \tag{B12}$$

Let

$$M_{31}^{(s)}[Y_n^m(\theta, \phi)] = \begin{cases} M_{29}[Y_n^m(\theta, \phi)], & s > 0\\ M_{30}[Y_n^m(\theta, \phi)], & s < 0.\end{cases} \tag{B13}$$

B.3. Dot Product of a Spheroidal Vector and a Toroidal Vector

The dot product of two spheroidal vectors is

$$\begin{aligned}B_1[\boldsymbol{S}_n^m(\theta, \phi), \boldsymbol{S}_a^b(\theta, \phi)] = {}& H_0[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]] * M_7[M_{12}[\boldsymbol{S}_a^b(\theta, \phi)]]\\ &+ H_1[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]] * M_{31}^{(m)}[M_{12}[\boldsymbol{S}_a^b(\theta, \phi)]].\end{aligned} \tag{B14}$$

For $\boldsymbol{v}_2 = \boldsymbol{T}_a^b(\theta, \phi) = c_2(a)\hat{r} \times \nabla_1 Y_a^b(\theta, \phi)$,

$$\nabla_1 \cos\theta \cdot [\hat{r} \times \nabla_1 Y_a^b(\theta, \phi)] = ibY_a^b(\theta, \phi), \tag{B15}$$

$$\begin{aligned}\nabla_1(\sin\theta e^{i\phi}) \cdot [\hat{r} \times \nabla_1 Y_a^b(\theta, \phi)] &= \tfrac{ie^{i\phi}}{\sin\theta}\left[\sin\theta \tfrac{\partial Y_a^b(\theta,\phi)}{\partial\theta} - b\cos\theta Y_a^b(\theta, \phi)\right]\\ &= -ic_3(a, b) P_a^{b+1}(\cos\theta) e^{i(b+1)\phi}\\ &= -iM_1[M_{25}[M_2[Y_a^b(\theta, \phi)]\\ &\equiv M_{34}[Y_a^b(\theta, \phi)],\end{aligned} \tag{B16}$$

$$\begin{aligned}\nabla_1(\sin\theta e^{-i\phi}) \cdot 000[\hat{r} \times \nabla_1 Y_a^b(\theta, \phi)] &= \frac{-ie^{i\phi}}{\sin\theta}\left[\sin\theta \frac{\partial Y_a^b(\theta, \phi)}{\partial\theta} + b\cos\theta Y_a^b(\theta, \phi)\right]\\ &= -ic_3(a, b) c_{11}(a, b) P_a^{b-1}(\cos\theta) e^{i(b-1)\phi}\\ &= -ic_{11}(a.b) M_1[M_{26}[M_2[Y_a^b(\theta, \phi)]\\ &\equiv M_{35}[Y_a^b(\theta, \phi)].\end{aligned} \tag{B17}$$

Let

$$M_{36}^{(s)}[Y_n^m(\theta, \phi)] = \begin{cases} M_{34}[Y_n^m(\theta, \phi)], & s > 0\\ M_{35}[Y_n^m(\theta, \phi)], & s < 0,\end{cases} \tag{B18}$$

then the dot product of a spheroidal vector and a toroidal vector is

$$\begin{aligned}B_1[\boldsymbol{S}_n^m(\theta, \phi), \boldsymbol{T}_a^b(\theta, \phi)] = {}& H_0[M_{12}[\boldsymbol{S}_a^b(\theta, \phi)]] * \{ib * M_{13}[\boldsymbol{T}_a^b(\theta, \phi)]\}\\ &+ H_1[M_{12}[\boldsymbol{S}_a^b(\theta, \phi)]] * M_{36}^{(m)}[M_{13}[\boldsymbol{T}_a^b(\theta, \phi)]].\end{aligned} \tag{B19}$$

Using the relation below,

$$(\boldsymbol{v}_1 \times \boldsymbol{v}_2) \cdot (\boldsymbol{v}_3 \times \boldsymbol{v}_4) = (\boldsymbol{v}_1 \cdot \boldsymbol{v}_3) * (\boldsymbol{v}_2 \cdot \boldsymbol{v}_4) - (\boldsymbol{v}_1 \cdot \boldsymbol{v}_4) * (\boldsymbol{v}_2 \cdot \boldsymbol{v}_3), \tag{B20}$$

the dot product of two toroidal vectors is

$$\begin{aligned} B1[\boldsymbol{T}_a^b(\theta,\phi),\boldsymbol{T}_n^m(\theta,\phi)] &= \boldsymbol{T}_a^b(\theta,\phi)\cdot\boldsymbol{T}_n^m(\theta,\phi)\\ &= c_2(a)c_2(n)[\hat{r}\times\nabla_1 Y_a^b(\theta,\phi)]\cdot[\hat{r}\times\nabla_1 Y_n^m(\theta,\phi)]\\ &= c_2(a)c_2(n)\{[\hat{r}\cdot\hat{r}]*[\nabla_1 Y_a^b(\theta,\phi)\cdot\nabla_1 Y_n^m(\theta,\phi)]\\ &\quad -[\hat{r}\cdot\nabla_1 Y_n^m(\theta,\phi)]*[\hat{r}\cdot\nabla_1 Y_a^b(\theta,\phi)]\}\\ &= c_2(a)c_2(n)\nabla_1 Y_a^b(\theta,\phi)\cdot\nabla_1 Y_n^m(\theta,\phi)\\ &= \boldsymbol{S}_a^b(\theta,\phi)\cdot\boldsymbol{S}_n^m(\theta,\phi). \end{aligned} \tag{B21}$$

B.4. Cross Product of Two VSHs

$\boldsymbol{R}_n^m(\theta,\phi)\times\boldsymbol{\Psi}_a^b(\theta,\phi)$ is

$$\begin{aligned} &\boldsymbol{R}_n^m(\theta,\phi)\times\boldsymbol{R}_a^b(\theta,\phi)=0\\ &\boldsymbol{R}_n^m(\theta,\phi)\times\boldsymbol{S}_a^b(\theta,\phi)=Y_n^m(\theta,\phi)*\hat{r}\times\boldsymbol{S}_a^b(\theta,\phi)=-M_{23}^{(n,m)}[\boldsymbol{T}_a^b(\theta,\phi)]\\ &\boldsymbol{R}_n^m(\theta,\phi)\times\boldsymbol{T}_a^b(\theta,\phi)=Y_n^m(\theta,\phi)*\hat{r}\times\boldsymbol{T}_a^b(\theta,\phi)=M_{23}^{(n,m)}[\boldsymbol{S}_a^b(\theta,\phi)]. \end{aligned} \tag{B22}$$

With Equation (B8), the cross product of two spheroidal vectors is

$$\begin{aligned} &\boldsymbol{S}_n^m(\theta,\phi)\times\boldsymbol{S}_a^b(\theta,\phi)=c_1(n)c_1(a)\nabla_1 Y_n^m(\theta,\phi)\times\nabla_1 Y_a^b(\theta,\phi)\\ &=c_1(n)c_1(a)\{H_0[Y_n^m(\theta,\phi]*\nabla_1\cos\theta\times\nabla_1 Y_a^b(\theta,\phi)\\ &\quad+H_1[Y_n^m(\theta,\phi]*\nabla_1(\sin\theta e^{\zeta(m)i\phi})\times\nabla_1 Y_a^b(\theta,\phi)\}, \end{aligned} \tag{B23}$$

where

$$\begin{aligned} \nabla_1\cos\theta\times\nabla_1 Y_a^b(\theta,\phi) &= -ibY_a^b(\theta,\phi)\hat{r}=-ib\boldsymbol{R}_n^m(\theta,\phi)\\ \nabla_1(\sin\theta e^{i\phi})\times\nabla_1 Y_a^b(\theta,\phi) &= -ie^{i\phi}\hat{r}\left[\frac{\partial Y_a^b(\theta,\phi)}{\partial\theta}-b\frac{\cos\theta}{\sin\theta}Y_a^b(\theta,\phi)\right]\\ &= ic_3(a,b)c_3^{-1}(a,b+1)\boldsymbol{R}_a^{b+1}(\theta,\phi)\\ \nabla_1(\sin\theta e^{-i\phi})\times\nabla_1 Y_a^b(\theta,\phi) &= ie^{-i\phi}\hat{r}\left[\frac{\partial Y_a^b(\theta,\phi)}{\partial\theta}+b\frac{\cos\theta}{\sin\theta}Y_a^b(\theta,\phi)\right]\\ &= ic_{11}(a,b)c_3^{-1}(a,b-1)\boldsymbol{R}_a^{b-1}(\theta,\phi). \end{aligned} \tag{B24}$$

Finally, the cross product of a spheroidal vector with a toroidal vector is

$$\begin{aligned} \boldsymbol{S}_n^m(\theta,\phi)\times\boldsymbol{T}_a^b(\theta,\phi) &= c_1(n)c_2(a)\nabla_1 Y_n^m(\theta,\phi)\times[\hat{r}\times\nabla_1 Y_a^b(\theta,\phi)]\\ &= c_1(n)c_2(a)\hat{r}[\nabla_1 Y_n^m(\theta,\phi)\cdot\nabla_1 Y_a^b(\theta,\phi)]. \end{aligned} \tag{B25}$$

The cross product of two toroidal vectors is

$$\begin{aligned} \boldsymbol{T}_n^m(\theta,\phi)\times\boldsymbol{T}_a^b(\theta,\phi) &= c_2(n)c_2(a)[\hat{r}\times\nabla_1 Y_n^m(\theta,\phi)]\times[\hat{r}\times\nabla_1 Y_a^b(\theta,\phi)]\\ &= c_1(n)c_2(a)\{\hat{r}\cdot[\nabla_1 Y_n^m(\theta,\phi)\times\nabla_1 Y_a^b(\theta,\phi)]\}\hat{r}. \end{aligned} \tag{B26}$$

Appendix C
Divergence of a Dyadic Tensor

The divergence of the tensor is

$$\nabla\cdot\overleftrightarrow{S}=\nabla\cdot[\lambda(\nabla\cdot\boldsymbol{u})\overleftrightarrow{I}]+\nabla\cdot\{\mu[\nabla\boldsymbol{u}+(\nabla\boldsymbol{u})^T]\}. \tag{C1}$$

The displacement vector $\boldsymbol{u}$ has the form (see Equation (A2)),

$$\boldsymbol{u} = \sum_{n,m}[U_n^m(r)\boldsymbol{R}_n^m(\theta, \phi) + V_n^m(r)\boldsymbol{S}_n^m(\theta, \phi) + W_n^m(r)\boldsymbol{T}_n^m(\theta, \phi)]. \tag{C2}$$

It is easy to obtain

$$\nabla \cdot [\chi_n^m(r)\boldsymbol{\Psi}_n^m(\theta, \phi)] = \begin{cases} \left[\partial_r\chi_n^m(r) + \dfrac{2}{r}\chi_n^m(r)\right] * Y_n^m(\theta, \phi), & \text{if } \boldsymbol{\Psi}_n^m(\theta, \phi) = \boldsymbol{R}_n^m(\theta, \phi) \\ -\dfrac{1}{r}\chi_n^m(r) * Y_n^m(\theta, \phi), & \text{if } \boldsymbol{\Psi}_n^m(\theta, \phi) = \boldsymbol{S}_n^m(\theta, \phi) \\ 0, & \text{if } \boldsymbol{\Psi}_n^m(\theta, \phi) = \boldsymbol{T}_n^m(\theta, \phi). \end{cases} \tag{C3}$$

The divergence of a second-order tensor in a spherical coordinate basis is

$$\begin{aligned} \nabla \cdot \overleftrightarrow{T} = &\left\{\partial_r T_{rr} + r^{-1}\left[\partial_\theta T_{\theta r} + \frac{1}{\sin\theta}\partial_\phi T_{\phi r} + 2T_{rr} - T_{\theta\theta} - T_{\phi\phi} + \cot T_{\theta r}\right]\right\}\hat{r} \\ &+ \left\{\partial_r T_{r\theta} + \frac{1}{r}\left[\partial_\theta T_{\theta\theta} + \frac{1}{\sin\theta}\partial_\phi T_{\phi\theta} + 2T_{r\theta} + T_{\theta r} + \cot\theta(T_{\theta\theta} - T_{\phi\phi})\right]\right\}\hat{\theta} \\ &+ \left\{\partial_r T_{r\phi} + \frac{1}{r}\left[\partial_\theta T_{\theta\phi} + \frac{1}{\sin\theta}\partial_\phi T_{\phi\phi} + 2T_{r\phi} + T_{\phi r} + \cot\theta(T_{\theta\phi} + T_{\phi\theta})\right]\right\}\hat{\phi}. \end{aligned} \tag{C4}$$

Let

$$\lambda(\nabla \cdot \boldsymbol{u}) = \gamma = \sum_n^m \gamma_n^m(r)Y_n^m(\theta, \phi), \tag{C5}$$

the first term on the right hand of Equation (C1) is

$$\begin{aligned} &\nabla \cdot [\lambda(\nabla \cdot \boldsymbol{u})\overleftrightarrow{I}] = \nabla \cdot (\gamma\overleftrightarrow{I}) \\ &= \hat{r}\partial_r\gamma + \hat{\theta}\frac{1}{r}\partial_\theta\gamma + \hat{\phi}\frac{1}{r * \sin\theta}\partial_\phi\gamma \\ &= \sum_n^m\left\{\partial_r\gamma_n^m(r)\boldsymbol{R}_n^m(\theta, \phi) + \frac{1}{r}\gamma_n^m(r)\left[\partial_\theta Y_n^m(\theta, \phi)\hat{\theta} + \frac{\partial\phi Y_n^m(\theta, \phi)}{\sin\theta}\hat{\phi}\right]\right\} \\ &= \sum_n^m\left\{\partial_r\gamma_n^m(r)\boldsymbol{R}_n^m(\theta, \phi) + \frac{1}{r}\gamma_n^m(r)c_1^{-1}(n)\boldsymbol{S}_n^m(\theta, \phi)\right\}. \end{aligned} \tag{C6}$$

The second term on the right hand side of Equation (C1) is

$$\begin{aligned} &\nabla \cdot \{\mu[\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T]\} = \mu\nabla \cdot [\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T] + \nabla\mu \cdot [\nabla\boldsymbol{u} + (\nabla\boldsymbol{u})^T] \\ &= \mu\nabla \cdot (\nabla\boldsymbol{u}) + \mu\nabla \cdot (\nabla\boldsymbol{u})^T + \nabla\mu \cdot (\nabla\boldsymbol{u}) + \nabla\mu \cdot (\nabla\boldsymbol{u})^T. \end{aligned} \tag{C7}$$

C.1. The First Term of Equation (C7)

There is a useful equation to compute the first term on the right hand of Equation (C7), which is

$$\nabla \cdot \nabla\boldsymbol{u} = \nabla(\nabla \cdot \boldsymbol{u}) - \nabla \times \nabla \times \boldsymbol{u}. \tag{C8}$$

With the following relations

$$\begin{aligned} \hat{r} \times \nabla_1\psi &= -\nabla \times (\hat{r}\psi) \\ \nabla_1 \times \nabla_1 &= \hat{r} \times \nabla_1 \\ \nabla_1 \times (\hat{r} \times \nabla_1) &= \hat{r}\nabla_1^2 - \nabla_1, \end{aligned} \tag{C9}$$

the curls of VSHs are obtained,

$$\begin{aligned}\nabla\times[\chi_n^m(r)*\boldsymbol{R}_n^m(\theta,\phi)]&=\frac{-c_2^{-1}(n)}{r}\chi_n^m(r)*\boldsymbol{T}_n^m(\theta,\phi)\\ \nabla\times[\chi_n^m(r)*\boldsymbol{S}_n^m(\theta,\phi)]&=-\left[\frac{\partial\chi_n^m(r)}{\partial r}+\frac{1}{r}\chi_n^m(r)\right]\boldsymbol{T}_n^m(\theta,\phi)\\ \nabla\times[\chi_n^m(r)*\boldsymbol{T}_n^m(\theta,\phi)]&=\frac{\chi_n^m(r)}{r}\boldsymbol{R}_n^m(\theta,\phi)+\left[\frac{\partial\chi_n^m(r)}{\partial r}+\frac{\chi_n^m(r)}{r}\right]\boldsymbol{S}_n^m(\theta,\phi).\end{aligned}\tag{C10}$$

Using Equations (C3), (C8) and (C10) leads to those useful equations,

$$\begin{aligned}\nabla_1\cdot\nabla_1\boldsymbol{R}_n^m(\theta,\phi)&=-(n^2+n+2)\boldsymbol{R}_n^m(\theta,\phi)+2c_1^{-1}(n)\boldsymbol{S}_n^m(\theta,\phi)\\ \nabla_1\cdot\nabla_1\boldsymbol{S}_n^m(\theta,\phi)&=2\boldsymbol{R}_n^m(\theta,\phi)-c_1^{-1}(n)\boldsymbol{S}_n^m(\theta,\phi)\\ \nabla_1\cdot\nabla_1\boldsymbol{T}_n^m(\theta,\phi)&=-n(n+1)\boldsymbol{T}_n^m(\theta,\phi).\end{aligned}\tag{C11}$$

C.2. The Second Term of Equation (C7)

The second term of Equation (C7), $\nabla\cdot(\nabla\boldsymbol{u})^T$, is

$$\begin{aligned}&\nabla\cdot\{\nabla[\chi_n^m(r)\boldsymbol{\Psi}_n^m(\theta,\phi)]\}^T\\ &=\hat{r}\cdot\{\partial_r[\partial_r\chi_n^m(r)\boldsymbol{\Psi}_n^m(\theta,\phi)\otimes\hat{r}]\}+\hat{r}\cdot\left\{\partial_r\left[\frac{1}{r}\chi_n^m(r)(\nabla_1\boldsymbol{\Psi}_n^m(\theta,\phi)\right]^T\right\}\\ &\quad+\frac{1}{r}\nabla_1\cdot[\partial_r\chi_n^m(r)\boldsymbol{\Psi}_n^m(\theta,\phi)\otimes\hat{r}]+\frac{1}{r}\nabla_1\cdot\left[\frac{1}{r}\chi_n^m(r)(\nabla_1\boldsymbol{\Psi}_n^m(\theta,\phi))^T\right]\\ &=\partial_{rr}^2[\chi_n^m(r)]*[\hat{r}\cdot\boldsymbol{\Psi}_n^m(\theta,\phi)\otimes\hat{r}]+\partial_r\left[\frac{1}{r}\chi_n^m(r)\right]*[\hat{r}\cdot(\nabla_1\boldsymbol{\Psi}_n^m(\theta,\phi))^T]\\ &\quad+\frac{1}{r}\partial_r\chi_n^m(r)\nabla_1\cdot[\boldsymbol{\Psi}_n^m(\theta,\phi)\otimes\hat{r}]+\frac{1}{r^2}\chi_n^m(r)\nabla_1\cdot[\nabla_1\boldsymbol{\Psi}_n^m(\theta,\phi)]^T.\end{aligned}\tag{C12}$$

We now process the four terms of the above equation one by one.

The second part of the first term in Equation (C12) is

$$\hat{r}\cdot\boldsymbol{\Psi}_n^m(\theta,\phi)\otimes\hat{r}=\begin{cases}\boldsymbol{R}_n^m(\theta,\phi), & \text{if } \boldsymbol{\Psi}_n^m(\theta,\phi)=\boldsymbol{R}_n^m(\theta,\phi)\\ 0, & \text{if } \boldsymbol{\Psi}_n^m(\theta,\phi)=\boldsymbol{S}_n^m(\theta,\phi)\text{or}\boldsymbol{T}_n^m(\theta,\phi).\end{cases}\tag{C13}$$

For the second term of Equation (C12), $\hat{r}\cdot(\nabla_1\boldsymbol{\Psi}_n^m(\theta,\phi))^T$, we first list the spherical surface (or horizontal) gradients of VSHs on a unit sphere:

$$\begin{aligned}\nabla_1\boldsymbol{R}_n^m(\theta,\phi)&=\hat{\theta}\partial_\theta[\boldsymbol{R}_n^m(\theta,\phi)]+\hat{\phi}\tfrac{1}{\sin\theta}\partial_\phi[\boldsymbol{R}_n^m(\theta,\phi)]\\ &=\partial_\theta Y_n^m(\theta,\phi)\hat{\theta}\hat{r}+\tfrac{1}{\sin\theta}\partial_\phi Y_n^m(\theta,\phi)\hat{\phi}\hat{r}+Y_n^m(\theta,\phi)\hat{\theta}\hat{\theta}+Y_n^m(\theta,\phi)\hat{\phi}\hat{\phi},\end{aligned}\tag{C14}$$

$$\begin{aligned}\nabla_1\boldsymbol{S}_n^m(\theta,\phi)&=c_1(n)\nabla_1\nabla_1Y_n^m(\theta,\phi)\\ &=c_1(n)\Big\{-\hat{\theta}\hat{r}\partial_\theta-\hat{\phi}\hat{r}\tfrac{1}{\sin\theta}\partial_\phi+(\hat{\theta}\hat{\phi}+\hat{\phi}\hat{\theta})\tfrac{1}{\sin\theta}(\partial_\theta\partial_\phi-\cot\theta\partial_\phi)\\ &\quad+\hat{\theta}\hat{\theta}\partial_\theta^2+\hat{\phi}\hat{\phi}\left[\tfrac{1}{\sin^2\theta}\partial_\phi^2+\cot\theta\partial_\theta\right]\Big\}Y_n^m(\theta,\phi),\end{aligned}\tag{C15}$$

$$\begin{aligned}\nabla_1\boldsymbol{T}_n^m(\theta,\phi)&=-c_2(n)\nabla_1[-\hat{r}\times\nabla_1Y_n^m(\theta,\phi)]\\ &=-c_2(n)\left\{-\hat{\theta}\hat{r}\frac{1}{\sin\theta}\partial_\phi+\hat{\phi}\hat{r}\partial_\theta+(\hat{\theta}\hat{\theta}-\hat{\phi}\hat{\phi})\frac{1}{\sin\theta}(\partial_\theta\partial_\phi-\cot\theta\partial_\phi)\right.\\ &\quad\left.-\hat{\theta}\hat{\phi}\partial_\theta^2+\hat{\phi}\hat{\theta}\left[\frac{1}{\sin^2\theta}\partial_\phi^2+\cot\theta\partial_\theta\right]\right\}Y_n^m(\theta,\phi).\end{aligned}\tag{C16}$$

So, the second term of Equation (C12), $\hat{r} \cdot [\nabla_1 \boldsymbol{\Psi}_n^m(\theta, \phi)]^T$, can become

$$\begin{aligned}\hat{r} \cdot [\nabla_1 \boldsymbol{R}_n^m(\theta, \phi)]^T &= \hat{r} \cdot \left[\partial_\theta Y_n^m(\theta, \phi)\hat{\theta}\hat{r} + \tfrac{1}{\sin\theta}\partial_\phi Y_n^m(\theta, \phi)\hat{\phi}\hat{r}\right]^T \\ &= \partial_\theta Y_n^m(\theta, \phi)\hat{\theta} + \tfrac{1}{\sin\theta}\partial_\phi Y_n^m(\theta, \phi)\hat{\phi} \\ &= c_1^{-1}(n)\boldsymbol{S}_n^m(\theta, \phi), \end{aligned} \tag{C17}$$

$$\begin{aligned}\hat{r} \cdot [\nabla_1 \boldsymbol{S}_n^m(\theta, \phi)]^T &= c_1(n)\hat{r} \cdot \left[-\hat{\theta}\hat{r}\partial_\theta Y_n^m(\theta, \phi) - \hat{\phi}\hat{r}\tfrac{1}{\sin\theta}\partial_\phi Y_n^m(\theta, \phi)\right]^T \\ &= -\boldsymbol{S}_n^m(\theta, \phi),\end{aligned} \tag{C18}$$

$$\begin{aligned}\hat{r} \cdot [\nabla_1 \boldsymbol{T}_n^m(\theta, \phi)]^T &= -c_2(n)\hat{r} \cdot \left[-\hat{\theta}\hat{r}\frac{1}{\sin\theta}\partial_\phi Y_n^m(\theta, \phi) + \hat{\phi}\hat{r}\partial_\theta Y_n^m(\theta, \phi)\right]^T \\ &= -\boldsymbol{T}_n^m(\theta, \phi).\end{aligned} \tag{C19}$$

Now, let us process the third term of Equation (C12). According to Equation (C4), spherical surface divergence of a second-order tensor is

$$\begin{aligned}\nabla_1 \cdot \overleftrightarrow{T} = &\left\{\partial_\theta T_{\theta r} + \frac{1}{\sin\theta}\partial_\phi T_{\phi r} + 2T_{rr} - T_{\theta\theta} - T_{\phi\phi} + \cot\theta T_{\theta r}\right\}\hat{r} \\ &+ \left\{\partial_\theta T_{\theta\theta} + \frac{1}{\sin\theta}\partial_\phi T_{\phi\theta} + 2T_{r\theta} + T_{\theta r} + \cot\theta(T_{\theta\theta} - T_{\phi\phi})\right\}\hat{\theta} \\ &+ \left\{\partial_\theta T_{\theta\phi} + \frac{1}{\sin\theta}\partial_\phi T_{\phi\phi} + 2T_{r\phi} + T_{\phi r} + \cot\theta(T_{\theta\phi} + T_{\phi\theta})\right\}\hat{\phi}.\end{aligned} \tag{C20}$$

For $\overleftrightarrow{T} = \boldsymbol{\Psi}_n^m(\theta, \phi) \otimes \hat{r}$,

$$\nabla_1 \cdot \overleftrightarrow{T} = \left\{\partial_\theta T_{\theta r} + \frac{1}{\sin\theta}\partial_\phi T_{\phi r} + 2T_{rr} + \cot\theta T_{\theta r}\right\}\hat{r} + T_{\theta r}\hat{\theta} + T_{\phi r}\hat{\phi}. \tag{C21}$$

So, the third term of Equation (C12), $\nabla_1 \cdot [\boldsymbol{\Psi}_n^m(\theta, \phi) \otimes \hat{r}]$, now turns into

$$\nabla_1 \cdot [\boldsymbol{R}_n^m(\theta, \phi) \otimes \hat{r}] = 2T_{rr}\hat{r} = 2\boldsymbol{R}_n^m(\theta, \phi), \tag{C22}$$

$$\begin{aligned}&\nabla_1 \cdot [\boldsymbol{S}_n^m(\theta, \phi) \otimes \hat{r}] \\ &= c_1(n)\left[\frac{d^2 Y_n^m(\theta, \phi)}{d\theta^2} + \cot\theta\frac{dY_n^m(\theta, \phi)}{d\theta} - \frac{m^2}{\sin^2\theta}Y_n^m(\theta, \phi)\right]\hat{r} + \boldsymbol{S}_n^m(\theta, \phi) \\ &= -c_1(n)n(n+1)\boldsymbol{R}_n^m(\theta, \phi) + \boldsymbol{S}_n^m(\theta, \phi).\end{aligned} \tag{C23}$$

$$\begin{aligned}&\nabla_1 \cdot [\boldsymbol{T}_n^m(\theta, \phi) \otimes \hat{r}] \\ &= c_2(n)\left[-im\partial_\theta\frac{Y_n^m(\theta, \phi)}{\sin\theta} + \frac{im}{\sin\theta}\partial_\theta Y_n^m(\theta, \phi) - \cot\theta\frac{imY_n^m(\theta, \phi)}{\sin\theta}\right]\hat{r} \\ &\quad + \boldsymbol{T}_n^m(\theta, \phi) = \boldsymbol{T}_n^m(\theta, \phi).\end{aligned} \tag{C24}$$

For the fourth term of Equation (C12), $\nabla_1 \cdot [\nabla_1 \boldsymbol{\Psi}_n^m(\theta, \phi)]^T$. As $\boldsymbol{\Psi}_n^m(\theta, \phi) = \boldsymbol{R}_n^m(\theta, \phi)$,

$$\begin{aligned}[\nabla_1 \boldsymbol{R}_n^m(\theta, \phi)]^T &= \{\nabla_1 [Y_n^m(\theta, \phi) \otimes \hat{r}]\}^T \\ &= \frac{\partial Y_n^m(\theta, \phi)}{\partial\theta}\hat{r}\hat{\theta} + \frac{1}{\sin\theta}\frac{\partial Y_n^m(\theta, \phi)}{\partial\phi}\hat{r}\hat{\phi} + Y_n^m(\theta, \phi)\hat{\theta}\hat{\theta} + Y_n^m(\theta, \phi)\hat{\phi}\hat{\phi}.\end{aligned} \tag{C25}$$

$\nabla_1 \cdot [\nabla_1 \boldsymbol{R}_n^m(\theta, \phi)]^T$ then becomes

$$\nabla_1 \cdot [\nabla_1 \boldsymbol{R}_n^m(\theta, \phi)]^T = -2\boldsymbol{R}_n^m(\theta, \phi) + 3c_1^{-1}(n)\boldsymbol{S}_n^m(\theta, \phi). \tag{C26}$$

With the following useful relation (Dahlen & Tromp 1998),

$$(\nabla_1\nabla_1)^T = \nabla_1\nabla_1 - \hat{r}\nabla_1 + (\hat{r}\nabla_1)^T, \tag{C27}$$

we can get,

$$\begin{aligned}\nabla_1\cdot[\nabla_1 \boldsymbol{S}_n^m(\theta,\phi)]^T &= c_1(n)\nabla_1\cdot[\nabla_1\nabla_1 Y_n^m(\theta,\phi)]^T\\ &= c_1(n)\nabla_1\cdot\{\nabla_1\nabla_1 Y_n^m(\theta,\phi)-\hat{r}\nabla_1 Y_n^m(\theta,\phi)+[\nabla_1 Y_n^m(\theta,\phi)]\hat{r}\}\\ &= \nabla_1^2 \boldsymbol{S}_n^m(\theta,\phi)-2\boldsymbol{S}_n^m(\theta,\phi)-c_1(n)n(n+1)\boldsymbol{R}_n^m(\theta,\phi)+\boldsymbol{S}_n^m(\theta,\phi)\\ &= \boldsymbol{R}_n^m(\theta,\phi)-(n^2+n+1)\boldsymbol{S}_n^m(\theta,\phi).\end{aligned} \tag{C28}$$

With the relation,

$$[\nabla_1(\hat{r}\times\nabla_1)]^T=(\hat{r}\times\nabla_1)\nabla_1-\hat{r}(\hat{r}\times\nabla_1)+[\hat{r}(\hat{r}\times\nabla_1)]^T, \tag{C29}$$

we get

$$\begin{aligned}\nabla_1\cdot(\nabla_1 \boldsymbol{T}_n^m(\theta,\phi))^T &= c_2(n)\nabla_1\cdot[\nabla_1(\hat{r}\times\nabla_1 Y_n^m(\theta,\phi)]^T\\ &= c_2(n)\nabla_1\cdot[(\hat{r}\times\nabla_1)\nabla_1 Y_n^m(\theta,\phi)]-\nabla_1\cdot[\hat{r}\otimes(\hat{r}\times\nabla_1 Y_n^m(\theta,\phi))]\\ &\quad+\nabla_1\cdot[(\hat{r}\times\nabla_1 Y_n^m(\theta,\phi))\otimes\hat{r}]\\ &= c_2(n)\nabla_1\cdot[(\hat{r}\times\nabla_1)\nabla_1 Y_n^m(\theta,\phi)]-\nabla_1\cdot[\hat{r}\otimes \boldsymbol{T}_n^m(\theta,\phi)]+\nabla_1\cdot[\boldsymbol{T}_n^m(\theta,\phi)\otimes\hat{r}]\\ &= 0-2\boldsymbol{T}_n^m(\theta,\phi)+\boldsymbol{T}_n^m(\theta,\phi)=-\boldsymbol{T}_n^m(\theta,\phi).\end{aligned} \tag{C30}$$

C.3. The Third Term of Equation (C7)

For an asymmetrical Earth model,

$$\mu=\mu_0(r)+\sum_{a,b}\mu_a^b(r)Y_a^b(\theta,\phi), \tag{C31}$$

the third term of Equation (C7) is

$$\begin{aligned}&\nabla\mu\cdot\nabla[\chi_n^m(r)\boldsymbol{\Psi}_n^m(\theta,\phi)]\\ &=\left[\frac{\partial\mu}{\partial r}\hat{r}+\frac{1}{r}\nabla_1\mu\right]\cdot\left[\partial_r\chi_n^m(r)\hat{r}\otimes\boldsymbol{\Psi}_n^m(\theta,\phi)+\frac{1}{r}\chi_n^m(r)\nabla_1\boldsymbol{\Psi}_n^m(\theta,\phi)\right]\\ &=\frac{\partial\mu}{\partial r}\partial_r\chi_n^m(r)\boldsymbol{\Psi}_n^m(\theta,\phi)+\frac{1}{r^2}\chi_n^m(r)\nabla_1\mu\cdot\nabla_1\boldsymbol{\Psi}_n^m(\theta,\phi)\\ &=\frac{\partial\mu}{\partial r}\partial_r\chi_n^m(r)\boldsymbol{\Psi}_n^m(\theta,\phi)+\frac{1}{r^2}\chi_n^m(r)\sum_{a,b}\mu_a^b(r)\nabla_1 Y_a^b(\theta,\phi)\cdot\nabla_1\boldsymbol{\Psi}_n^m(\theta,\phi).\end{aligned} \tag{C32}$$

For $\boldsymbol{\Psi}_n^m(\theta,\phi)=\boldsymbol{R}_n^m(\theta,\phi)$,

$$\begin{aligned}\nabla_1\boldsymbol{R}_n^m(\theta,\phi)&=\nabla_1 Y_n^m(\theta,\phi)\otimes\hat{r}+Y_n^m(\theta,\phi)\nabla_1\hat{r}\\ &=\nabla_1 Y_n^m(\theta,\phi)\otimes\hat{r}+Y_n^m(\theta,\phi)(\hat{\theta}\hat{\theta}+\hat{\phi}\hat{\phi}),\end{aligned} \tag{C33}$$

and the second term of Equation (C32) is

$$\begin{aligned}&\nabla_1 Y_a^b(\theta,\phi)\cdot\nabla_1\boldsymbol{R}_n^m(\theta,\phi)\\ &=[\nabla_1 Y_a^b(\theta,\phi)\cdot\nabla_1 Y_n^m(\theta,\phi)]\hat{r}+Y_n^m(\theta,\phi)\nabla_1 Y_a^b(\theta,\phi)\\ &=M_9[\nabla_1 Y_a^b(\theta,\phi)\cdot\nabla_1 Y_n^m(\theta,\phi)]+c_1^{-1}(a)Y_n^m(\theta,\phi)*\boldsymbol{S}_a^b(\theta,\phi).\end{aligned} \tag{C34}$$

For $\boldsymbol{\Psi}_n^m(\theta,\phi)=\boldsymbol{S}_n^m(\theta,\phi)$,

$$\begin{aligned}\nabla_1\boldsymbol{S}_n^m(\theta,\phi)&=c_1(n)\nabla_1\nabla_1 Y_n^m(\theta,\phi)\\ &=c_1(n)\sum_{s=0}^{[(n-|m|)/2]}c_{10}(n,m,s)\nabla_1\nabla_1[(\cos\theta)^{n-2s-|m|}(\sin\theta e^{i\zeta(m)\phi})^{|m|}].\end{aligned} \tag{C35}$$

Using the Leibniz rule twice, we get

$$\begin{aligned}\nabla_1\nabla_1[(\cos\theta)^k\,(\sin\theta e^{\zeta(m)i\phi})^l\,] &= \nabla_1[(\sin\theta e^{\zeta(m)i\phi})^l\ *k*(\cos\theta)^{k-1}\nabla_1\cos\theta]\\ &\quad+\nabla_1[(\cos\theta)^k\ *l*(\sin\theta e^{\zeta(m)i\phi})^{l-1}\nabla_1(\sin\theta e^{\zeta(m)i\phi})]\\ &= H_2^{(k,l)}\nabla_1\nabla_1\cos\theta + H_3^{(k,l)}\nabla_1(\sin\theta e^{\zeta(m)i\phi})\otimes\nabla_1\cos\theta\\ &\quad+H_4^{(k,l)}\nabla_1\cos\theta\otimes\nabla_1\cos\theta + H_5^{(k,l)}\nabla_1\nabla_1(\sin\theta e^{\zeta(m)i\phi})\\ &\quad+H_6^{(k,l)}\nabla_1\cos\theta\otimes\nabla_1(\sin\theta e^{\zeta(m)i\phi})\\ &\quad+H_7^{(k,l)}\nabla_1(\sin\theta e^{\zeta(m)i\phi})\otimes\nabla_1(\sin\theta e^{\zeta(m)i\phi}),\end{aligned}\tag{C36}$$

where

$$\begin{aligned} H_2^{(k,l)} &= (\sin\theta e^{\zeta(m)i\phi})^l\ *k*(\cos\theta)^{k-1}\\ H_3^{(k,l)} &= k*l*(\cos\theta)^{k-1}(\sin\theta e^{\zeta(m)i\phi})^{l-1}\\ H_4^{(k,l)} &= k*(k-1)(\sin\theta e^{\zeta(m)i\phi})^l(\cos\theta)^{k-2}\\ H_5^{(k,l)} &= (\cos\theta)^k\ *l*(\sin\theta e^{\zeta(m)i\phi})^{l-1}\\ H_6^{(k,l)} &= k*l(\cos\theta)^{k-1}(\sin\theta e^{\zeta(m)i\phi})^{l-1}\\ H_7^{(k,l)} &= (\cos\theta)^k\ *l*(l-1)(\sin\theta e^{\zeta(m)i\phi})^{l-2}\end{aligned}\ ,\tag{C37}$$

$$\begin{aligned}\nabla_1\nabla_1\cos\theta &= -\nabla_1\cos\theta\otimes\hat r - \cos\theta(\hat\theta\hat\theta+\hat\phi\hat\phi)\\ \nabla_1\nabla_1(\sin\theta e^{i\phi}) &= -\nabla_1(\sin\theta e^{i\phi})\otimes\hat r - \sin\theta e^{i\phi}(\hat\theta\hat\theta+\hat\phi\hat\phi)\\ \nabla_1\nabla_1(\sin\theta e^{-i\phi}) &= -\nabla_1(\sin\theta e^{-i\phi})\otimes\hat r - \sin\theta e^{-i\phi}(\hat\theta\hat\theta+\hat\phi\hat\phi).\end{aligned}\tag{C38}$$

To get the VSH representation of $\nabla_1 Y_a^b(\theta,\phi)\cdot\nabla_1\boldsymbol{S}_n^m(\theta,\phi)$, we need

$$\begin{aligned}\nabla_1 Y_n^m(\theta,\phi)\cdot\nabla_1\nabla_1\cos\theta &= -(\nabla_1 Y_n^m(\theta,\phi)\cdot\nabla_1\cos\theta)\hat r\\ &\quad-\cos\theta*\nabla_1 Y_n^m(\theta,\phi)\\ \nabla_1 Y_n^m(\theta,\phi)\cdot\nabla_1\nabla_1\sin\theta e^{i\phi} &= -(\nabla_1 Y_n^m(\theta,\phi)\cdot\nabla_1(\sin\theta e^{i\phi}))\hat r\\ &\quad-\sin\theta e^{i\phi}*\nabla_1 Y_n^m(\theta,\phi)\\ \nabla_1 Y_n^m(\theta,\phi)\cdot\nabla_1\nabla_1\sin\theta e^{-i\phi} &= -(\nabla_1 Y_n^m(\theta,\phi)\cdot\nabla_1(\sin\theta e^{-i\phi}))\hat r\\ &\quad-\sin\theta e^{-i\phi}*\nabla_1 Y_n^m(\theta,\phi).\end{aligned}\tag{C39}$$

For $\boldsymbol{\Psi}_n^m(\theta,\phi)=\boldsymbol{T}_n^m(\theta,\phi)$,

$$\begin{aligned}&\nabla_1\{\hat r\times\nabla_1[(\cos\theta)^k\,(\sin\theta e^{\zeta(m)i\phi})^l\,]\}\\ &\ =\nabla_1[(\sin\theta e^{\zeta(m)i\phi})^l\ *k*(\cos\theta)^{k-1}\hat r\times\nabla_1\cos\theta]\\ &\quad+\nabla_1[(\cos\theta)^k\ *l*(\sin\theta e^{\zeta(m)i\phi})^{l-1}\hat r\times\nabla_1(\sin\theta e^{\zeta(m)i\phi})]\\ &\ =H_2^{(k,l)}\nabla_1(\hat r\times\nabla_1\cos\theta)+H_3^{(k,l)}\nabla_1(\sin\theta e^{\zeta(m)i\phi})\otimes(\hat r\times\nabla_1\cos\theta)\\ &\quad+H_4^{(k,l)}\nabla_1\cos\theta\otimes(\hat r\times\nabla_1\cos\theta)+H_5^{(k,l)}\nabla_1[\hat r\times\nabla_1(\sin\theta e^{\zeta(m)i\phi})]\\ &\quad+H_6^{(k,l)}\nabla_1\cos\theta\otimes(\hat r\times\nabla_1(\sin\theta e^{\zeta(m)i\phi}))\\ &\quad+H_7^{(k,l)}\nabla_1(\sin\theta e^{\zeta(m)i\phi})\otimes[\hat r\times\nabla_1(\sin\theta e^{\zeta(m)i\phi})],\end{aligned}\tag{C40}$$

where

$$\begin{aligned}\nabla_1(\hat r\times\nabla_1\cos\theta) &= -\hat r\times\nabla_1\cos\theta\otimes\hat r+\cos\theta(\hat\phi\hat\theta-\hat\theta\hat\phi)\\ \nabla_1[\hat r\times\nabla_1(\sin\theta e^{i\phi})] &= -\hat r\times\nabla_1(\sin\theta e^{i\phi})\otimes\hat r+\sin\theta e^{i\phi}(\hat\phi\hat\theta-\hat\theta\hat\phi)\\ \nabla_1[\hat r\times\nabla_1(\sin\theta e^{-i\phi})] &= -\hat r\times\nabla_1(\sin\theta e^{-i\phi})\otimes\hat r+\sin\theta e^{-i\phi}(\hat\phi\hat\theta-\hat\theta\hat\phi).\end{aligned}\tag{C41}$$

With

$$\begin{aligned}
&\nabla_1 Y_n^m(\theta,\phi)\cdot\nabla_1(\hat{r}\times\nabla_1\cos\theta)\\
&\quad=-[\nabla_1 Y_n^m(\theta,\phi)\cdot(\hat{r}\times\nabla_1\cos\theta)]\hat{r}-\cos\theta*[\hat{r}\times\nabla_1 Y_n^m(\theta,\phi)]\\
&\nabla_1 Y_n^m(\theta,\phi)\cdot\nabla_1[\hat{r}\times\nabla_1(\sin\theta e^{i\phi})]\\
&\quad=-\{\nabla_1 Y_n^m(\theta,\phi)\cdot[\hat{r}\times\nabla_1(\sin\theta e^{i\phi})]\}\hat{r}-\sin\theta e^{i\phi}*[\hat{r}\times\nabla_1 Y_n^m(\theta,\phi)]\\
&\nabla_1 Y_n^m(\theta,\phi)\cdot\nabla_1[\hat{r}\times\nabla_1(\sin\theta e^{-i\phi})]\\
&\quad=-\{\nabla_1 Y_n^m(\theta,\phi)\cdot[\hat{r}\times\nabla_1(\sin\theta e^{-i\phi})]\}\hat{r}-\sin\theta e^{-i\phi}*[\hat{r}\times\nabla_1 Y_n^m(\theta,\phi)],
\end{aligned}\tag{C42}$$

the VSH representation of $\nabla_1 Y_a^b(\theta,\phi)\cdot\nabla_1\boldsymbol{T}_n^m(\theta,\phi)$ is obtained.

C.4. The Last Term of Equation (C7)

The last term of Equation (C7) is

$$\begin{aligned}
&\nabla\mu\cdot\{\nabla[\chi_n^m(r)\boldsymbol{\Psi}_n^m(\theta,\phi)]\}^T\\
&\quad=\left[\frac{\partial\mu}{\partial r}\hat{r}+\frac{1}{r}\nabla_1\mu\right]\cdot[\partial_r\chi_n^m(r)\boldsymbol{\Psi}_n^m(\theta,\phi)\otimes\hat{r}+\frac{1}{r}\chi_n^m(r)[\nabla_1\boldsymbol{\Psi}_n^m(\theta,\phi)]^T\\
&\quad=\frac{\partial\mu}{\partial r}\partial_r\chi_n^m(r)\hat{r}\cdot\boldsymbol{\Psi}_n^m(\theta,\phi)\otimes\hat{r}+\frac{1}{r}\frac{\partial\mu}{\partial r}\chi_n^m(r)\hat{r}\cdot[\nabla_1\boldsymbol{\Psi}_n^m(\theta,\phi)]^T\\
&\quad\quad+\frac{1}{r}\partial_r\chi_n^m(r)\nabla_1\mu\cdot\boldsymbol{\Psi}_n^m(\theta,\phi)\otimes\hat{r}+\frac{1}{r^2}\chi_n^m(r)\nabla_1\mu\cdot[\nabla_1\boldsymbol{\Psi}_n^m(\theta,\phi)]^T.
\end{aligned}\tag{C43}$$

For $\boldsymbol{\Psi}_n^m(\theta,\phi)=\boldsymbol{R}_n^m(\theta,\phi)$, the fourth term of Equation (C43) is

$$\nabla_1\mu\cdot[\nabla_1\boldsymbol{R}_n^m(\theta,\phi)]^T=Y_n^m(\theta,\phi)*\nabla_1\mu.\tag{C44}$$

For $\boldsymbol{\Psi}_n^m(\theta,\phi)=\boldsymbol{S}_n^m(\theta,\phi)$, as

$$[\nabla_1\nabla_1 Y_n^m(\theta,\phi)]^T=\nabla_1\nabla_1 Y_n^m(\theta,\phi)-\hat{r}\nabla_1 Y_n^m(\theta,\phi)+[\nabla_1 Y_n^m(\theta,\phi)]\hat{r},\tag{C45}$$

the fourth term of Equation (C43) turns into

$$\nabla_1\mu\cdot[\nabla_1\nabla_1 Y_n^m(\theta,\phi)]^T=\nabla_1\mu\cdot\nabla_1\nabla_1 Y_n^m(\theta,\phi)+\nabla_1\mu\cdot[\nabla_1 Y_n^m(\theta,\phi)]\otimes\hat{r}.\tag{C46}$$

Also, the terms on the right hand side above are already obtained.

For $\boldsymbol{\Psi}_n^m(\theta,\phi)=\boldsymbol{T}_n^m(\theta,\phi)$, $[\nabla_1\boldsymbol{\Psi}_n^m(\theta,\phi)]^T$ turns into

$$\begin{aligned}
&(\nabla_1\{\hat{r}\times\nabla_1[(\cos\theta)^k(\sin\theta e^{\zeta(m)i\phi})^l]\})^T\\
&\quad=\{\nabla_1[(\sin\theta e^{\zeta(m)i\phi})^l*k*(\cos\theta)^{k-1}\hat{r}\times\nabla_1\cos\theta]\}^T\\
&\quad\quad+\{\nabla_1[(\cos\theta)^k*l*(\sin\theta e^{\zeta(m)i\phi})^{l-1}\hat{r}\times\nabla_1(\sin\theta e^{\zeta(m)i\phi})]\}^T\\
&\quad=H_2^{(k,l)}[\nabla_1(\hat{r}\times\nabla_1\cos\theta)]^T+H_3^{(k,l)}(\hat{r}\times\nabla_1\cos\theta)\otimes\nabla_1(\sin\theta e^{\zeta(m)i\phi})\\
&\quad\quad+H_4^{(k,l)}(\hat{r}\times\nabla_1\cos\theta)\otimes\nabla_1\cos\theta+H_5^{(k,l)}\{\nabla_1[\hat{r}\times\nabla_1(\sin\theta e^{\zeta(m)i\phi})]\}^T\\
&\quad\quad+H_6^{(k,l)}(\hat{r}\times\nabla_1(\sin\theta e^{\zeta(m)i\phi}))\otimes\nabla_1\cos\theta\\
&\quad\quad+H_7^{(k,l)}[\hat{r}\times\nabla_1(\sin\theta e^{\zeta(m)i\phi})]\otimes\nabla_1(\sin\theta e^{\zeta(m)i\phi}),
\end{aligned}\tag{C47}$$

where

$$\begin{aligned}
[\nabla_1(\hat{r}\times\nabla_1\cos\theta)]^T&=-\hat{r}\otimes\hat{r}\times\nabla_1\cos\theta-\cos\theta(\hat{\phi}\hat{\theta}-\hat{\theta}\hat{\phi})\\
\{\nabla_1[\hat{r}\times\nabla_1(\sin\theta e^{i\phi})]\}^T&=-\hat{r}\otimes\hat{r}\times\nabla_1(\sin\theta e^{i\phi})-\sin\theta e^{i\phi}(\hat{\phi}\hat{\theta}-\hat{\theta}\hat{\phi})\\
\{\nabla_1[\hat{r}\times\nabla_1(\sin\theta e^{-i\phi})]\}^T&=-\hat{r}\otimes\hat{r}\times\nabla_1(\sin\theta e^{-i\phi})-\sin\theta e^{-i\phi}(\hat{\phi}\hat{\theta}-\hat{\theta}\hat{\phi}).
\end{aligned}\tag{C48}$$

With

$$\begin{aligned}\nabla_1 Y_n^m(\theta, \phi) \cdot [\nabla_1(\hat{r} \times \nabla_1 \cos\theta)]^T &= +\cos\theta * [\hat{r} \times \nabla_1 Y_n^m(\theta, \phi)]\\ \nabla_1 Y_n^m(\theta, \phi) \cdot \{\nabla_1[\hat{r} \times \nabla_1(\sin\theta e^{i\phi})]\}^T &= +\sin\theta e^{i\phi} * [\hat{r} \times \nabla_1 Y_n^m(\theta, \phi)]\\ \nabla_1 Y_n^m(\theta, \phi) \cdot \{\nabla_1[\hat{r} \times \nabla_1(\sin\theta e^{-i\phi})]\}^T &= +\sin\theta e^{-i\phi} * [\hat{r} \times \nabla_1 Y_n^m(\theta, \phi)],\end{aligned} \tag{C49}$$

the VSH representation of $\nabla_1\mu \cdot [\nabla_1 \boldsymbol{T}_n^m(\theta, \phi)]^T$ is obtained.

After all these procedures in this section, $\nabla \cdot \overleftrightarrow{S}$ can be finally represented in VSHs. $\boldsymbol{n} \cdot \overleftrightarrow{S}$ in Section 3 can be computed with a similar procedure as $\nabla\mu \cdot \overleftrightarrow{T}$ that has been computed in this section.

Appendix D
The Final Equations

By the LOM shown in the above sections, the VSH representation of the equations on an asymmetric 3D model is obtained. In this approach, we do not need GSSHs or Wigner 3-j symbols. Finally the dynamic equations will neither have $\cos\theta$ nor $\sin\theta e^{\pm i\phi}$ explicitly, and they can be transformed into the following form

$$\sum_{n,m} U_n^{*m}(r) * \boldsymbol{R}_n^m(\theta, \phi) + V_n^{*m}(r) * \boldsymbol{S}_n^m(\theta, \phi) + W_n^{*m}(r) * \boldsymbol{T}_n^m(\theta, \phi) = 0, \tag{D1}$$

where

$$\begin{aligned}U_n^{*m}(r) &= \sum_{i,j} L_{(1,1)}^{(n,m,i,j)}[U_i^j(r)] + L_{(1,2)}^{(n,m,i,j)}[V_i^j(r)] + L_{(1,3)}^{(n,m,i,j)}[W_i^j(r)]\\ V_n^{*m}(r) &= \sum_{i,j} L_{(2,1)}^{(n,m,i,j)}[U_i^j(r)] + L_{(2,2)}^{(n,m,i,j)}[V_i^j(r)] + L_{(2,3)}^{(n,m,i,j)}[W_i^j(r)]\\ W_n^{*m}(r) &= \sum_{i,j} L_{(3,1)}^{(n,m,i,j)}[U_i^j(r)] + L_{(3,2)}^{(n,m,i,j)}[V_i^j(r)] + L_{(3,3)}^{(n,m,i,j)}[W_i^j(r)],\end{aligned} \tag{D2}$$

and

$$L_{(a,b)}^{(n,m,i,j)}[f_i^j(r)] = x_{(a,b)}^{(n,m,i,j)} f_i^j(r) + y_{(a,b)}^{(n,m,i,j)} \frac{\partial f_i^j(r)}{\partial r} + z_{(a,b)}^{(n,m,i,j)} \frac{\partial^2 f_i^j(r)}{\partial r^2}. \tag{D3}$$

There are several methods to solve Equation (D1), such as Runge–Kutta integration (Smith 1974), collocation method and Galerkin method.

Appendix E
Definitions of Coefficients c_n

$$c_1(n) = \frac{1}{n(n+1)}, \tag{E1}$$

$$c_2(n) = -\frac{1}{n(n+1)}, \tag{E2}$$

$$c_3(n, m) = (-1)^m \left(\frac{2n+1}{4\pi}\right)^{1/2} \left[\frac{(n-m)!}{(n+m)!}\right]^{1/2}, \tag{E3}$$

$$c_4(n, m) = \frac{n-m+1}{2n+1}, \tag{E4}$$

$$c_5(n, m) = \frac{n+m}{2n+1}, \tag{E5}$$

$$c_6(n, m) = \frac{1}{n+m+1}, \tag{E6}$$

$$c_7(n, m, s) = \frac{(-1)^s(2n-2s)!}{2^n s!(n-s)!(n-2s-m)!}, \tag{E7}$$

$$c_8(n, m, s) = c_3(n, m)c_7(n, m, s), \tag{E8}$$

$$c_9(n, m) = (-1)^m \frac{(n+m)!}{(n-m)!}, \tag{E9}$$

$$c_{10}(n, m, s) = \begin{cases} c_8(n, m, s), & m > 0 \\ c_3(n, m)c_9(n, m)c_7(n, |m|, s), & m < 0 \end{cases}, \tag{E10}$$

$$c_{11}(n, m) = (n+m)(n-m+1). \tag{E11}$$

Appendix F
Definitions of Main Linear Operators

$$M_0[P_n^m(\cos\theta)] = c_4(n, m)P_{n+1}^m(\cos\theta) + c_5(n, m)P_{n-1}^m(\cos\theta), \tag{F1}$$

$$M_1[P_n^m(\cos\theta)] = c_3^{-1}(n, m)Y_n^m(\theta, \phi), \tag{F2}$$

$$M_2[Y_n^m(\theta, \phi)] = c_3(n, m)P_n^m(\cos\theta), \tag{F3}$$

$$M_3[P_n^m(\cos\theta)] = c_6(n, m)\{P_{n+1}^{m+1}(\cos\theta) - M_0[P_n^{m+1}(\cos\theta)]\}, \tag{F4}$$

$$M_4[P_n^m(\cos\theta)] = (n+m)M_0[P_n^{m-1}(\cos\theta)] - (n-m+2)P_{n+1}^{m-1}(\cos\theta), \tag{F5}$$

$$M_5[P_n^m(\cos\theta)] = (n+m)P_{n-1}^m(\cos\theta) - nM_0[P_n^m(\cos\theta)], \tag{F6}$$

$$M_6[X] = \begin{cases} M_1[M_0[M_2[Y_n^m(\theta, \phi)]]], & X = Y_n^m(\theta, \phi) \\ M_{10}[M_6[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]]] + M_{16}[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]], & X = \boldsymbol{S}_n^m(\theta, \phi) \\ M_{17}[M_6[M_{18}[\boldsymbol{T}_n^m(\theta, \phi)]]], & X = \boldsymbol{T}_n^m(\theta, \phi) \end{cases} \tag{F7}$$

$$M_7[Y_n^m(\theta, \phi)] = M_1[M_5[M_2[Y_n^m(\theta, \phi)]]], \tag{F8}$$

$$M_8[\boldsymbol{R}_n^m(\theta, \phi)] = Y_n^m(\theta, \phi), \tag{F9}$$

$$M_9[Y_n^m(\theta, \phi)] = \boldsymbol{R}_n^m(\theta, \phi), \tag{F10}$$

$$M_{10}[Y_n^m(\theta, \phi)] = c_1^{-1}(n)\boldsymbol{S}_n^m(\theta, \phi), \tag{F11}$$

$$M_{11}[Y_n^m(\theta, \phi)] = c_2^{-1}(n)\boldsymbol{T}_n^m(\theta, \phi), \tag{F12}$$

$$M_{12}[\boldsymbol{S}_n^m(\theta, \phi)] = c_1(n)Y_n^m(\theta, \phi), \tag{F13}$$

$$M_{13}[\boldsymbol{T}_n^m(\theta, \phi)] = c_2(n)Y_n^m(\theta, \phi), \tag{F14}$$

$$M_{14}[Y_n^m(\theta, \phi)] = \boldsymbol{S}_n^m(\theta, \phi), \tag{F15}$$

$$M_{15}[Y_n^m(\theta, \phi)] = \boldsymbol{T}_n^m(\theta, \phi), \tag{F16}$$

$$M_{16}[Y_n^m(\theta, \phi)] = M_{14}\{-2M_6[Y_n^m(\theta, \phi)] + M_7[Y_n^m(\theta, \phi)]\} - M_{15}[imY_n^m(\theta, \phi)], \tag{F17}$$

$$M_{17}[\boldsymbol{u}] = \begin{cases} \boldsymbol{T}_n^m, & \text{if } \boldsymbol{u} = \boldsymbol{S}_n^m \\ -\boldsymbol{S}_n^m, & \text{if } \boldsymbol{u} = \boldsymbol{T}_n^m \end{cases}, \tag{F18}$$

$$M_{18}[\boldsymbol{T}_n^m] \equiv \boldsymbol{S}_n^m. \tag{F19}$$

$$M_{19}[X] = \begin{cases} M_1[M_3[M_2[Y_n^m(\theta, \phi)]]], & X = Y_n^m(\theta, \phi) \\ M_{14}[M_{24}[\boldsymbol{S}_n^m(\theta, \phi)]] + M_{15}[M_{27}[\boldsymbol{S}_n^m(\theta, \phi)]], & X = \boldsymbol{S}_n^m(\theta, \phi) \\ M_{17}[M_{19}[M_{18}[\boldsymbol{T}_n^m(\theta, \phi)]]], & X = \boldsymbol{T}_n^m(\theta, \phi) \end{cases} \tag{F20}$$

$$M_{20}[X] = \begin{cases} M_1[M_4[M_2[Y_n^m(\theta, \phi)]]], & X = Y_n^m(\theta, \phi) \\ (n^2 + n - m)M_{14}[M_{20}[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]]] & \\ -c_{11}(n, m)M_{14}[M_1[M_0[M_{26}[M_2[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]]]]]] & \\ -ic_{11}(n, m)M_{15}[M_1[M_{26}[M_2[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]]]]], & X = \boldsymbol{S}_n^m(\theta, \phi) \\ M_{17}[M_{20}[M_{18}[\boldsymbol{T}_n^m(\theta, \phi)]]], & X = \boldsymbol{T}_n^m(\theta, \phi) \end{cases} \tag{F21}$$

$$M_{21}^{(s)} = \begin{cases} M_{19}, & \text{if} s = 1 \\ M_{20}, & \text{if} s = -1 \end{cases}, \tag{F22}$$

$$M_{23}^{(n,m)}[X] = \begin{cases} \sum_{s=0}^{[(n-|m|)/2]} c_{10}(n, m, s)(M_6)^{n-2s-|m|}[(M_{21}^{(\varsigma(m))})^{|m|}[Y_a^b(\theta, \phi)]], & X = Y_a^b(\theta, \phi) \\ M_9[M_{23}^{(n,m)}[M_8[\boldsymbol{R}_a^b(\theta, \phi)]]], & X = \boldsymbol{R}_a^b(\theta, \phi) \\ \sum_{s=0}^{[(n-|m|)/2]} c_{10}(n, m, s)(M_6)^{n-2s-|m|}[(M_{21}^{(\varsigma(m))})^{|m|}[\boldsymbol{S}_a^b(\theta, \phi)]], & X = \boldsymbol{S}_a^b(\theta, \phi) \\ M_{17}[M_{23}^{(n,m)}[M_{18}[\boldsymbol{T}_a^b(\theta, \phi)]]], & X = \boldsymbol{T}_a^b(\theta, \phi) \end{cases} \tag{F23}$$

$$\begin{aligned} M_{24}[\boldsymbol{S}_n^m(\theta, \phi)] = & (n^2 + n + m)M_{19}[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]] \\ & + M_1[M_0[M_{25}[M_2[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]]]]] \end{aligned}, \tag{F24}$$

$$M_{25}[P_n^m(\cos\theta)] = P_n^{m+1}(\cos\theta), \tag{F25}$$

$$M_{26}[P_n^m(\cos\theta)] = P_n^{m-1}(\cos\theta), \tag{F26}$$

$$M_{27}[\boldsymbol{S}_n^m(\theta, \phi)] = +iM_1[M_{25}[M_2[M_{12}[\boldsymbol{S}_n^m(\theta, \phi)]]]]. \tag{F27}$$

Appendix G
Some Useful Equations of Three Basic Operators

We list some useful operations on VSHs, i.e., Y_n^m (which is similar to $\boldsymbol{R}_n^m$), $\boldsymbol{S}_n^m$ and $\boldsymbol{T}_n^m$.

$$\cos\theta * Y_n^m(\theta, \phi) = d_1(n, m) * Y_{n+1}^m(\theta, \phi) + d_2(n, m) * Y_{n-1}^m(\theta, \phi), \tag{G1}$$

where

$$\begin{aligned} d_1(n, m) &= c_3(n, m)c_4(n, m)c_3^{-1}(n + 1, m) \\ d_2(n, m) &= c_3(n, m)c_5(n, m)c_3^{-1}(n - 1, m). \end{aligned} \tag{G2}$$

$$\sin\theta\frac{\partial Y_n^m(\theta, \phi)}{\partial\theta} = d_3(n, m)Y_{n+1}^m(\theta, \phi) + d_4(n, m)Y_{n-1}^m(\theta, \phi), \tag{G3}$$

where

$$\begin{aligned} d_3(n, m) &= c_3(n, m) * n * c_4(n, m)c_3^{-1}(n + 1, m) \\ d_4(n, m) &= c_3(n, m) * n * c_5(n, m)c_3^{-1}(n - 1, m) \\ &\quad - c_3(n, m)(n + m)c_3^{-1}(n - 1, m). \end{aligned} \tag{G4}$$

$$\sin\theta * Y_n^m(\theta, \phi)\hat{\theta} = -im\boldsymbol{T}_n^m(\theta, \phi) + d_5(n, m)\boldsymbol{S}_{n+1}^m(\theta, \phi) + d_6(n, m)\boldsymbol{S}_{n-1}^m(\theta, \phi), \tag{G5}$$

where

$$d_5(n, m) = -2d_1(n, m) - d_3(n, m)$$
$$d_6(n, m) = -2d_2(n, m) - d_4(n, m). \quad \text{(G6)}$$

$$\cos\theta * \boldsymbol{S}_n^m(\theta, \phi) = d_7(n, m) * \boldsymbol{S}_{n+1}^m(\theta, \phi) + d_8(n, m) * \boldsymbol{S}_{n-1}^m(\theta, \phi) + d_9(n, m) * \boldsymbol{T}_n^m(\theta, \phi), \quad \text{(G7)}$$

where

$$d_7(n, m) = c_1(n)[d_1(n, m)c_1^{-1}(n + 1) + d_5(n, m)]$$
$$d_8(n, m) = c_1(n)[d_2(n, m)c_1^{-1}(n - 1) + d_6(n, m)$$
$$d_9(n, m) = -imc_1(n). \quad \text{(G8)}$$

$$\cos\theta * \boldsymbol{T}_n^m(\theta, \phi) = d_7(n, m) * \boldsymbol{T}_{n+1}^m(\theta, \phi) + d_8(n, m) * \boldsymbol{T}_{n-1}^m(\theta, \phi) - d_9(n, m) * \boldsymbol{S}_n^m(\theta, \phi), \quad \text{(G9)}$$

$$\sin\theta e^{i\phi} * Y_n^m(\theta, \phi) = d_{10}(n, m)Y_{n+1}^{m+1}(\theta, \phi) + d_{11}(n, m)Y_{n-1}^{m+1}(\theta, \phi), \quad \text{(G10)}$$

where

$$d_{10}(n, m) = c_3(n, m)c_3^{-1}(n + 1, m + 1)c_6(n, m)[1 - c_4(n, m + 1)]$$
$$d_{11}(n, m) = -c_3(n, m)c_3^{-1}(n - 1, m + 1)c_6(n, m)c_5(n, m + 1). \quad \text{(G11)}$$

$$\sin\theta e^{-i\phi} * Y_n^m(\theta, \phi) = d_{12}(n, m)Y_{n+1}^{m-1}(\theta, \phi) + d_{13}(n, m)Y_{n-1}^{m-1}(\theta, \phi), \quad \text{(G12)}$$

where

$$d_{12}(n, m) = c_3(n, m)c_3^{-1}(n + 1, m - 1)[(n + m)c_4(n, m - 1) - (n - m + 2)]$$
$$d_{13}(n, m) = c_3(n, m)c_3^{-1}(n - 1, m - 1)(n + m)c_5(n, m - 1). \quad \text{(G13)}$$

$$\sin\theta e^{i\phi} * \boldsymbol{S}_n^m(\theta, \phi) = d_{14}(n, m) * \boldsymbol{S}_{n+1}^{m+1}(\theta, \phi) + d_{15}(n, m) * \boldsymbol{S}_{n-1}^{m+1}(\theta, \phi) + d_{16}(n, m) * \boldsymbol{T}_n^{m+1}(\theta, \phi), \quad \text{(G14)}$$

where

$$d_{14}(n, m) = c_1(n)(n^2 + n + m)d_{10}(n, m) + c_1(n)c_3(n, m)c_3^{-1}(n, m + 1)d_1(n, m + 1)$$
$$d_{15}(n, m) = c_1(n)(n^2 + n + m)d_{11}(n, m) + c_1(n)c_3(n, m)c_3^{-1}(n, m + 1)d_2(n, m + 1)$$
$$d_{16}(n, m) = ic_1(n)c_3(n, m)c_3^{-1}(n, m + 1). \quad \text{(G15)}$$

$$\sin\theta e^{i\phi} * \boldsymbol{T}_n^m(\theta, \phi) = \begin{array}{l} d_{14}(n, m) * \boldsymbol{T}_{n+1}^{m+1}(\theta, \phi) + d_{15}(n, m) * \boldsymbol{T}_{n-1}^{m+1}(\theta, \phi) \\ -d_{16}(n, m) * \boldsymbol{S}_n^{m+1}(\theta, \phi) \end{array}, \quad \text{(G16)}$$

$$\sin\theta e^{-i\phi} * \boldsymbol{S}_n^m(\theta, \phi) = d_{17}(n, m) * \boldsymbol{S}_{n+1}^{m-1}(\theta, \phi) + d_{18}(n, m) * \boldsymbol{S}_{n-1}^{m-1}(\theta, \phi) + d_{19}(n, m) * \boldsymbol{T}_n^{m-1}(\theta, \phi), \quad \text{(G17)}$$

where

$$d_{17}(n, m) = c_1(n)(n^2 + n - m)d_{12}(n, m) - c_1(n)c_{11}(n, m)c_3(n, m)c_3^{-1}(n, m - 1)d_1(n, m - 1)$$
$$d_{18}(n, m) = c_1(n)(n^2 + n - m)d_{13}(n, m) - c_1(n)c_{11}(n, m)c_3(n, m)c_3^{-1}(n, m - 1)d_2(n, m - 1)$$
$$d_{19}(n, m) = -ic_1(n)c_3(n, m)c_{11}(n, m)c_3^{-1}(n, m - 1). \quad \text{(G18)}$$

$$\sin\theta e^{-i\phi} * \boldsymbol{T}_n^m(\theta,\phi) = d_{17}(n,m) * \boldsymbol{T}_{n+1}^{m-1}(\theta,\phi) + d_{18}(n,m) * \boldsymbol{T}_{n-1}^{m-1}(\theta,\phi) - d_{19}(n,m) * \boldsymbol{S}_n^{m-1}(\theta,\phi). \tag{G19}$$

Appendix H
Rotation Actions on VSHs Around z-axis

$$\hat{z} = \hat{r}\cos\theta + \nabla_1\cos\theta. \tag{H1}$$

$$\sin\theta * Y_n^m(\theta,\phi)\hat{\phi} = -im\boldsymbol{S}_n^m(\theta,\phi) + d_{20}(n,m)\boldsymbol{T}_{n+1}^m(\theta,\phi) + d_{21}(n,m)\boldsymbol{T}_{n-1}^m(\theta,\phi), \tag{H2}$$

where

$$\begin{aligned} d_{20}(n,m) &= +2d_1(n,m) + d_3(n,m) \\ d_{21}(n,m) &= +2d_2(n,m) + d_4(n,m). \end{aligned} \tag{H3}$$

$$\begin{aligned} \hat{z}\times\boldsymbol{R}_n^m(\theta,\phi) &= -im\boldsymbol{S}_n^m(\theta,\phi) + d_{20}(n,m)\boldsymbol{T}_{n+1}^m(\theta,\phi) \\ &\quad + d_{21}(n,m)\boldsymbol{T}_{n-1}^m(\theta,\phi) \end{aligned}, \tag{H4}$$

$$\begin{aligned} \hat{z}\times\boldsymbol{S}_n^m(\theta,\phi) &= -d_7(n,m) * \boldsymbol{T}_{n+1}^m(\theta,\phi) - d_8(n,m) * \boldsymbol{T}_{n-1}^m(\theta,\phi) \\ &\quad + d_9(n,m) * \boldsymbol{S}_n^m(\theta,\phi) - imc_1(n)\boldsymbol{R}_n^m(\theta,\phi) \end{aligned}, \tag{H5}$$

$$\begin{aligned} \hat{z}\times\boldsymbol{T}_n^m(\theta,\phi) &= d_7(n,m) * \boldsymbol{S}_{n+1}^m(\theta,\phi) + d_8(n,m) * \boldsymbol{S}_{n-1}^m(\theta,\phi) \\ &\quad + d_9(n,m) * \boldsymbol{T}_n^m(\theta,\phi) + d_{22}(n,m) * \boldsymbol{R}_{n+1}^m(\theta,\phi), \\ &\quad + d_{23}(n,m) * \boldsymbol{R}_{n-1}^m(\theta,\phi), \end{aligned} \tag{H6}$$

where

$$\begin{aligned} d_{22}(n,m) &= -c_2(n)d_3(n,m) \\ d_{23}(n,m) &= -c_2(n)d_4(n,m). \end{aligned} \tag{H7}$$

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References

Alterman, Z., Jarosch, H., & Pekeris, C. 1959, RSPSA, 252, 80
Dahlen, F. A., & Tromp, J. 1998, Theoretical Global Seismology (Princeton, NJ: Princeton Univ. Press)
Dehant, V., & Defraigne, P. 1997, JGRB, 102, 27659
Dziewonski, A. M., & Anderson, D. L. 1981, PEPI, 25, 297
Edmonds, A. 2016, Angular Momentum in Quantum Mechanics (Princeton, NJ: Princeton Univ. Press), 68
Hedman, M., & Nicholson, P. 2013, AJ, 146, 12
Huang, C.-L., Dehant, V., & Liao, X.-H. 2004, GeoJI, 157, 831
Huang, C.-L., Dehant, V., Liao, X.-H., Van Hoolst, T., & Rochester, M. G. 2011, JGRB, 116, A03309
Huang, C.-L., Jin, W.-J., & Liao, X.-H. 2001, GeoJI, 146, 126
Huang, C.-L., & Liao, X.-H. 2003, GeoJI, 155, 669
Huang, C.-L., Liu, Y., Liu, C.-J., & Zhang, M. 2019, JGeod, 93, 297
Le Bihan, B., & Burrows, A. 2013, ApJ, 764, 18
Moritz, H. 1990, The Figure of the Earth: Theoretical Geodesy and the Earth's Interior (Karlsruhe: Wichmann)
Phinney, R. A., & Burridge, R. 1973, GeoJI, 34, 451
Rochester, M. G. 1989, Continuum Mechanics and its Applications (Washington, DC: Hemisphere), 797
Rochester, M. G., Crossley, D. J., & Zhang, Y.-L. 2014, GeoJI, 198, 1848
Rogister, Y. 2001, GeoJI, 144, 459
Schastok, J. 1997, GeoJI, 130, 137
Seyed-Mahmoud, B. 1994, PhD thesis, Memorial Univ. of Newfoundland
Seyed-Mahmoud, B., & Rochester, M. 2006, PEPI, 156, 143
Smith, M. L. 1974, GeoJI, 37, 491
Smith, M. L. 1977, GeoJI, 50, 103
Vorontsov, S., & Zharkov, V. 1981, AZh, 58, 1101
Vorontsov, S., Zharkov, V., & Lubimov, V. 1976, Icar, 27, 109
Wahr, J. M. 1981, GeoJI, 64, 705
Zhang, M., & Huang, C.-l. 2018, ChA&A, 42, 129