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A semi-analytic model for the study of 1/1 resonant dynamics of the planar elliptic restricted co-orbital problem

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Received 2020 February 7; accepted 2020 July 21

Abstract Mean motion resonances (MMRs) are widespread in our Solar System. Moreover, resonant dynamics has always been an essential topic in planetary research. Recently, the research about exoplanets and the potential Planet Nine with large eccentricity has given rise to our interests in the secular dynamics inside MMRs in the elliptic model. In this paper, we study the fixed points of the averaged Hamiltonian and the long-term stable regions of the 1/1 resonance (or co-orbital motion) in the elliptic restricted three-body problem (ERTBP) systematically. Numerical integrations prove those test particles trapped in "apsidal co-rotation", where both the resonant angle $\phi_{\rm res}$ and the secular angle $\Delta \varpi$ (or apsidal longitude differences) librate simultaneously, always survive the long-term simulations. Furthermore, utilizing a semianalytical method based on the adiabatic approach, three families of long-term fixed points of the averaged Hamiltonian of the planar ERTBP inside the 1/1 resonance have been found. We call them QS-points, Hpoints, and T-points here, whose values of the (ϕ_{res} , $\Delta \varpi$) are (0°, 180°), (180°, 0°), and ($\pm 60^{\circ}$, $\pm 60^{\circ}$), respectively. All the fixed points of the averaged Hamiltonian of the co-orbital motion in the ERTBP are presented in the e - e' plane (' represents the elements of the planet in this paper). We find that QS-points and T-points always exist for the arbitrary eccentricity of a planet, while H-points only exist for the cases of low e' and very high e. Furthermore, we measure the libration width in terms of eccentricity, Δe , around these stable equilibrium points in the e- $\Delta \varpi$ phase-space portraits. The "apsidal co-rotation" around all the stable equilibrium points is presented in the e - e' plane. All these results are effectively confirmed by numerical experiments. The long-term stable zones around these periodic orbits in the e-e' plane are significant for the research of the co-orbital motion in the ERTBP. Above all, these practical approaches that we proposed can also be used to study the secular dynamics of other MMRs.

Key words: celestial mechanics — minor planets, asteroids — planets and satellites: dynamical evolution and stability

1 INTRODUCTION

Mean motion resonances (MMRs) have been confirmed to play a pivotal role in the dynamical evolution and stability of the Solar System. They greatly influence the orbital architecture of the minor and major objects (Murray & Dermott 1999; Morbidelli 2002). There is a p/q MMR between the minor body and the planet when the $pn - qn' \approx 0$ holds, where p and q are both positive integers, and n and n' are the mean motion frequencies of the minor body and the planet, respectively. In this paper, we used the standard Keplerian elements to describe the orbits of the particles: a(semi-major axis), e(eccentricity), I(inclination), ω (argument of the pericenter), Ω (longitude of the ascending node), $\varpi = \omega + \Omega$ (longitude of the pericenter), f(true anomaly), M(mean anomaly), $\lambda = \omega +$ $\Omega + M$ (mean longitude), and n(mean motion frequency). The corresponding elements of the planet's orbit are denoted by superscript symbol '. The essential parameter of the p/q MMR is the resonant argument $\phi_{\rm res}$, and its classical form in the planar restricted problem is $\phi_{\rm res} =$ $p\lambda' - q\lambda + (q-p)\varpi$. One of the most prominent influences of MMRs is the phase protection mechanism from the planetary collision. Both numerical and theoretical research confirm that resonances can extend the lifetime of minor bodies by avoiding the configurations of the closest approach (Morbidelli et al. 1995; Bailey & Malhotra 2009; Namouni & Morais 2015; Morais & Namouni 2017; Li et al. 2019). However, in the elliptic problem, secular dynamics inside MMRs also significantly affect the long-term stability of the objects. For the elliptic problem, apsidal resonance is generated when the apsidal longitudes difference ($\Delta \varpi = \varpi' - \varpi$) of the two orbits librates around one certain value (Murray & Dermott 1999; Morbidelli 2002; Zhou & Sun 2003). The apsidal resonance can be divided into two cases: aligned resonance $(\triangle \varpi$ librates around 0°) and antialigned resonance $(\triangle \varpi \text{ librates around } 180^\circ)$. In addition, Malhotra (2002) suggested that the apsidal resonance can prevent close encounters by the phase-protection mechanism. Ji et al. (2003b) numerically found two mechanisms of stabilizing the HD82943 planetary system: the 2/1 MMR and apsidal resonance. Hadjidemetriou & Psychoyos (2003) confirmed that the alignment of the line of apsides of the planetary orbits also plays a stabilizing role. Goździewski (2003) studied the stability of the HD 12661 planetary system and believed that the crucial factor for system stability is the presence of the apsidal resonance. Lee & Peale (2003) investigated apsidal resonance of hierarchical planetary systems where the ratio of the orbital semimajor axes $\alpha = a_1/a_2$ is small. Ji et al. (2003a) emphasized that the MMR and apsidal locking can act as two important mechanisms for stabilizing the planetary system. Ji et al. (2003c) investigated the apsidal motion for the multiplanet systems, and found that some planets undergo the apsidal alignment or antialignment.

For the 1/1 resonance in planetary system or Trojan planets, there are some literatures providing some simulations on the specific systems. Laughlin & Chambers (2002) explored the viability and detectability of extrasolar planets in 1/1 MMR. Ji et al. (2005) numerically investigated the dynamical architecture of 47 UMa and confirmed that Trojan planets with low eccentricities and inclinations can secularly stay around the triangular equilibrium points of the two massive planets. Ji et al. (2007) performed numerical simulations to study the secular orbital evolution and dynamical structure in the HD 69830 planetary system. Meanwhile, minor bodies inside the 1/1 resonance (or co-orbital resonance) with the planet are common in our Solar System. To date, more than 7000 Jupiter Trojans have been discovered, locked in stable co-orbital motions with Jupiter. Trojans of other planets such as Earth (Connors et al. 2011), Mars (de la Fuente Marcos & de la Fuente Marcos 2013), Uranus (de la Fuente Marcos & de la Fuente Marcos 2017), and Neptune (Chiang & Lithwick 2005) have also been discovered¹.

There are even several asteroids that are identified trapped in retrograde co-orbital motions with planets. Wiegert et al. (2017) proved that 2015 BZ509 is the first asteroid trapped in the retrograde 1/1 resonance with the giant planet. Morais & Namouni (2019) studied the families of periodic orbits in the planar retrograde coorbital problem and analysed their stability and bifurcations into 3-D periodic orbits. Kotoulas & Voyatzis (2020) studied planar resonant retrograde periodic orbits, using the model of the restricted three-body problem with the Sun and Jupiter as primaries. Recently, we have even identified four Centaurs potentially in retrograde co-orbital motions with Saturn (Li et al. 2018).

In recent years, there has been a great deal of research on the potential Planet Nine (Batygin et al. 2019; Bromley & Kenyon 2016; Batygin & Brown 2016b; de la Fuente Marcos et al. 2016), and the existence of this hypothetical distant massive planet may explain some interesting observations around the outer Solar System, such as the existence of the most distant Kuiper belt objects (KBOs) and their orbital alignment phenomena (Batygin & Morbidelli 2017; Batygin & Brown 2016a; Beust 2016). Researchers believe that this potential distant planet has a high eccentric orbit whose eccentricity is around 0.4 - 0.6, and is estimated to have 5 to 10 times the mass of Earth. Additionally, the most distant detached KBOs may participate in MMRs with the potential planet (Malhotra et al. 2016; Millholland & Laughlin 2017; Beust 2016). Moreover, Batygin & Brown (2016a) suggested that MMRs with the Planet Nine may be responsible for the existence of the apsidally clustered KBOs. Meanwhile, the long-lived particles in their numerical simulations are always trapped in MMRs with the potential planet (Batygin & Morbidelli 2017). All of these clues inspire our interest in studying the dynamics of the resonant configuration in the elliptic model. In this paper, we focus on the fixed points of the averaged Hamiltonian of the 1/1 MMR and the long-term stable zones around them in the elliptic restricted three-body problem (ERTBP).

As we know, there are three kinds of periodic orbits for co-orbital motions in the planar circular restricted three-body problem (CRTBP) (Morais & Namouni 2016, 2017). They are Tadpole orbits (T), Quasi-satellite orbits (QS), and Horseshoe orbits (H), whose resonant angle $\phi_{\rm res}$ librates around $\pm 60^{\circ}$, 0° and 180° respectively. However, to the best of our knowledge, there is no systematic review of the fixed points of the averaged Hamiltonian of the co-orbital system and the long-term stable zones around them in the planar ERTBP. The families of the

¹ More specifically, until now, there are one Earth Trojan, nine Mars Trojans, 7311 Jupiter Trojans, one Uranus Trojan, and 24

Neptune Trojans. These data are taken from the website of Minor Planet Center. https://www.minorplanetcenter.net/iau/lists/Trojans.html

periodic orbits in 3/1 resonance (Hadjidemetriou 1992) and 3/2 resonance (Varadi 1999) have been studied in the planar ERTBP. Hadjidemetriou (1992) presented the families of the periodic orbits of the planar ERTBP at the 3/1 resonance based on the Sun-Jupiter-asteroid system. However, the maximal eccentricity of the planet (or Jupiter) he concerned is very limited. Antoniadou & Libert (2018) studied the origin and continuation of 3/2, 5/2, 3/1, 4/1, and 5/1 resonant periodic orbits from the CRTBP to ERTBP. Especially, for $\mu = 5.178 \times$ 10^{-5} families of periodic orbits associated with the dynamics of resonances in Edgeworth-Kuiper belt were computed in Kotoulas & Hadjidemetriou (2002); Voyatzis & Kotoulas (2005); Kotoulas & Voyatzis (2005) and in other similar works. The previous research on the 1/1 MMR focused more on the planetary problem, particularly the exoplanet system. Hadjidemetriou et al. (2009) have presented both stable and unstable periodic orbits of the extrasolar planetary system at the 1/1 MMR in the rotating frame. Hadjidemetriou & Voyatzis (2011) proved that a periodic orbit in planetary type might end up as a satellite orbit. Giuppone et al. (2010) studied the stable regions and equilibrium points of two planets in 1/1 MMR, but discussed only the quasi-satellite orbits and the classical equilibrium Lagrangian points. Some traditional methods have studied the quasi-satellite periodic motion in ERTBP. Pousse et al. (2017) revisited the quasi-satellite motion and presented the families of fixed points by numerical explorations. Voyatzis & Antoniadou (2018) determined quasi-satellite periodic motion in the elliptic and spatial model by the analytical continuation of periodic orbits. In this work, we attempt to obtain all the fixed points of the averaged Hamiltonian of the co-orbital motions in the ERTBP by means of a more straightforward and intuitive approach.

The long-term stable region is helpful in the research of the co-orbital motion in ERTBP. There is some literature about the resonant width (generally, in terms of the semimajor axis $\triangle a$). Murray & Dermott (1999) provided a plot of width when the eccentricities of the particles are smaller than 0.3 by perturbative analysis. Wang & Malhotra (2017) measured the 2/1 and 3/2 resonant width of the libration domains in the surface-of-section. Nesvorný et al. (2002) studied the resonant and secular dynamics of the co-orbital motion by the Hamiltonian function, and they calculated the stable libation centers and the resonant width of the 1/1 MMR in the CRTBP. Meanwhile, there are some papers studied the dynamics of Trojan asteroids in the elliptic model. Morais (1999) derived a secular theory for Trojan-type motion in the framework of the 3-D restricted three-body problem. Moreover, Morais (2001) re-derived the secular theory using a Hamiltonian formulation and considered the effect of an oblate central mass and the secular perturbations from additional bodies in a rigorous way. Lhotka et al. (2008) studied the analytical determination of stability region around the L_4, L_5 points using Nekhoroshev theory.

In this paper, we systematically study the periodic orbits and the long-term stable regions of the co-orbital motion in the ERTBP. The article is structured as follows: In Section 2, we depict the results of the longterm numerical simulations. The characteristics of the survivals in the long-term integrations are generalized. In Section 3, we introduce a semi-analytical theory for 1/1 resonance in the ERTBP and the methods of studying the secular dynamics inside a co-orbital configuration. All the stable periodic orbits of the elliptic co-orbital motion are presented systematically in Section 4. In Section 5, we measure the long-term stable regions around the periodic orbits in the e - e' plane. Section 6 contains a summary and discussion of our work.

2 NUMERICAL SIMULATIONS

In order to search for the long-term stable co-orbital motions in the planar ERTBP, numerous simulations are performed. We investigate the evolution and lifetimes of 5000 test particles, considering the influence of the Sun and a massive elliptic planet. We start our numerical experiments by using Jupiter as the primary planet of the planar ERTBP (e' = 0.0489, $\mu = 10^{-3}$).

The 5000 coplanar test particles (inclination $i=0^{\circ}$) are initialized with the exact resonant semi-major axis a = a' and eccentricity e uniformly distributed in the range of [0,0.99], where the values of other orbital elements are given randomly. Numerical integrations are performed using the MERCURY6 (Chambers 1999) package with an accuracy parameter of 10^{-12} . Because the objects we concerned may have large eccentric orbits, the general Bulirsch-Stoer integrator is employed. The whole integration timespan is 1 million years (Myr) with a time step of 20 days. Particles hitting the planet or whose heliocentric distance is either below the radius of the Sun or above 1000 au, will be removed from the integrations. Our primary concern is to observe the characteristics of particles that can survive the 1 Myr integrations.

Because both of the MMR and apsidal resonance can prevent orbits from close encounters with the planet (Murray & Dermott 1999; Morbidelli 2002; Malhotra 2002), the resonant motions that are trapped in the "apsidal co-rotation" belong to the most stable zones in the ERTBP. We define these long-term stable zones where $\phi_{\rm res}$ and $\Delta \varpi$ are locked in the libration state simultaneously as "apsidal co-rotation". In this numerical experiments (e' =0.0489, $\mu = 10^{-3}$), there are 424 of the most stable



Fig. 1 Stable resonant configurations of the representative surviving particles in the $(\Delta \varpi, \phi_{res})$ plane. They are divided into three parts (I-III) which are presented by different colors. These survivors are trapped into the regions $(\Delta \varpi, \phi_{res}) = (180^\circ, 0^\circ)$ (type I, *red part*), $(0^\circ, 180^\circ)$ (type II, *yellow part*), $(\pm 60^\circ, \pm 60^\circ)$ (type III, *green part*), respectively.

survivors among the 5000 test particles belonging to the "apsidal co-rotation". To illustrate this phenomenon more clearly, as shown in Figure 1, we choose 10 representative particles from the most stable survivors. In the same way, as depicted in Figure 2, all the surviving particles are trapped in 1/1 MMR with the resonant angle $\phi_{\rm res}$ oscillating, and the semi-major axes of survivors remain nearly constant during the whole integrations. These results are consistent with the numerical simulations of Batygin & Morbidelli (2017); Murray & Dermott (2000); Morbidelli (2002). Meanwhile, as shown in Figure 3, these survivors are trapped in apsidal resonance simultaneously with oscillating apsidal difference $\triangle \varpi$. We find that the particles with circulating $\Delta \varpi$ will be removed from the simulations as the integration proceeds, even if they are trapped in 1/1 MMR with the planet.

As shown in Figure 1, the motions of the surviving particles can be classified into three types I-III. Type I (red) is trapped in the region $(\Delta \varpi, \phi_{\rm res},) = (180^\circ, 0^\circ)$. This type is equivalent to the QS orbits in the planar CRTBP, whose resonant angle $\phi_{\rm res}$ librates around 0° . Type II (yellow) presents the long-term stable motions where $(\Delta \varpi, \phi_{\rm res})$ librates around $(180^\circ, 0^\circ)$. Type III (green) presents the asymmetric quasi-periodic orbits that librate around Lagrangian points L_4 or L_5 . Types II and III here are just like H orbits and T orbits in the planar CRTBP, respectively. This color classification is maintained throughout the article.

Identical numerical experiments for other eccentricities e' (e.g., e' = 0.2, e' = 0.6 or 0.9) are also



Fig. 2 Stable resonant configurations of the representative surviving particles in the (a, ϕ_{res}) plane.



Fig. 3 Stable resonant configurations of the representative surviving particles in the $(e, \Delta \varpi)$ plane.

performed, and the main results are similar. In general, the test particles trapped in the "apsidal co-rotation" always survive in the long-term simulations. These results agree with the findings in Antoniadou & Libert (2018) in other MMRs, and similar phenomena have been found in the general three-body problem (Antoniadou & Voyatzis 2016; Michtchenko et al. 2006). Our numerical results are also consistent with the results of Batygin & Morbidelli (2017), and could be used to understand the clustering of the distant KBOs in arguments of perihelion if Neptune is the dominant planet.

3 SEMI-ANALYTICAL MODEL

Numerical experiments are not sufficient to obtain a comprehensive understanding of the dynamics of the coorbital motion because they are time-consuming and nonsystematic. The classical analytical methods for addressing the celestial mechanics are based on the series expansions and have been introduced in detail by Murray & Dermott (1999). However, the series expansion is only valid if there is no intersection between the orbit of the particle and that of the planet (Mardling 2013; Batygin & Morbidelli 2017). It is suitable to analyze the dynamics of some normal mean motion commensurabilities such as 2/1 or 3/2 resonance and the eccentricities of the bodies under consideration must be small. The series expansion of the disturbing function is no longer valid for the eccentric co-orbital system. Considering the deficiency of the series expansion approach, the semi-analytical theory has been introduced to study the structure of MMRs in the framework of CRTBP (Morbidelli 2002). This has also been used to analyze the resonant dynamics like 2/1 resonance in the planetary problem (Michtchenko et al. 2008a,b). In recent years, the semi-analytic method has commonly been used to study the retrograde 1/1 resonance in CRTBP (Morais & Namouni 2013; Huang et al. 2018a; Li et al. 2018). This section is devoted to constructing the semi-analytical theory and the phase-space portraits applicable to the elliptic co-orbital motion.

3.1 1/1 MMR in the ERTBP

The semi-analytical model includes two main steps. Firstly, choosing the adequate canonical action-angle variables to reduce the degrees of freedom (d.o.f) of the system. Secondly, numerically evaluating the value of the Hamiltonian by averaging over all fast angles (Morbidelli 2002; Giuppone et al. 2010; Huang et al. 2018b). Further detail will be described below.

In order to study the dynamics of the different systems, various sets of canonical variables are defined. For instance, Beaugé & Michtchenko (2003) introduced a set of canonical variables to study the highly eccentric planetary three-body problem, and researchers have also proposed a set of canonical variables for retrograde motions (Gayon et al. 2009; Namouni & Morais 2015; Huang et al. 2018a). For the planar restricted three-body problem (e.g., Sun-Planet-particle), Murray & Dermott (1999) defined the classical canonical Poincare variables for the test particle as follows:

$$\Lambda = \sqrt{GM_{\odot}a} \qquad \lambda = M + \omega + \Omega,$$

$$\Gamma = \sqrt{GM_{\odot}a}(1 - \sqrt{1 - e^2}) \qquad \gamma = -\omega - \Omega,$$
(1)

where G is the gravitational constant, and M_{\odot} is the mass of the Sun. The canonical variables Λ , λ , Γ , γ are usually called modified Delaunay variables (Morbidelli 2002).

The Hamiltonian of the restricted three-body problem can be written in Poincare variables as (Morbidelli 2002)

$$\mathcal{H} = \mathcal{H}_{\text{kep}} + \mathcal{H}_{\text{pert}} = -\frac{1}{2} \left(\frac{GM_{\odot}}{\Lambda} \right)^2 - Gm' \left(\frac{1}{\boldsymbol{r} - \boldsymbol{r}'} - \frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{|\boldsymbol{r}'|^3} \right),$$
(2)

where r and r' denote the heliocentric position of the particle and planet respectively. Note that Hamiltonian (2) consists of a Keplerian part (\mathcal{H}_{kep}) and a disturbing function (\mathcal{H}_{pert}) , where $\mathcal{H}_{pert} = -Gm' \left(\frac{1}{r-r'} - \frac{r \cdot r'}{|r'|^3}\right)$ plays the role of a perturbation.

As suggested by Morbidelli (Morbidelli 2002), we extend the phase space by introducing a couple of conjugate action-angle variables for each independent time-frequency to make Hamiltonian (2) autonomous. Following the method of Batygin & Morbidelli (2017), we add the term $(\lambda'\Lambda' - \dot{\varpi}'\Gamma')$ to make the Hamiltonian independent of time. This does not alter the dynamics of the Hamiltonian. Now the Hamiltonian can be rewritten as

$$\mathcal{H} = -\frac{1}{2} \left(\frac{GM_{\odot}}{\Lambda} \right)^2 + (\lambda' \Lambda' - \dot{\varpi}' \Gamma') - Gm' \left(\frac{1}{\boldsymbol{r} - \boldsymbol{r}'} - \frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{|\boldsymbol{r}'|^3} \right).$$
(3)

If there is a 1/1 resonance between the orbits, i.e., if $n - n' \approx 0$ is satisfied, we can introduce another appropriate set of planar elliptic canonical action-angle variables for this resonance (Morbidelli 2002; Batygin & Morbidelli 2017):

$$\Lambda^* = \Lambda \qquad \phi^*_{\rm res} = \lambda - \lambda',
\Lambda'^* = \Lambda' + \Lambda \qquad \lambda'^* = \lambda',
\Gamma^* = \Gamma \qquad \gamma^* = \gamma - \gamma' = -\Delta \varpi,
\Gamma'^* = \Gamma' + \Gamma \qquad \gamma'^* = \gamma',$$
(4)

where $\phi_{\rm res}^*$ is the resonant angle of the 1/1 MMR.

Note that the following contact transformation relationship is satisfied:

$$\Lambda^* \phi_{\rm res}^* + \Lambda'^* \lambda'^* + \Gamma^* \gamma^* + \Gamma'^* \gamma'^* = \Lambda \lambda + \Gamma \gamma + \Lambda' \lambda' + \Gamma' \gamma'.$$
(5)

In other words, the new set of variables (4) is a set of canonical action-angle variables. From now on, we remove the symbol * from the new set of variables (4) for the sake of simplicity. We can numerically calculate the value of the perturbation ($\overline{\mathcal{H}}_{pert}$) by averaging over the fast angle as follows:

$$\overline{\mathcal{H}}_{pert} = -\frac{Gm'}{2\pi} \int_0^{2\pi} \left(\frac{1}{\boldsymbol{r} - \boldsymbol{r}'} - \frac{\boldsymbol{r} \cdot \boldsymbol{r}'}{|\boldsymbol{r}'|^3}\right) d\lambda'.$$
(6)

According to the d'Almebert rules, for different values of the planet's eccentricity e', the averaged Hamiltonian has two degrees of freedom and depends only on (Γ, γ) and (Λ, ϕ_{res}) (Nesvorný et al. 2002; Batygin & Morbidelli 2017). Hamiltonian (3) in the new variables turns out to be:

$$\overline{\mathcal{H}} = \mathcal{H}_{kep}(\Gamma, \Lambda, \phi_{res}) + \overline{\mathcal{H}}_{pert}(\Gamma, \gamma, \Lambda, \phi_{res}; e').$$
(7)

This system is still non-integrable. However, as the frequency of the resonant angle ϕ_{res} is at least a factor of 20 faster than that of the secular angle $\Delta \varpi$ (Nesvorný



Fig. 4 The resonant phase-space portrait in the $a - \phi_{\rm res}$ plane of the 1/1 MMR, where the secular variables $(e, \Delta \varpi) = (0.6, 180^\circ)$ are freezed. The variance of the color and contours represent the level curves of the Hamiltonian in Eq. (7). The *thick dark lines* present the collision curves with the planet where the values of Hamiltonian tend to infinity.

et al. 2002), we can analyze the resonant dynamics using the adiabatic invariant approximation. In other words, due to the separation of the timescale of the two degrees of freedom, we can calculate the value of the resonant Hamiltonian in the a- ϕ_{res} plane by freezing the secular angle-action variables (Γ , γ). For example, Figure 4 presents the 1/1 resonance phase portrait in the a- ϕ_{res} plane, where e' = 0.0489 and $(e, \Delta \varpi) = (0.6,$ $180^{\circ})$. Figure 4 depicts the short-term resonant dynamics, including collision curves and equilibrium points of the 1/1 resonance. However, this diagram cannot describe the secular dynamics inside the 1/1 resonance.

3.2 Secular Dynamics inside 1/1 MMR

To study the secular dynamics of the co-orbital motion, we focus on the evolution of the secular canonical variables $(e, \Delta \varpi)$. Similarly, to obtain the $e \cdot \Delta \varpi$ phase-space portrait, we need to fix the values of resonant canonical variables (a, ϕ_{res}) . Taking into consideration the vast gap in the frequencies of the angles ϕ_{res} and $\Delta \varpi$, the secular perturbation can also be treated by introducing an adiabatic invariant \mathcal{J} as follows (Wisdom 1985; Nesvorný et al. 2002; Batygin & Morbidelli 2017),

$$\mathcal{J} = -\frac{1}{2\pi} \int_0^{2\pi} \Lambda d\phi_{\rm res}.$$
 (8)

The value of \mathcal{J} is zero for resonant equilibrium points $(d\phi_{\rm res} = 0)$. Then the problem can be significantly simplified. In other words, we can obtain the secular phase-space portrait in the $e - \Delta \varpi$ plane at the resonant equilibrium points (Morbidelli & Moons 1993). Without



Fig. 5 The secular phase-space portrait in the $e-\Delta \varpi$ plane. The condition of the resonant variables $(a, \phi_{res}) = (1 \text{ au}, 180^\circ)$ is always met in this diagram. The variance of the color and contours represent the level curves of the Hamiltonian in Eq. (7). The *solid black dot* marks the position of the ideal equilibrium point of the averaged Hamiltonian.

loss of generality, the exact resonance location $a_{res} = 1$ au is considered as the location of 1/1 resonant equilibrium points on the (a, ϕ_{res}) plane. Therefore, we can fix the value of the semi-major axis a = 1 au during the averaging process. For example, if we focus on the secular dynamics around the apocentric resonant equilibrium point ($\phi_{res} =$ 180°), the Hamiltonian in Equation (7) can be calculated by freezing $(a, \phi_{res}) = (1 \text{ au}, 180^\circ)$. Figure 5 is the secular phase-space diagram in the e- $\Delta \varpi$ plane where the assumption $(a, \phi_{res}) = (1 \text{ au}, 180^{\circ})$ is always held. This diagram depicts the evolution of the secular canonical variables (e, $\Delta \varpi$). Because the fixed point in Figure 5 meets the conditions $\partial \mathcal{H}/\partial \Lambda = \partial \mathcal{H}/\partial \phi_{res} = \partial \mathcal{H}/\partial \Gamma =$ $\partial \mathcal{H}/\partial \gamma = 0$ simultaneously, it is an ideal periodic orbit of the co-orbital motion for the planar elliptic problem. We will discuss this in detail in the following section.

4 FAMILIES OF EQUILIBRIUM POINTS OF THE AVERAGED HAMILTONIAN

The equilibrium points of the averaged Hamiltonian are equivalent to periodic orbits of the original co-orbital system (Hadjidemetriou 1991, 1992). We can obtain the long-term stable equilibrium points of the averaged Hamiltonian via the method introduced in the previous section. Using the semi-analytical method introduced above, we obtain the families of the equilibrium points in the e-e' plane systematically.

We first introduce another useful phase-space portrait in the $\Delta \varpi$ - ϕ_{res} plane. Just like the method we get the e- $\Delta \varpi$ phase-space diagram, we can obtain the phasespace portrait in $\Delta \varpi$ - ϕ_{res} plane by freezing the value of



Fig. 6 The phase-space portrait on the $\Delta \varpi$ - $\phi_{\rm res}$ plane by freezing *a* and *e* = *e'*. The variance of the color and contours represent the level curves of the Hamiltonian in Eq. (7). There are a stable symmetric libration center point at ($\Delta \varpi$, $\phi_{\rm res}$) = (180°, 0°) and two asymmetric libration center points.



Fig. 7 The phase-space portrait on the $\Delta \varpi \cdot \phi_{res}$ plane by freezing (a, e, e') = (1 au, 0.9, 0.1). This kind of configuration only exists in situations with low e' and very high e. The variance of the color and contours represent the level curves of the Hamiltonian in Eq. (7). There are two stable symmetric libration center points at $(\Delta \varpi, \phi_{res})$ $= (180^\circ, 0^\circ), (0^\circ, 180^\circ).$

(a, e, e') on the domain. Because both $\Delta \varpi$ and ϕ_{res} are angle variables, the $\Delta \varpi$ - ϕ_{res} diagram cannot be used to analyze the dynamics of the averaged Hamiltonian directly. However, since the two action variables (a, e) are fixed, we can obtain all possible combinations of $\Delta \varpi$ and ϕ_{res} of the ideal equilibrium points.

Through numbers of experiments with different values of the (e', e), we find that there are only two types of the $\Delta \varpi - \phi_{res}$ phase-space portraits. These are presented in Figures 6 and 7, respectively. Figure 6 is the most common configuration, while Figure 7 only happens in the cases with low e' and very large e. We find that the value of the $(\triangle \varpi, \phi_{\rm res})$ for symmetric stable equilibrium points can be $(180^\circ, 0^\circ)$ or $(0^\circ, 180^\circ)$. The other two symmetric points $(\triangle \varpi, \phi_{\rm res}) = (0^\circ, 0^\circ), (180^\circ, 180^\circ)$ are unstable, due to the close encounters with the planet. It is worth noting that, as shown in Figure 1, there are some stable quasi-periodic orbits around the unstable equilibrium point ($\Delta \varpi, \phi_{\rm res}$) $= (0^{\circ}, 0^{\circ})$. A similar phenomenon has been found in the planetary system of the 1/1 resonance (Giuppone et al. 2010). As for the asymmetric periodic orbits, they are exactly located at the asymmetric stable Lagrangian points L_4 and L_5 , corresponding to $(\triangle \varpi, \phi_{\rm res}) = (\pm 60^\circ, \pm 60^\circ)$ in elliptic co-orbital system. Taken all together, there are three families of the stable equilibrium points of the 1/1 MMR in the ERTBP, which are characterized by $(\Delta \varpi, \phi_{\rm res}) = (180^\circ, 0^\circ), (0^\circ, 180^\circ), \text{ and } (\pm 60^\circ, \pm 60^\circ).$ With reference to the definitions of the quasi-periodic orbits of the co-orbital motion in CRTBP, we call these three families of equilibrium points QS-points, H-points, and T-points, respectively.

4.1 QS-Points

To obtain the QS-points in the whole e - e' plane, for each certain value of e' in the range of (0, 1), we calculate the e- $\Delta \varpi$ phase-space diagrams by freezing $(a,\phi_{
m res})=(1\,$ au, $0^\circ).$ We find that all the $e{-}\Delta \varpi$ diagrams share similar structures as shown in Figure 8, and the stable libration center $(e', e, triangle \varpi, \phi_{res}) =$ $(0.4, 0.6468, 180^\circ, 0^\circ)$ should be a QS-point. To verify this result, numerical integration is performed, including the initial orbit elements $(a, e, \Delta \varpi, \phi_{res}) = (1 \text{ au},$ $0.6468, 180^{\circ}, 0^{\circ}$) and the eccentricity of the planet e' =0.4. The dynamical evolution of the particle's orbit for 500000 years is depicted in Figure 9. The orbit of the QS-point looks like a satellite of the planet in the rotating frame, and the four variables $(a, e, \Delta \varpi, \phi_{res})$ remain unchanged over the whole integration timespan. This meets the conditions of a periodic orbit of the original system $\partial \mathcal{H} / \partial \Lambda = \partial \mathcal{H} / \partial \phi_{\rm res} = \partial \mathcal{H} / \partial \Gamma = \partial \mathcal{H} / \partial \gamma =$ 0. The numerical result demonstrates that (e', e) = (0.4, e)0.6468) is one of QS-points in the whole e - e' plane. By this mean, we can obtain other QS-points of the planar elliptic co-orbital motion. Figure 10 depicts all QS-points in the e - e' plane. We find that the QS-point always exists for an arbitrary eccentricity of the planet e'. Moreover, eccentricities e of these QS-points first decrease and then increase with an increase of e'. The minimum value of the eccentricity is $e \approx 0.4752$ when $e' \approx 0.8$.



Fig. 8 The secular phase-space portrait on the $e - \Delta \varpi$ plane by freezing $(a, \phi_{res}) = (1 \text{ au}, 0^{\circ})$ with e' = 0.4. The variance of the color and contours represent the level curves of the Hamiltonian in Eq. (7). The *solid black dot* denotes a QS-point.

4.2 H-Points

We now focus on another type of symmetric long-term stable equilibrium points where $(\Delta \varpi, \phi_{\rm res}) = (0^{\circ}, 180^{\circ})$. For the co-orbital motion in the CRTBP, Horseshoe orbits are enclosed by the separatrix originating at Lagrangian point $L_1(L_2)$ (Morbidelli 2002). That is to say the resonant angle $\phi_{\rm res}$ of the Horseshoe orbit librates around 180° . Nesvorný et al. (2002) suggested that the centers of the Horseshoe orbits are always unstable saddle points for the cases of e = 0, 0.2, 0.4, and 0.8 in the planar CRTBP. As shown in Figure 7, we have confirmed that the stable libration center at $(\Delta \varpi, \phi_{\rm res}) = (0^{\circ}, 180^{\circ})$ exists only for the cases with low e' and very large e. This section systematically presents the H-points of the elliptic coorbital system.

Similarly to the previous method to search for QSpoints, for a certain value of e' in the range of (0, 1), we calculate the $e \cdot \Delta \varpi$ phase-space portrait by freezing $(a, \phi_{res}) = (1 \text{ au}, 180^\circ)$. We can obtain the value of eccentricity e of the stable libration center at $(\Delta \varpi, \phi_{res})$ $= (0^\circ, 180^\circ)$ in the $e \cdot \Delta \varpi$ diagram. Considering that Hpoints do not always exist, we should check whether or not there is a stable libration center at $(\Delta \varpi, \phi_{res}) = (0^\circ, 180^\circ)$ in the corresponding $\Delta \varpi \cdot \phi_{res}$ phase-space diagram.

For example, as shown in Figure 11, the stable libration center is located at $(e, \Delta \varpi) = (0.9636, 0^{\circ})$ in the $e \cdot \Delta \varpi$ plane (panel a), and $(\Delta \varpi, \phi_{res}) = (0^{\circ}, 180^{\circ})$ is exactly a stable libration center in the corresponding $\Delta \varpi - \phi_{res}$ plane by freezing (e', e) = (0.15, 0.9636) (panel b). So the point $(e', e, \Delta \varpi, \phi_{res}) = (0.15, 0.9636, 0^{\circ}, 180^{\circ})$ is one of H-points, and this result is also verified by numerical experiments. The dynamical evolution in



Fig. 9 Dynamical evolution of the numerical integration in 500 000 years about the QS-point in Fig. 8. Panel a: *red path* presents the orbit of the particle in the rotating frame of the planet. The *pink* and *blue lines* are the orbits of the Lagrangian points and the planet, respectively. The *solid dots* are their respective initial position in numerical integration. Panel b: the evolution of the elements $a, e, \phi_{res}, \Delta \varpi$ respectively over the integration timespan.



Fig. 10 Family of the QS-points of the planar elliptic coorbital motion.

500 000 years integration of the particle at this Hpoint (e', e) = (0.15, 0.9636) is depicted in Figure 12. The particle's orbit is close to the Sun in the rotating frame due to the large eccentricity e, and all the variables $(a, e, \Delta \varpi, \phi_{res})$ remain constant over the whole integration timespan. As shown in Figure 13, we obtain M. Li et al.: Dynamics of the Elliptic Co-orbital Problem



Fig. 11 Panel a: The secular phase-space portrait on the $e - \Delta \varpi$ plane by freezing $(a, \phi_{res}) = (1 \text{ au}, 180^\circ)$ with e' = 0.15. Panel b: The $\Delta \varpi - \phi_{res}$ phase-space diagram obtained by freezing (e, e') = (0.9636, 0.15). The variance of the color and contours represent the level curves of the Hamiltonian in Eq. (7). Through these two diagrams, we find an H-point of the elliptic co-orbital motion.



Fig. 12 Dynamical evolution in 500 000 years integration of the H-point in Fig. 11. The basic meaning of this figure is same as Fig. 9.

all H-points in the e-e' plane by this process. Unlike QSpoints, H-points only exist in a much smaller region in the e-e' plane. The maximal value of the eccentricity e' of the H-points is around 0.22, while e is always above 0.95.

4.3 T-Points

The orbits librating around the stable Lagrangian points L_4 or L_5 in the rotating frame are known as Tadpole orbits in the CRTBP. We now focus on these asymmetric stable



Fig. 13 Family of the H-points of the planar elliptic coorbital motion.

equilibrium points of the elliptic co-orbital motion, and we call them T-points. T-points in the elliptic co-orbital system are precisely equivalent to the stable Lagrangian points L_4 and L_5 . As shown in Figure 14, for an arbitrary eccentricity e', if we fix the value $(a, \phi_{res}) = (1 \text{ au}, \pm 60^\circ)$, then there will be a stable libration center at $\Delta \varpi =$ $\pm 60^\circ$ and the corresponding eccentricity e = e' in the $e - \Delta \varpi$ phase-space portrait. The dynamical evolution of this T-point is presented in Figure 15, which is identical to the Lagrangian L_4 over the whole integration. This is a long-term stable asymmetric equilibrium point in the eccentric co-orbital motion, which meets the conditions of $\partial \mathcal{H}/\partial \Lambda = \partial \mathcal{H}/\partial \phi_{res} = \partial \mathcal{H}/\partial \Gamma = \partial \mathcal{H}/\partial \gamma = 0$. For the co-orbital motion in the planar ERTBP, the family of T-points in the e - e' plane is depicted in Figure 16.



Fig. 14 The secular phase-space portrait on the $e - \Delta \varpi$ plane by freezing $(a, \phi_{\text{res}}) = (1 \text{ au}, 60^\circ)$ with e' = 0.3. The variance of the color and contours represent the level curves of the Hamiltonian in Eq. (7). The *solid black dot* denotes the position of a T-point.



Fig. 15 Dynamical evolution in 500 000 years integration of the T-point in Fig. 14. The basic meaning of this figure is same as Fig. 9.

5 STABLE REGIONS AROUND THE EQUILIBRIUM POINTS

It is significant to study the long-term stable domains of the elliptic co-orbital motion. Based on the numerical results in Section 2, we find that all the test particles trapped in the "apsidal co-rotation" always survive in the long-term simulations. In Section 4, we present three families of the long-term fixed points of the averaged Hamiltonian of the



Fig. 16 Family of the T-points of the planar elliptic coorbital motion.

elliptic co-orbital motion in the *e-e'* plane, including QSpoints, H-points, and T-points. As shown in Figure 2, our numerical results confirm that the libration amplitude of the semi-major axis of the stable quasi-periodic orbits is very small in the long-term simulation. In this section, we attempt to provide the stable domains in the *e-e'* plane for the particles staying at the nominal resonance location. We know that the Hamiltonian at the stable equilibrium points satisfies the conditions $\partial \mathcal{H}/\partial \phi_{res} = \partial \mathcal{H}/\partial \gamma = 0$. In the vicinity of the fixed points of the averaged Hamiltonian, both the resonant angle ϕ_{res} and apsidal difference $\Delta \varpi$ are locked in the libration state (Antoniadou & Libert 2018).

We now focus on the libration width in the eccentricity, $\triangle e$, around the equilibrium points in the $e \cdot \triangle \varpi$ phasespace portraits. Due to the existence of the collision curves, Morbidelli (2002) defined that the width of the libration is the trajectory of the largest librational amplitude that does not cross the collision curves. Similarly, in the e- $\Delta \varpi$ phase-space portrait, we measure the libration width with eccentricity, $\triangle e$, defined as the maximum range of the eccentricity where the $\triangle \varpi$ is in the libration state. Because the $e - \Delta \varpi$ phase-space portrait is obtained by freezing the values of the resonant variables (a, ϕ_{res}) , the libration zones we measured in e- $\Delta \varpi$ phase-space portraits should belong to the "apsidal co-rotation". By this means, we can obtain the long-term stable domains in the e-e' plane. For example, Figure 17 presents the width of the eccentricity, $\triangle e$, of a T-point when e' = 0.6. For all the stable equilibrium points, we measure the libration width in the eccentricity, and the results are shown in Figure 18.

The three regions in different colors represent the stable domains around different types of stable equilibrium points. The stable zone around QS-points (red area) is the most abundant, while stable zone around T-points (green area) is slightly smaller. Moreover, the stable zone around H-points (yellow area) is much smaller than the



Fig. 17 The secular phase-space portrait on the $e - \Delta \varpi$ plane by freezing $(a, \phi_{\text{res}}) = (1 \text{ au}, 60^\circ)$ with e' = 0.6. We measure the maximum amplitude in eccentricity around the T-point where the $\Delta \varpi$ is locked in an oscillation state.



Fig. 18 Long-term stable zones around the stable equilibrium points. The *blue dash-dotted lines* represent the positions of the stable equilibrium points. Three different colors represent the stable regions around different types of the equilibrium points ($red \rightarrow QS$ -points, $yellow \rightarrow H$ points, $green \rightarrow T$ -points).

other two. It should be noted that we only focus on the "apsidal co-rotation" around the stable equilibrium points. There are some other stable regions that do not have a stable equilibrium point. For example, as shown in Figure 1, the blue part presents the survivors librating around $(\Delta \varpi, \phi_{res}) = (0^\circ, 0^\circ)$, even though no stable equilibrium point exists here. We will study these stable regions without stable equilibrium points in future work.

Numerical experiments are performed to demonstrate the validity of Figure 18. Particles in the condition of the three stable zones can always survive the long-term numerical simulations. Moreover, as shown in Figure 18, there are three types (red, yellow and green parts) of stable orbits when e' = 0.0489, which is consistent with the numerical results in Figures 1-3. These phenomena further illustrate the reliability of our analytical results. Our study is very helpful in evaluating the stability of the particles in eccentric 1/1 resonance. It is also the first time that the long-term stable zones are given systematically for the elliptic co-orbital motion.

6 CONCLUSIONS

This paper discusses the secular dynamics inside the 1/1 resonance in the ERTBP. We present the families of periodic orbits and the long-term stable regions around them. Considering the protection mechanisms of MMRs and apsidal resonance (Morbidelli 2002; Malhotra 2002), we define the "apsidal co-rotation" where both the resonant angle $\phi_{\rm res}$ and apsidal difference $\Delta \varpi$ are simultaneously locked in the libration state. Numerical results in Section 2 demonstrate that test particles trapped in the "apsidal co-rotation" can always survive the long-term simulations.

Based on the adiabatic approach, we constructed a semi-analytical model for the co-orbital motion in ERTBP. Through the $\Delta \varpi$ - $\phi_{\rm res}$ phase-space portraits such as that given in Figure 6, we find that the families of long-term stable equilibrium points of the elliptic 1/1 resonance can be divided into three types: $(\triangle \varpi, \phi_{\rm res}) = (0^\circ, 180^\circ),$ $(180^\circ, 0^\circ)$, and $(\pm 60^\circ, \pm 60^\circ)$, corresponding to QS-points, H-points and T-points, respectively. By introducing the adiabatic invariant approximation, we obtain the phasespace diagram in the e- $\Delta \varpi$ plane by limiting the libration amplitude of the resonant variables (a, ϕ_{res}) to zero. By means of the e- $\Delta \varpi$ phase-space portraits, we can obtain the position of the stable equilibrium points in the ee' plane. This semi-analytical method is practical and reliable, as verified by numerical experiments. Moreover, it is more intuitive than the analytical continuation and purely numerical explorations (Pousse et al. 2017; Voyatzis & Antoniadou 2018). In this way, all the stable equilibrium points of the planar elliptic co-orbital system are presented, and our results are consistent with the numerical explorations of Pousse et al. (2017). As shown in Figures 10, 13 and 16, the QS-points and T-points always exist in the e - e' plane with an arbitrary e', while the H-points only exist in the cases with low e' and very large e.

We measure the libration width in eccentricity, $\triangle e$, around the stable equilibrium points in the $e-\triangle \varpi$ plane. These libration zones are stable regions because the vicinity of the stable equilibrium points belongs to the "apsidal co-rotation". To the best of our knowledge, it is the first time that the long-term stable zones are given for the elliptic co-orbital motion. These results are significant to study the stability of the co-orbital motion in the planar ERTBP. Our results are also helpful for further understanding of the dynamics of the elliptic system, like the potential Planet Nine. Above all, these approaches can be used to study the stable equilibrium points and the longterm stable regions of the ERTBP at other MMRs similarly.

Acknowledgements This work was supported by the National Natural Science Foundation of China (Grant No. 11772167).

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