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Effects of rotation and tidal distortions on the shapes of radial velocity curves of polytropic models of pulsating variable stars

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Abstract Anharmonic oscillations of rotating stars have been studied by various authors in literature to explain the observed features of certain variable stars. However, there is no study available in literature that has discussed the combined effect of rotation and tidal distortions on the anharmonic oscillations of stars. In this paper, we have created a model to determine the effect of rotation and tidal distortions on the anharmonic radial oscillations associated with various polytropic models of pulsating variable stars. For this study we have used the theory of Rosseland to obtain the anharmonic pulsation equation for rotationally and tidally distorted polytropic models of pulsating variable stars. The main objective of this study is to investigate the effect of rotation and tidal distortions on the shapes of the radial velocity curves for rotationally and tidally distorted polytropic models of pulsating variable stars. The results of the present study show that the rotational effects cause more deviations in the shapes of radial velocity curves of pulsating variable stars as compared to tidal effects.

Key words: stars: rotation — stars: oscillations — stars: binaries — technique: radial velocities

1 INTRODUCTION

The development of the theory of anharmonic oscillations is a vital part of oscillation theory in order to understand the form of light and velocity curves, and the evolutionary history of pulsating stars. A theory of anharmonic oscillation was developed by Rosseland (1943) to explain the skewness of the radial velocity curve and also longer period oscillations than predicted by the standard model. Sen (1948) studied the anharmonic pulsations of Cepheid variable stars and showed that only homogeneous stars are capable of uniform, radial and adiabatic pulsations. Further, Rosseland (1949) developed the theory of anharmonic pulsation to study the effect of higher modes and higher order on the shape of a radial velocity curve for a pulsating stellar model. Since then, several investigators, such as Bhatnagar & Kothari (1944), Schwarzschild & Savedoff (1949), Prasad (1949a), Gurm (1963), van der Borght & Murphy (1966) and Prasad & Mohan (1969), have used this theory for studying various problems related to pulsating stars. Prasad (1949a) used the concept of Fourier series to solve the anharmonic equation in general and developed a method to evaluate coefficients in the series. He applied it to the standard model of a star with two overtones. This theory of Rosseland (1949) has further been utilized by Prasad (1949b), Bhatnagar & Kushwaha (1951), Jain & Lal (1955) and Gurm (1963) for the homogenous and inverse square model, Roche model, two phase model and composite model respectively.

The radial anharmonic pulsations of the standard model have been considered by Eddington (1918) and Schwarzschild (1941), while those of polytropes with polytropic indices, 1.5 and 1, have been considered by Chatterji (1952) and Lucas (1956). Miller (1929) investigated the pulsations of polytropic indices 2 and 4 for the ratio of specific heats $\gamma = \frac{10}{7}(\alpha = 0.2)$. Stempels et al. (2007) studied variations in the radial velocity of the classical T Tauri star RU Lupi. They considered binarity and pulsations as possible sources of the radial velocity variations, however, their study showed that these variations are probably related to spots on the stellar surface. The third order effects of rotation on the oscillations of a β-Cepheid star were examined by Karami (2009). Pandey et al. (2012) studied the anharmonic vibration in pulsating stars by using the Hamiltonian formulation of Newtonian dynamics. Using the equation of motion, they investigated a theoretical model of a star that can produce the anharmonic nature of pulsation in radial velocity, luminosity and apparent magnitude with respect to time. In order to understand the pulsation mechanism of Cepheid variables, Dehnen et al. (2016) presented a model based on primary physical principles, namely energy conservation, mechanical adiabaticity and hydrodynamical equilibrium. They found that a star pulsates anharmonically if the parameter a takes values close to 1 and harmonically when $a \approx 0$. Kjurkchieva et al. (2017) conducted intensive photometric and spectral observations of the variable star V2551 Cyg. They also studied its radial velocity curves and found it to be a pulsating star that pulsates with the fundamental mode.

As most of the theoretical studies in literature on anharmonic radial oscillations have focused on undistorted or rotating stars, we thought of further extending the anharmonic theory of pulsating stars – as proposed by Rosseland (1949) and using the methodology given by Prasad (1949b) – by taking into account the effects of both rotation and tidal distortions. We have utilized the methodology of Mohan & Saxena (1985) to incorporate the effects of rotation and tidal distortions in our study. So, in the present paper we have tried to investigate the effects of rotation and tidal distortions on the anharmonic radial oscillations for certain polytropic models of pulsating variable stars. The objective of this study is also to examine how the inclusion of rotational and tidal effects in the anharmonic theory of pulsating variable stars affects the shapes of their radial velocity curves.

We have used the methodology of Mohan & Saxena (1985), which utilizes the averaging technique of Kippenhahn & Thomas (1970) and results of Roche equipotentials as given by Kopal (1972), to incorporate the effects of rotations and/or tidal distortions in our present study. The various assumptions that have been made in this methodology are: (i) The star is highly centrally condensed so that the actual equipotential surface of a rotationally and tidally distorted star can be approximated by a Roche equipotential surface. (ii) Binary systems are considered to be circular, synchronous and aligned so that the axis of rotation of the star under investigation (primary component) is at a right angle to the line joining the center of the primary and secondary component of the binary system. (iii) The rotational velocity as well as ratio of the mass of the companion causing tidal distortions to the mass of the primary star is not so large that models do not deviate too much from spherical symmetry and, hence, the terms beyond second order of smallness in rotational distortion parameter and tidal distortion parameter can be neglected in various mathematical expressions used. (iv) The distorted model of the star is well within its Roche lobe. (v) The equipotential surface of the distorted star is also the surfaces of equipressure and equidensity. (vi) Oscillations are barotropic so that fluid elements on the equipotential surface of the star in its equilibrium position always remain on it during the entire period of small adiabatic oscillations. So in view of these assumptions, our present analysis is applicable to highly centrally condensed pulsating variable stars with small oscillations that are (i) single and rotating slowly; (ii) slowly rotating primary components of synchronous, circular and aligned binary systems in which the mass of the secondary is very much less than the mass of the primary star.

The paper is organized as follows: in Section 2 we formulate the equation of anharmonic pulsation for rotationally and/or tidally distorted (hereafter RTD) stellar models. In Section 3 the anharmonic pulsation equation for the RTD polytropic model of stars is obtained. A successive approximation method is discussed in Section 4 to solve the anharmonic pulsation equation as developed in Section 3. In Section 5, numerical computation is performed to obtain the solution of the anharmonic equation of certain RTD polytropic models (with polytropic indices $N = 1.5, 3.0$) of stars. Numerical results thus obtained are analyzed and certain conclusions are discussed in Section 6.

2 FORMULATION OF ANHARMONIC RADIAL PULSATION EQUATION FOR A ROTATIONALLY AND TIDALLY DISTORTED STELLAR MODEL

Following the approach used by Rosseland (1949), the equation of anharmonic pulsation of a rotationally and tidally distorted stellar model can be written by applying it to the topologically equivalent spherical model. Using the averaging concept of Kippenhahn & Thomas (1970) as used in Mohan & Saxena (1985), we assume $r_{1\psi}$ to be the average displacement on the equipotential surface ($\psi = \text{constant}$) and write it as

$$r_{1\psi} = \eta_1(r_{0\psi})q_1(t) + \eta_2(r_{0\psi})q_2(t) + \eta_3(r_{0\psi})q_3(t) + \dots, \tag{1}$$

where η_1, η_2, η_3, ... are the solutions for various modes of the pulsation equation as discussed by Mohan & Saxena (1985) and $q_1(t)$, $q_2(t)$, $q_3(t)$... are functions of time to be determined by substituting in the exact equation of motion.

Let r_ψ denote radius of the topologically equivalent spherical model which corresponds to an equipotential surface ψ=constant of the rotationally and tidally distorted model. Also, let R_ψ be the value of r_ψ on the outermost equipotential surface of the model. So, the equation of motion for a rotationally and tidally distorted gaseous sphere can be written as

$$\ddot{r}_\psi = -g - \frac{1}{\rho_\psi}\frac{dP_\psi}{dr_\psi}\,. \tag{2}$$

On substituting $r_\psi = r_{0\psi}(1 + r_{1\psi})$ in Equation (2) and using the adiabatic condition $P_\psi = \text{constant} \times \rho_\psi^\gamma$ and the equation of continuity $\rho_\psi {r_\psi}^2 dr_\psi = \rho_{0\psi} r_{0\psi}^2 dr_{0\psi}$ (subscript zero refers to the equilibrium value), we get after some simplification

$$\begin{aligned} r_{0\psi}\ddot{r}_{1\psi} = &-\Big(1 + r_{1\psi}\Big)^{-2} g_0 - \frac{\Big(1 + r_{1\psi}\Big)^2}{\rho_{0\psi}}\frac{\partial}{\partial r_{0\psi}} \\ &\times \Big[P_{0\psi}\Big(1 + r_{1\psi}\Big)^{-2\gamma} \\ &\times \Big(1 + r_{1\psi} + r_{0\psi}\frac{\partial r_{1\psi}}{\partial r_{0\psi}}\Big)^{-\gamma}\Big], \end{aligned} \tag{3}$$

where $P_{0\psi}$, $\rho_{0\psi}$ and $r_{0\psi}$ are the equilibrium values of pressure P_ψ, density ρ_ψ and distance from the center r_ψ respectively, γ is the ratio of specific heat and $r_{1\psi}$ is the relative displacement $\frac{r - r_{0\psi}}{r_{0\psi}}$. The dots denote differentiation with respect to time t.

The terms q_1, q_2, q_3,... appearing in Equation (1) can be separated by using orthogonality

$$\int_0^{R_\psi} \rho_{0\psi} r_{0\psi}^4 \eta_j \eta_k dr_{0\psi} = 0, \quad j \neq k\,. \tag{4}$$

On expanding the right hand side and retaining terms up to second order, Equation (3) can be written as

$$\begin{aligned} \rho_{0\psi} r_{0\psi}\ddot{r}_\psi = &P'_{0\psi}\Big[(3\gamma - 4)r_{1\psi} + \gamma r_{0\psi} r'_{1\psi}\Big] \\ &+ \gamma P_{0\psi}\Big(4r'_{1\psi} + r_{0\psi} r''_{1\psi}\Big) \\ &- P'_{0\psi}\Big[\frac{1}{2}(3\gamma - 4)(3\gamma + 1)r_{1\psi}^2 \\ &+ \gamma(3\gamma - 1)r_{0\psi} r_{1\psi} r'_{1\psi} + \frac{1}{2}\gamma(\gamma + 1)r_{0\psi}^2 {r'_{1\psi}}^2\Big] \\ &- \gamma P_{0\psi}\Big[4(3\gamma - 1)r_{1\psi} r'_{1\psi} + 2(2\gamma + 1)r_{0\psi}^2 {r'_{1\psi}}^2 \\ &+ (3\gamma - 1)r_{0\psi} r_{1\psi} r''_{1\psi} + (\gamma + 1)r_{0\psi}^2 r'_{1\psi} r''_{1\psi}\Big]\,, \end{aligned} \tag{5}$$

where the prime symbols denote differentiation with respect to $r_{0\psi}$. On multiplying Equation (5) by $r_{0\psi}^3 \eta_k dr_{0\psi}$, integrating over the entire star and then using Equation (1), the left hand side of Equation (5) gives

$$\int_0^{R_\psi} \rho_{0_\psi} r_{0\psi}^4 \eta_k \left(\sum_j (\eta_j \ddot{q}_j)\right) dr_{0\psi} = I_k \ddot{q}_k, \tag{6}$$

where

$$I_k = \int_0^{R\psi} \rho_{0\psi} r_{0\psi}^4 \eta_k^2 dr_{0\psi}. \tag{7}$$

The other terms vanish due to orthogonality in Equation (4). On the right hand side of Equation (5), the linear terms yield

$$\int_0^{R\psi} \gamma P_{0\psi} r_{0\psi}^4 \eta_k \left[r''_{1\psi} + \frac{4 - \mu}{r_{0\psi}} r'_{1\psi} - \Big(3 - \frac{4}{\gamma}\Big)\frac{\mu}{r_{0\psi}^2} r_{1\psi}\right] dr_{0\psi}\,, \tag{8}$$

where

$$\mu = -\frac{r_{0\psi}}{P_{0\psi}}\frac{dP_{0\psi}}{dr_{0\psi}}\,.$$

Substituting for $r_{1\psi}$ from Equation (1) and using the fact that η_1, η_2, η_3,... are eigenfunctions of the pulsation equation discussed by Mohan & Saxena (1985) with corresponding periods ($\frac{2\pi}{\sigma_1}, \frac{2\pi}{\sigma_2}, \frac{2\pi}{\sigma_3}, \dots$), Equation (8) can further be written as

$$\begin{aligned} &-\int_0^{R\psi} \gamma P_{0\psi} r_{0\psi}^4 \eta_k \sum_j \frac{\sigma_j^2 \rho_{0\psi}}{\gamma P_{0\psi}} (\eta_j q_j) dr_{0\psi} \\ &= -I_k {\sigma_k}^2 q_k. \end{aligned} \tag{9}$$

Now, the second order terms on the right hand side of Equation (5), after integrating the last two terms by parts to get rid of $r''_{1\psi}$, gives

$$-\frac{1}{2}(3\gamma-4)(3\gamma+1)\int_0^{R\psi} P'_{0\psi} r^3_{0\psi}\eta_k r^2_{1\psi} dr_{0\psi}$$
$$+\frac{1}{2}\gamma(3\gamma-1)\int_0^{R_\psi} P_{0\psi} r^4_{0\psi}\eta_k {r'_{1\psi}}^2 dr_{0\psi}$$
$$+\gamma(3\gamma-1)\int_0^{R_\psi} p_{0\psi} r^4_{0\psi}\eta_k r_{1\psi} r'_{1\psi} dr_{0\psi}$$
$$+\frac{1}{2}\gamma(\gamma+1)\int_0^{R_\psi} P_{0\psi} r^5_{0\psi}\eta'_k {r'_{1\psi}}^2 dr_{0\psi}\,. \qquad (10)$$

Substituting this expression (1) for $r_{1\psi}$, it may further be expressed as

$$\gamma\Big[\sum_i D_{ii,k} q_i^2 + 2\sum_{\substack{i,j\\ i\neq j}} D_{ij,k} q_i q_j\Big], \qquad (j \geq i), \quad (11)$$

where

$$D_{ij,k} = -\frac{1}{2}\Big(3-\frac{4}{\gamma}\Big)(3\gamma+1)\int_0^{R_\psi} P'_{0\psi} r^3_{0\psi}\eta_i\eta_j\eta_k dr_{0\psi}$$
$$+\frac{1}{2}(3\gamma-1)\int_0^{R_\psi} P_{0\psi} r^4_{0\psi}$$
$$(\eta_i\eta'_j\eta'_k + \eta'_i\eta_j\eta'_k + \eta'_i\eta'_j\eta_k) dr_{0\psi}$$
$$+\frac{1}{2}(\gamma+1)\int_0^{R\psi} P_{0\psi} r^5_{0\psi}\eta'_i\eta'_j\eta'_k dr_{0\psi}\,. \qquad (12)$$

Thus, the pulsation equation breaks up into the equations

$$\ddot{q_k} + {\sigma_k}^2 q_k = \frac{\gamma}{I_k}\Big[\sum_i D_{ii,k} q_i^2 + 2\sum_{\substack{i,j\\ i\neq j}} D_{ij,k} q_i q_j\Big],$$
$$(j \geq i)\,, \qquad (13)$$

where $k = 1, 2, 3 \ldots$. It is more convenient to take the time variable τ instead of t such that

$$\tau = \sigma_1 t\,,$$

where σ_1 is eigenfrequency of the fundamental mode for the stellar model. This makes the period of oscillation in the fundamental mode 2π in the variable τ. So using $\tau = \sigma_1 t$, Equation (13) becomes

$$\frac{d^2 q_k}{d\tau^2} + \beta_k q_k = \frac{\gamma}{I_k\sigma_1^2}\Big[\sum_i D_{ii,k} q_i^2 + 2\sum_{\substack{i,j\\ i\neq j}} D_{ij,k} q_i q_j\Big],$$
$$(j \geq i) \text{ for } k = 1, 2, 3\ldots\,, \qquad (14)$$

where

$$\beta_k = \frac{\sigma_k^2}{\sigma_1^2}\,. \qquad (15)$$

Using r_ψ, ρ_ψ and P_ψ (that denote the equilibrium values on the equipotential surface) in place of $r_{0\psi}$, $\rho_{0\psi}$ and $P_{0\psi}$ respectively, the expressions for I_K and $D_{ij,k}$ finally become

$$I_k = \int_0^{R\psi} \rho_\psi r^4_\psi \eta_k^2 dr_\psi\,, \qquad (16)$$

and

$$D_{ij,k} = -\frac{1}{2}\Big(3-\frac{4}{\gamma}\Big)(3\gamma+1)\int_0^{R_\psi} P'_\psi r^3_\psi \eta_i\eta_j\eta_k dr_\psi$$
$$+\frac{1}{2}(3\gamma-1)\int_0^{R_\psi} P_\psi r^4_\psi$$
$$(\eta_i\eta'_j\eta'_k + \eta'_i\eta_j\eta'_k + \eta'_i\eta'_j\eta_k) dr_\psi$$
$$+\frac{1}{2}(\gamma+1)\int_0^{R\psi} P_\psi r^5_\psi \eta'_i\eta'_j\eta'_k dr_\psi\,. \qquad (17)$$

So, Equations (14)–(17) constitute the equations of anharmonic radial pulsations for a rotationally and tidally distorted stellar model correct up to second order. The coefficients $D_{ij,k}$ and I_k are constants which can be computed for a given stellar model. In calculating $D_{ij,k}$ and I_k we take the amplitude of pulsation to be normalized to unity at the star's surface, so that the displacement q_b at the surface is given by

$$q_b = q_1 + q_2 + q_3 + \cdots\,. \qquad (18)$$

3 ANHARMONIC RADIAL PULSATION EQUATION OF A ROTATIONALLY AND TIDALLY DISTORTED POLYTROPIC MODEL OF A STAR

Suppose a polytropic model of a star is subject to rotation and tidal distortions. Then the structure of such a star will be rotationally and tidally distorted. Following the approach adopted by Mohan & Saxena (1985) we approximate the equipotential surface of this distorted model by Roche equipotential. Let P_ψ and ρ_ψ denote the pressure and density on the equipotential surface (ψ =constant) of the distorted model respectively. Then the values of P_ψ and ρ_ψ of this distorted model are connected through polytropic relations of the type

$$P_\psi = P_{c\psi}{\theta_\psi}^{N+1}, \quad \rho_\psi = \rho_{c\psi}{\theta_\psi}^N\,, \qquad (19)$$

where $P_{c\psi}$ and $\rho_{c\psi}$ are values of P_ψ and ρ_ψ at the center respectively, θ_ψ is the value of θ $(0 \leq \theta \leq 1)$ on the equipotential surface (ψ =constant) and N is the polytropic index. Substituting the values of Equation (19) in Equation (14), we obtain the anharmonic pulsation equation for RTD polytropic model of stars as

$$\frac{d^2q_k}{d\tau^2}+\beta_k q_k=\frac{\gamma}{I_k^*\omega_1^2}\Bigg[\sum_i A^*_{ii,k}q_i^2+2\sum_{\substack{i,j\\ i\neq j}}A^*_{ij,k}q_iq_j\Bigg],$$
$$(j\geq i)\ \text{ for }\ k=1,2,3\cdots, \tag{20}$$

where

$$\beta_k=\frac{\omega_k^2}{\omega_1^2};\quad A^*_{ii,k}=\frac{D^*_{ii,k}}{\omega_1^2I_k^*};\quad A^*_{ij,k}=\frac{D^*_{ij,k}}{\omega_1^2I_k^*}.$$

$$I_k^*=\int_0^1\theta^Nx^4\eta_k^2dx.$$

$$\begin{aligned}D^*_{ij,k}\ =\ &-\frac{(N+1)}{2}\Big(3-\frac{4}{\gamma}\Big)(3\gamma+1)\\ &\int_0^1\theta^N\theta' x^3\eta_i\eta_j\eta_k dx\\ &+\frac{1}{2}(3\gamma-1)\int_0^1x^4\theta^{(N+1)}\\ &\qquad(\eta_i\eta_j'\eta_k'+\eta_i'\eta_j\eta_k'+\eta_i'\eta_j'\eta_k)dx\\ &+\frac{1}{2}(\gamma+1)\int_0^1x^5\theta^{(N+1)}\eta_i'\eta_j'\eta_k'dx.\end{aligned}$$

Here, γ is the ratio of specific heat, ω_1^2 is eigenfrequency of the fundamental radial mode, x is a non-dimensional variable of displacement varying from the center to surface for a given polytropic model of a star and η_1, η_2, η_3, ... are the eigenfunctions of various modes of radial oscillation in RTD polytropic models for stars. So, Equation (20) governs the anharmonic radial pulsations for RTD polytropic models of stars. The coefficients $D^*_{ij,k}$ and I^*_k are constants which can be computed for a given RTD polytropic model of a star. Also, the displacement q_b at the surface can be obtained by using Equation (18).

The eigenfunctions and eigenfrequencies have been obtained using the methodology as discussed by Mohan & Saxena (1985). The equations governing the radial oscillations of an RTD polytropic model of a star, as obtained by Mohan & Saxena (1985), in a nondimensional form are given by

$$H_1\frac{d^2\eta}{dr_0^2}\ +\ H_2\frac{d\eta}{dr_0^2}\ +\ (H_3\omega^2-H_4)\eta=0\,, \tag{21}$$

where

$$\begin{aligned}H_1=&1-\frac{16}{3}nr_0^3-\Big(\frac{56}{5}q^2+\frac{112}{15}nq+\frac{104}{45}n^2\Big)r_0^6\\ &-\frac{90}{7}q^2r_0^8-\frac{44}{3}q^2r_0^{10}+\ldots,\\ H_2=&\frac{1}{r_0}\Big[4-\frac{64}{3}nr_0^3-\Big(\frac{296}{5}q^2+\frac{592}{15}nq+\frac{1064}{45}n^2\Big)r_0^6\\ &-\frac{560}{7}q^2r_0^8-\frac{316}{3}q^2r_0^{10}+\ldots)\\ &+(N+1)\Big(\frac{1}{\theta_\psi}\frac{d\theta_\psi}{dx}\Big)r_0H_1\Big],\\ H_3=&\frac{(N+1)}{3\gamma r_{0s}^2}\xi_u^2K\frac{\bar\rho}{\rho_c}\frac{1}{\theta_\psi},\\ H_4=&-\Big(3-\frac{4}{\gamma_1}\Big)(N+1)\Big(\frac{1}{\theta_\psi}\frac{d\theta_\psi}{dr_0}\Big)\frac{1}{r_0}\Big[1-\frac{10n}{3}nr_0^3,\\ &-\Big(\frac{32}{5}q^2+\frac{64}{15}nq+\frac{188}{45}n^2\Big)r_0^6\\ &-\frac{50}{7}q^2r_0^8-8q^2r_0^{10}+\ldots\Big],\\ \omega^2=&\frac{D^3r_{0s}^2\sigma^2}{GM_0}\quad\text{and}\quad r_0=xr_{0s},\end{aligned}$$

n is a rotation parameter ($2n=\Omega^2$, Ω is the normalized angular velocity of rotation), q is the ratio of mass of the secondary to that of the primary star, D is the separation between the two components of a binary star system, G is the universal gravitational constant, M_0 is the total mass of the star, ω^2 is the non-dimensional form of the actual eigenfrequencies of oscillation σ, with r_{0s} being the value of r_0 on the outermost surface, N is the polytropic index, H_1, H_2, H_3 and H_4 are nonlinear functions of distortions parameters n and q, ρ_c represents density at the center, $\bar\rho$ is the average density for the undistorted polytropic model of the star and ξ_u is the value of ξ (where ξ is the Lane-Emden variable: specifically we have the value of $\xi_u=3.65375, 6.89685$ corresponding to polytropic indices $N=1.5, 3.0$ respectively) at the outermost surface of polytropic models.

4 METHOD FOR SOLVING THE EQUATION OF ANHARMONIC RADIAL OSCILLATION FOR ROTATIONALLY AND TIDALLY DISTORTED POLYTROPIC MODEL OF A STAR

In order to solve Equation (20), we consider the equations for q_1, q_2 and q_3. These may be written as

$$\begin{aligned}&\tfrac{d^2q_1}{d\tau^2}+q_1=A_{11,1}q_1^2+2A_{12,1}q_1q_2+2A_{13,1}q_1q_3\,,\\ &\tfrac{d^2q_2}{d\tau^2}+\beta_2q_2=A_{11,2}q_1^2+2A_{12,2}q_1q_2+2A_{13,2}q_1q_3\,,\\ &\tfrac{d^2q_3}{d\tau^2}+\beta_3q_3=A_{11,3}q_1^2+2A_{12,3}q_1q_2+2A_{13,3}q_1q_3\,,\end{aligned} \tag{22}$$

where $A_{11,1}, A_{12,1}, A_{13,1}$ are constants that are to be determined. Following Prasad (1949b), we assume the solution for these equations of the form

$$\begin{aligned} q_1 =& a_{0,1} + a_{1,1}\cos n_1\tau + a_{2,1}\cos 2n_1\tau \\ &+ a_{3,1}\cos 3n_1\tau + \dots, \\ q_2 =& a_{0,2} + a_{1,2}\cos n_1\tau + a_{2,2}\cos 2n_1\tau \\ &+ a_{3,2}\cos 3n_1\tau + \dots, \\ q_3 =& a_{0,3} + a_{1,3}\cos n_1\tau + a_{2,3}\cos 2n_1\tau \\ &+ a_{3,3}\cos 3n_1\tau + \dots, \end{aligned} \tag{23}$$

where $a_{0,1}$, $a_{1,1}$, $a_{2,1}$, ..., $a_{0,2}$, $a_{1,2}$, $a_{2,2}$, ..., $a_{0,3}$, $a_{1,3}$, $a_{2,3}$, ... and n_1 are constants that are to be determined.

These values of q_1, q_2 and q_3 are substituted in Equation (22). On equating the constant terms and the coefficients of $\cos kn_1\tau$ (for different k) to zero we get

$$\begin{aligned} a_{0,1} =& A_{11,1}\left[a_{0,1}^2 + \frac{1}{2}a_{1,1}^2 + \frac{1}{2}a_{2,1}^2 + \frac{1}{2}a_{3,1}^2 + \dots\right] \\ &+ 2A_{12,1}\Big[a_{0,1}a_{0,2} + \frac{1}{2}a_{1,1}a_{1,2} + \frac{1}{2}a_{2,1}a_{2,2} \\ &+ \frac{1}{2}a_{3,1}a_{3,2} + \dots\Big] \\ &+ 2A_{13,1}\Big[a_{0,1}a_{0,3} + \frac{1}{2}a_{1,1}a_{1,3} + \frac{1}{2}a_{2,1}a_{2,3} \\ &+ \frac{1}{2}a_{3,1}a_{3,3} + \dots\Big], \end{aligned} \tag{24a}$$

$$\begin{aligned} \left(1 - n_1^2\right)a_{1,1} =& A_{11,1}[2a_{0,1}a_{1,1} + a_{1,1}a_{2,1} \\ &+ a_{2,1}a_{3,1} + \dots] \\ &+ 2A_{12,1}[a_{1,1}a_{0,2} + a_{0,1}a_{1,2} + a_{2,1}a_{0,2} \\ &+ \frac{1}{2}\left(a_{1,1}a_{2,2} + a_{2,1}a_{1,2}\right) \\ &+ \frac{1}{2}(a_{2,1}a_{3,2} + a_{3,1}a_{3,2}) + \dots] \\ &+ 2A_{13,1}\,[2a_{0,1}a_{1,3} + a_{1,1}a_{0,3} \\ &+ \frac{1}{2}a_{1,1}a_{2,3} \\ &+ \frac{1}{2}a_{2,1}a_{1,3} + \frac{1}{2}a_{3,1}a_{2,3} + \dots\Big], \end{aligned} \tag{24b}$$

$$\begin{aligned} \Big(1 - n_1^2k^2\Big)a_{k,1} =& A_{11,1}\Big[\frac{1}{2}\sum_{i=0}^{k} a_{i,1}a_{k-i,1} + \sum_{i=0}^{\infty} a_{i,1}a_{k+i,1}\Big] \\ &+ A_{12,1}\Big[\frac{1}{2}\sum_{i=0}^{k} a_{i,1}a_{k-i,2} \\ &+ \frac{1}{2}\sum_{i=0}^{\infty}(a_{i,1}a_{k+i,2} + a_{k+i,1}a_{i,2})\Big] \\ &+ A_{13,1}\Big[\frac{1}{2}\sum_{i=0}^{k} a_{i,1}a_{k-i,3} \\ &+ \frac{1}{2}\sum_{i=0}^{\infty}(a_{i,1}a_{k+i,3} + a_{k+i,1}a_{i,3})\Big]. \end{aligned} \tag{24c}$$

We derive similar equations from the second equation in Equation (22) which are the same as above but with $(\beta_2 - k^2n_1^2)a_{k,2}$ in place of $(1 - k^2n_1^2)a_{k,1}$ and $A_{11,2}$, $A_{13,2}$ and $A_{22,2}$ in place of $A_{11,1}$, $A_{12,1}$ and $A_{13,1}$ respectively.

In order to solve these algebraic equations, defined in Equations (24a)–(24c), we suppose that $a_{1,1}$ is a known small quantity and we determine the other a's in terms of $a_{1,1}$. Considering Equation (24a) we see that $a_{0,1}$ is a small quantity of the second order, and all other terms are square or products and they contain the term $a_{1,1}^2$. Similarly, we find that $a_{2,1}$ is a quantity of the second order, $a_{3,1}$ is of the third order and in general $a_{k,1} (k > 1)$ is of the k^{th} order. Considering the equation for $a_{1,2}$ is of third order and $a_{2,2}$ is of second order and in general $a_{k,2} (k > 1)$ is of k^{th} order, a first approximation to the solution of Equations (24a)–(24c) can be given as:

$$\begin{aligned} a_{0,1} &= \frac{1}{2}A_{11,1}a_{1,1}^2, \\ a_{2,1} &= -\frac{1}{6}A_{11,1}a_{1,1}^2, \\ a_{3,1} &= \frac{1}{16}\left[\frac{1}{3}A_{11,1}^2 + \frac{A_{12,1}}{(4-\beta_2)}A_{11,2}\right]a_{1,1}^3, \\ &\dots\dots\dots\dots \\ a_{0,2} &= \frac{1}{2}A_{11,2}a_{1,1}^2/\beta_2, \\ a_{1,2} &= \left[\frac{5}{6}A_{11,1} + \frac{(8-3\beta_2)}{2\beta_2(4-\beta_2)}A_{12,2}\right]\frac{A_{11,2}a_{1,1}^3}{(\beta_2 - 1)}, \\ a_{2,2} &= -\frac{1}{2}\frac{A_{11,2}a_{1,1}^2}{(4-\beta_2)}, \\ a_{3,2} &= \frac{1}{2}\left[\frac{1}{3}A_{11,1} + \frac{A_{12,2}}{(4-\beta_2)}\right]\frac{A_{11,2}a_{1,1}^3}{(9-\beta_2)}, \\ &\dots\dots\dots\dots \\ a_{0,3} &= \frac{1}{2}A_{11,3}a_{1,1}^2/\beta_3, \\ a_{1,3} &= \left[\frac{5}{6}A_{11,1} + \frac{(8-3\beta_3)}{2\beta_3(4-\beta_3)}A_{13,3}\right]\frac{A_{11,3}a_{1,1}^3}{(\beta_3 - 1)}, \\ a_{2,3} &= -\frac{1}{2}\frac{A_{11,3}a_{1,1}^2}{(4-\beta_3)}, \\ a_{3,3} &= \frac{1}{2}\left[\frac{1}{3}A_{11,1} + \frac{A_{13,3}}{(4-\beta_3)}\right]\frac{A_{11,3}a_{1,1}^3}{(9-\beta_3)}, \\ &\dots\dots\dots\dots \end{aligned}$$

and

$$\begin{aligned} {n_1}^2 = 1 - \Big[&\frac{5}{6}A_{11,1}^2 + (8-3\beta_2)A_{12,1}A_{11,2}/ \\ &\Big(2\beta_2(4-\beta_2)\Big)\Big]a_{1,1}^2. \end{aligned}$$

These values of $a_{i,1}$, $a_{i,2}$ and n_1 are substituted in Equations (24a)–(24c) and a better approximation is obtained. The new values are resubstituted in

Table 1 Various Parameters Used in the Manuscript

Parameter	Definition
n	The rotation parameter that represents distortions due to rotation ($2n = \Omega^2, \Omega$ is normalized angular velocity of rotation for a star)
q	The tidal parameter that represents distortion due to tidal effects ($q = \frac{m_2}{m_1}$, mass of secondary component to primary component for a binary system)
N	Polytropic index
$q_b(t)$	The displacement at the surface of a star at time t
τ	A variable for time ($\tau = \sigma_1 t$)
σ_1	Eigenfrequency of fundamental mode
η	The relative amplitude of displacement of an element at a distance r from the center of a star
K	The skewness coefficient

Table 2 Eigenvalues for Various Modes of Rotationally Distorted Polytropic Models of Stars

	$N = 1.5$			$N = 3.0$		
	ω_1^2	ω_2^2	ω_3^2	ω_1^2	ω_2^2	ω_3^2
($n = 0, q = 0$)	2.7058	12.5337	26.533	9.2545	16.9839	28.5540
($n = 0.01, q = 0.0$)	2.6442	12.0984	25.6116	9.1138	16.6188	27.8497
($n = 0.03, q = 0.0$)	2.5747	11.6478	24.5983	8.8233	15.8699	26.4272
($n = 0.05, q = 0.0$)	2.4865	11.0428	23.2970	8.5190	15.0990	24.5980
($n = 0.07, q = 0.0$)	2.3974	10.3877	21.98165	8.3061	14.6425	23.9485
($n = 0.09, q = 0.0$)	2.3072	9.7802	20.6470	7.8535	13.4209	21.1310

Table 3 Eigenvalues for Various Modes of Tidally Distorted Polytropic Models of Stars

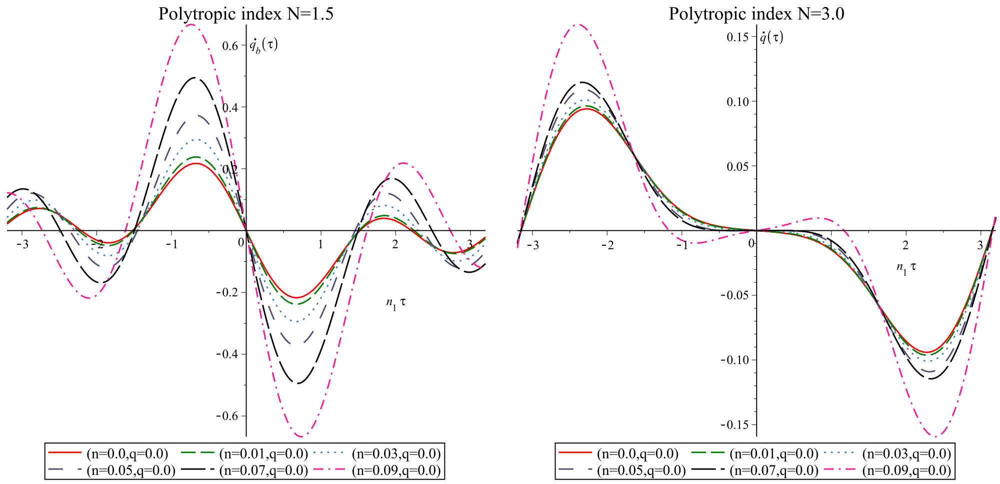
	$N = 1.5$			$N = 3.0$		
	ω_1^2	ω_2^2	ω_3^2	ω_1^2	ω_2^2	ω_3^2
($n = 0, q = 0$)	2.7058	12.5337	26.5330	9.2545	16.9839	28.5540
($n = 0.0, q = 0.1$)	2.7050	12.5260	26.5160	9.2522	16.9754	28.5377
($n = 0.0, q = 0.2$)	2.7024	12.5026	26.4640	9.2450	16.9497	28.4760
($n = 0.0, q = 0.3$)	2.6982	12.4636	26.3770	9.2333	16.9105	28.3776
($n = 0.0, q = 0.4$)	2.6931	12.3947	26.2333	9.2162	16.8496	28.2289
($n = 0.0, q = 0.5$)	2.6855	12.3378	26.1021	9.1941	16.7664	28.0370

Table 4 Eigenvalues for Various Modes of Rotationally and Tidally Distorted Polytropic Models of Stars

	$N = 1.5$			$N = 3.0$		
	ω_1^2	ω_2^2	ω_3^2	ω_1^2	ω_2^2	ω_3^2
($n = 0, q = 0$)	2.7058	12.5337	26.5330	9.2545	16.9839	28.5540
($n = 0.505, q = 0.01$)	2.4279	10.6362	22.4308	8.3099	14.5690	23.770
($n = 0.515, q = 0.03$)	2.4218	10.5909	22.3374	8.2874	14.5112	23.6428
($n = 0.525, q = 0.05$)	2.4182	10.5123	22.3225	8.2654	14.4514	23.5060
($n = 0.535, q = 0.07$)	2.4094	10.4994	22.1448	8.2418	14.3914	23.3520
($n = 0.55, q = 0.1$)	2.3999	10.4282	21.9953	8.2064	14.300	23.1500

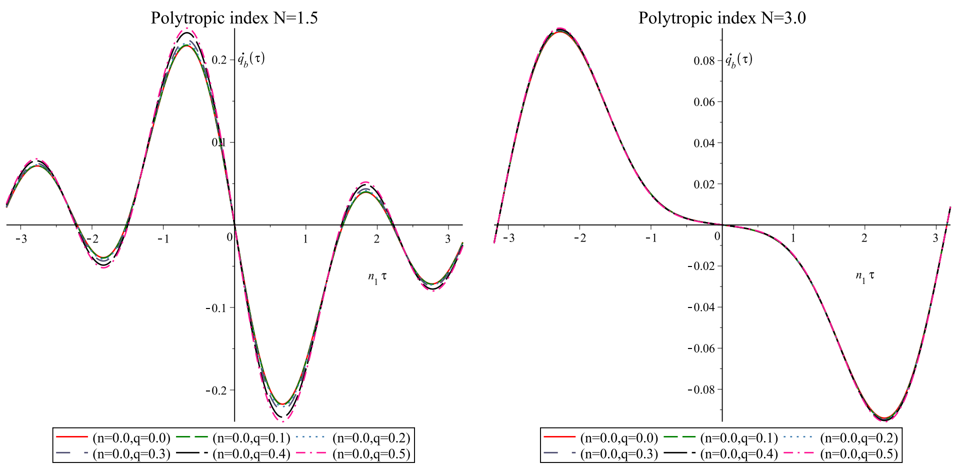
Table 5 Values of n_1^2 and Skewness Coefficient K for Rotationally Distorted Polytropic Models of Stars

	$N = 1.5$		$N = 3.0$	
	n_1^2	K	n_1^2	K
($n = 0, q = 0$)	0.90616	0.79	0.99468	0.29
($n = 0.01, q = 0.0$)	0.89568	0.79	0.99431	0.29
($n = 0.03, q = 0.0$)	0.88040	0.79	0.99353	0.28
($n = 0.05, q = 0.0$)	0.85341	0.78	0.99213	0.28
($n = 0.07, q = 0.0$)	0.80337	0.78	0.99235	0.27
($n = 0.1, q = 0.0$)	0.71363	0.77	0.98854	0.26



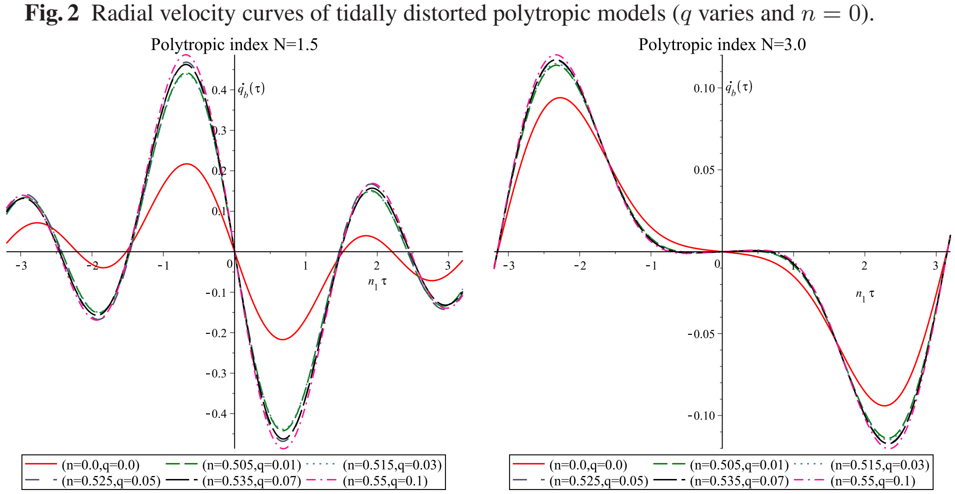
(a) (b)

Fig. 1 Radial velocity curves of rotationally distorted polytropic models (n varies and $q = 0$).



(a) (b)

Fig. 2 Radial velocity curves of tidally distorted polytropic models (q varies and $n = 0$).



(a) (b)

Fig. 3 Radial velocity curves of rotationally and tidally distorted polytropic models (both n and q vary).

Table 6 Values of n_1^2 and Skewness Coefficient K for Tidally Distorted Polytropic Models of Stars

	$N = 1.5$		$N = 3.0$	
	n_1^2	K	n_1^2	K
$(n = 0, q = 0)$	0.90616	0.79	0.99468	0.29
$(n = 0.0, q = 0.1)$	0.90595	0.79	0.99464	0.28
$(n = 0.0, q = 0.2)$	0.90534	0.79	0.99633	0.28
$(n = 0.0, q = 0.3)$	0.90430	0.79	0.99461	0.28
$(n = 0.0, q = 0.4)$	0.90212	0.79	0.99457	0.28
$(n = 0.0, q = 0.5)$	0.90067	0.79	0.99453	0.28

Table 7 Values of n_1^2 and Skewness Coefficient K for Rotationally and Tidally Distorted Polytropic Models of Stars

	$N = 1.5$		$N = 3.0$	
	n_1^2	K	n_1^2	K
$(n = 0, q = 0)$	0.90616	0.79	0.99468	0.29
$(n = 0.505, q = 0.01)$	0.82681	0.78	0.99166	0.27
$(n = 0.515, q = 0.03)$	0.82789	0.78	0.99156	0.27
$(n = 0.525, q = 0.05)$	0.82170	0.78	0.99143	0.27
$(n = 0.535, q = 0.07)$	0.81556	0.78	0.99133	0.27
$(n = 0.55, q = 0.1)$	0.80884	0.78	0.99113	0.27

Equations (24a)–(24c) and the process is repeated a number of times till we attain the desired accuracy.

5 NUMERICAL COMPUTATIONS

We have solved the equation of anharmonic radial oscillation for the first three modes of pseudo radial oscillations of the rotationally and tidally distorted polytropic models of stars for polytropic indices 1.5 and 3.0, and for different values of distortion parameters n and q with $\gamma = \frac{5}{3}$. Simpson's rule has been used to numerically evaluate the coefficients I_k^* and $D_{ij,k}^*$ of the anharmonic pulsation Equation (20). The numerical technique discussed in Section 4 has been used to solve the anharmonic Equation (20).

In Table 1, we have described the important parameters that have been used in the manuscript. While solving the anharmonic equation we require various eigenfrequencies of pseudo radial modes of oscillations of RTD polytropic models. These are obtained by solving Equation (21) and presented in Tables 2–4 for certain polytropic models of stars.

The skewness coefficient K (the ratio of the rise time, from minimum to the maximum, for the radial velocity to the total pulsation period) has also been computed in each case and is given in Tables 5–7. Finally, radial velocity curves for each model (graph of $\frac{dq_b}{d\tau}$ against $n_1\tau$) for polytropic indices $N = 1.5$ and 3.0 has been obtained, as shown in Figures 1–3.

In the present work, to study the effects of rotation and/or tidal distortion on the anharmonic radial oscillations and hence on the radial velocity curves of pulsating variable stars, we have considered the following polytropic models of pulsating variable stars: (i) undistorted star (single non-rotating star, no rotational or tidal effects ($n = 0$, $q = 0$)), (ii) single rotating star or rotationally distorted star ($q = 0$, n has some value), (iii) tidally distorted star ($n = 0$, q has some value, although stars with only tidal distortion and no rotation do not exist observationally, however, from a theoretical point of view and for the completeness of our study we have also considered the case of tidally distorted stars), (iv) a rotating star which is the primary component in a synchronous, circular and aligned binary system or a rotationally and tidally distorted star (n and q have some values; $2n = q + 1$ is the same as what was derived by Kopal (1972) for synchronous binary stars assuming Keplerian angular velocity).

6 CONCLUDING OBSERVATIONS

The results shown in Tables 2–4 represent the eigenfrequencies of the fundamental, first and second pseudo radial modes of oscillations, for rotationally and tidally distorted polytropic models with $\gamma = \frac{5}{3}$ and with polytropic indices $N = 1.5, 3.0$. The eigenfrequencies of radial modes for rotationally distorted, tidally distorted, and rotationally and tidally distorted polytropic models are smaller as compared to the values of the undistorted polytropic model. These results are in accordance with previous results of Mohan & Saxena (1985).

From Figure 1, we observe that with an increase in the value of n (or angular velocity of rotation of a star or rotational distortion) the radial velocity curves for rotating polytropic models of pulsating variable stars deviate from the radial velocity curve of an undistorted polytropic model and this effect is more appreciable near points of maxima and minima. Also, polytropes with index $N = 1.5$ show more deviation near points of maxima and minima with increasing value of n as compared to the polytropic model with index $N = 3.0$.

From Figure 2, we can conclude that with increase in the value of q (tidal distortions due to companion or secondary component) there is no appreciable effect on the shapes of radial velocity curves for tidally distorted polytropic models as compared to the radial velocity curve for the undistorted polytropic model.

It is clear from Figure 3 that when combined effects of rotation (n) and tidal distortions (q) are considered then again radial velocity curves of these rotationally and tidally distorted polytropic models of pulsating variable stars deviate from the radial velocity curve for the undistorted model and these deviations are larger near points of maxima and minima.

From Tables 5–7, it can be observed that, as compared to the undistorted model, there is a decrease in the value of skewness coefficient K under the effect of rotational distortions, tidal distortions, and rotational and tidal distortions. Also from Table 5, it can be concluded that the value of K decreases with increase in rotational distortions. However, from Tables 6–7 we can observe that with an increase in tidal distortions, and rotation and tidal distortions there is no change in the value of skewness coefficient K.

So from the present study we can conclude that rotational effects cause more deviations in the shapes of radial velocity curves for pulsating variables stars as compared to tidal effects. Also these deviations in the shapes of radial velocity curves increase with an increase in the effects of rotational distortions.

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