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EDITOR'S RECOMMENDATION

Saturn's gravitational field induced by its equatorially antisymmetric zonal winds

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Abstract The cloud-level zonal winds of Saturn are marked by a substantial equatorially antisymmetric component with a speed of about $50\,\mathrm{m\,s^{-1}}$ which, if they are sufficiently deep, can produce measurable odd zonal gravitational coefficients $\Delta J_{2k+1}, k = 1, 2, 3, 4$. This study, based on solutions of the thermal-gravitational wind equation, provides a theoretical basis for interpreting the odd gravitational coefficients of Saturn in terms of its equatorially antisymmetric zonal flow. We adopt a Saturnian model comprising an ice-rock core, a metallic dynamo region and an outer molecular envelope. We use an equatorially antisymmetric zonal flow that is parameterized, confined in the molecular envelope and satisfies the solvability condition required for the thermal-gravitational wind equation. The structure and amplitude of the zonal flow at the cloud level are chosen to be consistent with observations of Saturn. We calculate the odd zonal gravitational coefficients $\Delta J_{2k+1}, k = 1, 2, 3, 4$ by regarding the depth of the equatorially antisymmetric winds as a parameter. It is found that ΔJ_3 is -4.197×10^{-8} if the zonal winds extend about $13\,000\,\mathrm{km}$ downward from the cloud tops while it is -0.765×10^{-8} if the depth is about $4000\,\mathrm{km}$. The depth/profile of the equatorially antisymmetric zonal winds can eventually be estimated when the high-precision measurements of the Cassini Grand Finale become available.

Key words: gravitation — planets and satellites: individual (Saturn) — planets and satellites: interiors

1 INTRODUCTION

The axisymmetric gravitational potential V_g of Saturn can be written as

$$V_g(r,\theta) = -\frac{GM_\mathrm{S}}{r}\left\{1 - \sum_{n=1}^{\infty}\left[J_n + \Delta J_n\right] \times \left(\frac{R_\mathrm{S}}{r}\right)^n P_n(\cos\theta)\right\}, \quad r > R_\mathrm{S}, \tag{1}$$

where $GM_\mathrm{S} = 3.7931 \times 10^{16}\ \mathrm{m^3\ s^{-2}}$ (Williams 2016), (r, θ) are spherical polar coordinates with $\theta = 0$ at the axis of rotation, $R_\mathrm{S} = 60\,268\,\mathrm{km}$ is the equatorial radius of Saturn (Williams 2016), G is the universal gravitational constant ($G = 6.67384 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$), $P_n(\cos\theta)$ are the Legendre functions, n takes integer values, J_n represent the zonal gravitational coefficients in hydrostatic equilibrium, ΔJ_n denote the corrections due to the effects of the axially symmetric zonal flow and $(J_n + \Delta J_n)$ represent the measured gravitational coefficients of Saturn. It is anticipated that the high-precision measurements of the Cassini Grand Finale will determine the coefficients up to $(J_{10} + \Delta J_{10})$ or higher. The present study concentrates on how the lowermost odd gravitational coefficients ΔJ_{2k+1} with $k = 1, 2, 3, 4$ in expansion (1) are produced by the equatorially antisymmetric zonal winds of Saturn if they are sufficiently deep.

Two important features of the odd coefficients in expansion (1) should be particularly highlighted. First, the odd coefficients $(J_{2k+1} + \Delta J_{2k+1})$ with $k \geq 1$ differ profoundly from the even coefficients $(J_{2k} + \Delta J_{2k})$ with $k \geq 1$. Since the rotational distortion in hydrostatic equilibrium, because of its equatorial symmetry, does not contribute to the odd coefficients, we have $J_{2k+1} = 0$ with $k \geq 1$. This implies that the odd coefficients directly reflect the structure and amplitude of the equatorially antisymmetric zonal flow in the Saturnian interior (see, for example, Kaspi 2013; Liu et al. 2014; Kong et al. 2015). Second, the effect of Saturn's departure from spherical geometry is of secondary significance in estimating the lower-order odd coefficients ΔJ_{2k+1} with $k \geq 1$ caused by the equatorially antisymmetric zonal flow (Kong et al. 2016). It follows that spherical geometry is appropriate for computing the values of ΔJ_{2k+1} with $k = 1, 2, 3, 4$. The primary purpose of this study is to establish a theoretical relationship between the radial structure of the equatorially antisymmetric zonal winds and the value of the wind-induced odd coefficients ΔJ_{2k+1} with $k = 1, 2, 3, 4$ for Saturn, which enables estimating the depth of its cloud-level zonal winds when the high-precision measurements of the Cassini Grand Finale become available.

In comparison to the even coefficients ΔJ_{2k} with $k \geq 1$ in expansion (1), establishing the relationship between the winds and the odd coefficients ΔJ_{2k+1} with $k \geq 1$ is difficult in a different way. In the second paper of this series on the gravitational field, shape and zonal winds of Saturn (Kong et al. 2018b), we have employed a self-consistent perturbation approach to study the effect of the equatorially symmetric zonal winds on the even coefficients ΔJ_{2k} with $k \geq 1$. Since the leading-order problem accounts exactly for rotational distortion, the self-consistent perturbation allows us to solve the full Euler equation involving an equatorially symmetric zonal flow on cylinders, to uniquely determine the wind-induced density anomaly satisfying the proper physical boundary conditions, and to identify the variation of the gravitational field solely caused by the flow. For an equatorially antisymmetric zonal flow, the full Euler equation with the flow on cylinders cannot be employed, the wind-induced density anomaly cannot be uniquely determined and, consequently, different approaches have to be adopted for computing the odd coefficients ΔJ_{2k+1} with $k \geq 1$. An approach based on "solutions" of the thermal wind equation (TWE) was adopted by Kaspi (2013). Discussed below is an attempt to explain why the TWE is inappropriate for this purpose and why the thermal-gravitational wind equation (TGWE), derived by Zhang et al. (2015), should be adopted.

Since the TGWE – a two-dimensional inhomogeneous integral equation describing a relationship between the equatorially antisymmetric zonal winds and the wind-induced density anomalies – is infrequently encountered in geophysical and astrophysical problems, it is insightful to look at the analogous inhomogeneous ordinary differential equation,

$$\pi^2 u(x) + \frac{\mathrm{d}^2 u(x)}{\mathrm{d}x^2} = f(x), \tag{2}$$

subject to the boundary condition

$$u(0) = u(1) = 0,$$

where $f(x)$ is a given function of x, for the purpose of explaining why the TWE would lead to an inaccurate and spurious solution of the problem. Note that replacing the differential operator $\mathrm{d}^2 u(x)/\,\mathrm{d}x^2$ in Equation (2) with an integral operator results in a TGWE-like equation while removing the differential operator $\mathrm{d}^2 u(x)/\,\mathrm{d}x^2$ from Equation (2) gives a TWE-like equation.

There are two major deficiencies in the TWE approach for computing the odd gravitational coefficients induced by the equatorially antisymmetric zonal winds. First, neglecting the derivative term $\mathrm{d}^2 u(x)/\,\mathrm{d}x^2$ in Equation (2) always yields a "solution" in the form

$$u(x) = \frac{f(x)}{\pi^2}$$

for any given function $f(x)$. This solution might represent an approximation to the true solution of Equation (2) but it is certainly inaccurate because $\mathrm{d}^2 u(x)/\,\mathrm{d}x^2$ is generally of the same order as $\pi^2 u(x)$ and, hence, should not be neglected. Second, more significantly, the "solution" $u(x) = f(x)/\pi^2$ can be totally irrelevant to the true solution of Equation (2). This is because when the inhomogeneous term $f(x)$ does not satisfy the solvability condition

$$\int_0^1 \sin(\pi x) f(x) dx = 0 \tag{3}$$

required for Equation (2), its solution does not even exist. For example, when $f(x) = \sin(\pi x)$ and the solvability condition (3) in this case are violated, a mathematically valid solution for Equation (2) does not exist. It follows that (i) the "solution" $u(x) = \sin(\pi x)/\pi^2$ is spurious and does not offer an approximation to the true solution of Equation (2) and (ii) the differential operator $\mathrm{d}^2u(x)/\,\mathrm{d}x^2$ cannot, in general, be removed from Equation (2) for mathematical or numerical convenience.

The primary objective of this study, based on solutions of the TGWE in spherical geometry, is to provide a theoretical basis required for understanding the relationship between the odd coefficients ΔJ_{2k+1} with $k \geq 1$ and the depth of the equatorially antisymmetric zonal winds on Saturn. We adopt an interior model of Saturn consisting of an ice-rock core, a metallic region and an outer molecular envelope (see, for example, Stevenson 1982; Guillot 2005; Kong et al. 2018a) and assume that its zonal winds are confined in the molecular envelope (see, for example, Heimpel et al. 2005; Jones & Kuzanyan 2009). We choose the wind profiles that respect the solvability condition required for the TGWE in spherical geometry. As in many physical problems governed by inhomogeneous differential or integral equations, it is the required solvability condition that helps to select mathematically acceptable and physically relevant solutions. We calculate the odd zonal gravitational coefficients of Saturn, ΔJ_{2k+1} with $k = 1, 2, 3, 4$, using the profile of the zonal flow that is equatorially antisymmetric, parameterized and consistent with the observed cloud-level structure. We expect that, when the gravitational signals for the equatorially antisymmetric gravitational field of Saturn detected by the Cassini Grand Finale become available, the parameters of the Saturnian equatorially antisymmetric winds used in our model can be determined or constrained.

In what follows, we begin by presenting the governing equation for the density anomalies induced by the equatorially antisymmetric zonal winds in Section 2, which is followed by discussion of the results in Section 3 with a summary and some remarks given in Section 4.

2 THE GOVERNING EQUATION

Our model of Saturn assumes that (i) the planet with mass M_{S} is isolated and rotating about the z-axis with constant angular velocity $\Omega_{\mathrm{S}}\hat{z}$; (ii) the planet is axially symmetric and consists of fully compressible gases; (iii) the equatorially antisymmetric zonal winds observed at the cloud level are associated with the axially symmetric fluid motion taking place in the molecular region whose location is provided by the hydrostatic model of Saturn (Kong et al. 2018a); (iv) the effect of Saturn's departure from spherical geometry on the odd gravitational coefficients is negligibly small; (v) the Rossby number in connection with the Saturnian equatorially antisymmetric zonal flow is small and both the viscous and magnetic effects are weak; and (vi) the zonal flow is in a statistically steady state. The special case when the zonal winds are largely confined to a very top thin layer of the stably stratified atmosphere is mathematically trivial because the winds in this case cannot produce measurable external gravitational signatures.

The above assumptions lead to the following governing equations in the rotating frame of reference

$$2\Omega_{\mathrm{S}}\hat{\boldsymbol{z}} \times \boldsymbol{u}\,(\boldsymbol{r}) = -\frac{1}{\rho(\boldsymbol{r})}\nabla p(\boldsymbol{r}) + \boldsymbol{g}(\boldsymbol{r}) + \frac{\Omega_{\mathrm{S}}^2}{2}\nabla\left|\hat{\boldsymbol{z}}\times\boldsymbol{r}\right|^2, \qquad (4)$$

$$\nabla\cdot[\boldsymbol{u}(\boldsymbol{r})\rho(\boldsymbol{r})] = 0, \qquad (5)$$

where $\boldsymbol{u}(\boldsymbol{r})$ represents the velocity of the equatorially antisymmetric zonal flow, $\boldsymbol{r}$ is the position vector with the origin at the center of figure, $p(\boldsymbol{r})$ is the pressure and $\rho(\boldsymbol{r})$ is the density. The density $\rho(\boldsymbol{r})$ of the fully compressible barotropic fluid is assumed to be a function only of the pressure $p(\boldsymbol{r})$ obeying the polytropic law which is discussed in Kong et al. (2018a). Suppose that the typical speed U_0 of the zonal flow $\boldsymbol{u}$ in Saturn's molecular region is small compared to the rotational speed of the planet, $U_0/(\Omega_{\mathrm{S}}R_{\mathrm{S}}) \ll 1$. Under the axisymmetric assumption, Equations (4)–(5) can be solved by making use of the expansions (Zhang et al. 2017):

$$p = p_0(r,\theta) + p'(r,\theta), \qquad (6)$$

$$\rho = \rho_0(r,\theta) + \rho'(r,\theta), \qquad (7)$$

$$\boldsymbol{g} = \boldsymbol{g}_0(r,\theta) + \boldsymbol{g}'(r,\theta), \qquad (8)$$

where the leading-order solution $(p_0, \rho_0, \boldsymbol{g}_0)$ represents the hydrostatic state of Saturn while $(p', \rho', \boldsymbol{g}')$ denotes the perturbations arising from the effect of the equatorially antisymmetric zonal flow $\boldsymbol{u}$.

Upon substituting the expansions (6)–(8) into Equations (4)–(5) and neglecting the effect of rotational distortion, we obtain the leading-order problem governed by

$$\boldsymbol{0} = -\frac{1}{\rho_0}\nabla p_0 + \boldsymbol{g}_0, \qquad (9)$$

$$\boldsymbol{g}_0(r,\theta) = 2\pi G \boldsymbol{\nabla}\left[\int_0^\pi \int_0^{R_\mathrm{S}} \frac{\tilde{r}^2\rho_0(\tilde{r},\tilde{\theta})}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|} \times \sin\tilde{\theta}\,\mathrm{d}\tilde{r}\,\mathrm{d}\tilde{\theta}\right]. \quad (10)$$

With the equation of state provided by Kong et al. (2018a), Equations (9) – (10) can be readily solved to determine the density distribution $\rho_0(\boldsymbol{r})$ and the gravitational force $\boldsymbol{g}_0(\boldsymbol{r})$ for a spherical Saturn in its hydrostatic state.

The next-order problem, which describes the density anomaly $\rho'(r,\theta)$ induced by the zonal flow $\boldsymbol{u}(r,\theta)$ and the concomitant gravitational perturbation $\boldsymbol{g}'(r,\theta)$ produced by $\rho'(r,\theta)$, is governed by the equations

$$2\rho_0(r)\left[\Omega_\mathrm{S}\hat{\boldsymbol{z}}\times\boldsymbol{u}(r,\theta)\right] = -\boldsymbol{\nabla}p'(r,\theta) + \boldsymbol{g}_0(r)\rho'(r,\theta) + \boldsymbol{g}'(r,\theta)\rho_0(r), \quad (11)$$

$$0 = \boldsymbol{\nabla}\cdot\left[\boldsymbol{u}(r,\theta)\rho_0(r)\right]. \quad (12)$$

In deriving Equations (11)–(12), we have neglected the small high-order terms which are of $\mathrm{O}(|\boldsymbol{g}'\rho'|)$ and $\mathrm{O}(|\boldsymbol{u}\rho'\Omega_\mathrm{S}|)$. Note that the terms $\boldsymbol{g}_0\rho'$ and $\boldsymbol{g}'\rho_0$ in Equation (11) are generally of the same order of magnitude. Physically, when the internal density anomaly ρ' is induced by the deep flow $\boldsymbol{u}$, the hydrostatic gravitational force $\boldsymbol{g}_0$ must be also perturbed to yield the concomitant gravitational perturbation $\boldsymbol{g}'$.

Taking the curl of Equation (11), making use of Equation (12) and writing the equatorially antisymmetric flow $\boldsymbol{u} = U_\mathrm{A}(r,\theta)\hat{\boldsymbol{\phi}}$, the azimuthal component of the resulting equation can be expressed as

$$2\Omega_\mathrm{S}\left[\cos\theta\frac{\partial}{\partial r}(\rho_0 U_\mathrm{A}) - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}(\rho_0 U_\mathrm{A})\right] = \frac{g_0(r)}{r}\frac{\partial\rho'}{\partial\theta} - \frac{2\pi G(\mathrm{d}\rho_0/\mathrm{d}r)}{r} \times \frac{\partial}{\partial\theta}\left[\int_0^\pi\int_0^{R_\mathrm{S}}\frac{\tilde{r}^2\rho'(\tilde{r},\tilde{\theta})}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|}\sin\tilde{\theta}\,\mathrm{d}\tilde{r}\,\mathrm{d}\tilde{\theta}\right]. \quad (13)$$

Integrating (13) over θ gives rise to the TGWE

$$2\Omega_\mathrm{S}\int_{\pi/2}^{\theta}\left[\cos\tilde{\theta}\frac{\partial}{\partial r} - \frac{\sin\tilde{\theta}}{r}\frac{\partial}{\partial\tilde{\theta}}\right](\rho_0 U_\mathrm{A})\,\mathrm{d}\tilde{\theta} = \frac{g_0(r)}{r}\rho'(r,\theta) - \frac{2\pi G(\mathrm{d}\rho_0/\mathrm{d}r)}{r} \times \int_0^\pi\int_0^{R_\mathrm{S}}\frac{\tilde{r}^2\rho'(\tilde{r},\tilde{\theta})}{|\boldsymbol{r}-\tilde{\boldsymbol{r}}|}\sin\tilde{\theta}\,\mathrm{d}\tilde{r}\,\mathrm{d}\tilde{\theta}, \quad (14)$$

where $\boldsymbol{r} = \boldsymbol{r}(r,\theta)$ and $\tilde{\boldsymbol{r}} = \tilde{\boldsymbol{r}}(\tilde{r},\tilde{\theta})$. The value of the lower limit in the above integration is largely arbitrary and we choose $\pi/2$ because of the equatorial symmetry $\rho'(r,\pi/2) = 0$. It should be noticed that a solution $[\mathcal{F}(r)+\rho'(r,\theta)]$, where $\mathcal{F}(r)$ is an arbitrary function of r, can be another solution of the TGWE (14). However, $\mathcal{F}(r)$ does not affect the values of the odd gravitational coefficients ΔJ_{2k+1} with $k \geq 1$. This non-uniqueness makes the precise structure of the density anomaly $\rho'(r,\theta)$ less significant. For a prescribed equatorially antisymmetric flow $U_\mathrm{A}(r,\theta)$ together with the leading-order solution $\rho_0(r)$ and $\boldsymbol{g}_0(r)$, the TGWE (14) can be solved, via a spectral numerical method discussed in Zhang et al. (2015), for determining $\rho'(r,\theta)$ which can be then used to compute the coefficients ΔJ_{2k+1} with $k \geq 1$.

Two important features of the TGWE (14) should be underlined. First, the two terms on the right side of Equation (14) are generally comparable in size and, hence, the second integral term cannot be neglected. Some special profile $U_\mathrm{A}(r,\theta)$ might produce $\rho'(r,\theta)$ that has a special structure (for example, an alternating positive and negative pattern leading to an average cancelation) such that the integral term on the right side of Equation (14) is small compared to the first term and, thus, negligible. In this special case, the integral term can be neglected a posteriori. There are no physical or mathematical reasons to justify the a priori neglect of the integral term. Second, the inhomogeneous TGWE (14), analogous to the inhomogeneous ordinary differential equation (2), must satisfy the solvability condition discussed by Kong et al. (2016). A "solution" $\rho'(r,\theta)$ of the TWE, given by

$$\rho'(r,\theta) = \frac{2r\Omega_\mathrm{S}}{g_0}\int_{\pi/2}^{\theta}\left[\cos\tilde{\theta}\frac{\partial}{\partial r} - \frac{\sin\tilde{\theta}}{r}\frac{\partial}{\partial\tilde{\theta}}\right] \times (\rho_0 U_\mathrm{A})\,\mathrm{d}\tilde{\theta}, \quad (15)$$

which always exists for any given $U_\mathrm{A}(r,\theta)$, would be spurious if the flow $U_\mathrm{A}(r,\theta)$ does not respect the solvability condition. This is because the TWE (15) merely represents a diagnostic relation which always generates a "solution" for any given zonal flow $U_\mathrm{A}(r,\theta)$.

In the present study, we solve the TGWE (14), an inhomogeneous integral equation in spherical geometry, to determine the wind-induced density anomaly $\rho'(r,\theta)$ for a prescribed equatorially antisymmetric zonal flow $U_\mathrm{A}(r,\theta)$.

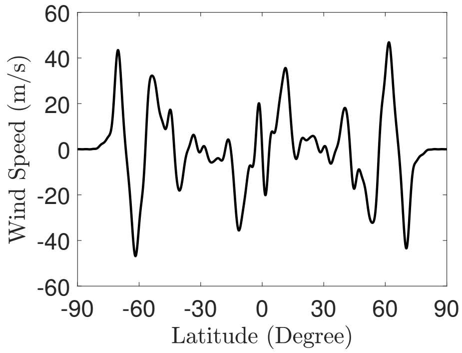

Fig. 1 The latitudinal profile of the observed equatorially antisymmetric component of the zonal winds at the cloud-level of Saturn (García-Melendo et al. 2011).

3 RESULTS AND DISCUSSIONS

The observed cloud-level zonal winds on Saturn (García-Melendo et al. 2011) can be decomposed into two different components: the equatorially symmetric winds $U_{\rm S}(R_{\rm S},\theta)$ obeying the parity $U_{\rm S}(R_{\rm S},\theta) = U_{\rm S}(R_{\rm S},\pi-\theta)$ and the equatorially antisymmetric winds $U_{\rm A}(R_{\rm S},\theta)$ obeying the parity $U_{\rm A}(R_{\rm S},\theta) = -U_{\rm A}(R_{\rm S},\pi-\theta)$.

Figure 1 shows the latitudinal profile $U_{\rm A}(R_{\rm S},\theta)$ of the equatorially antisymmetric zonal winds at the cloud level, where the speed of the antisymmetric winds can reach about $50\,{\rm m\,s^{-1}}$. This is sufficiently strong to yield gravitational signals detectable by the high-precision measurements of the Cassini Grand Finale.

Since the structure of the zonal flow $U_{\rm A}(r,\theta)$ in the Saturnian interior is unknown, a parameterized flow is needed in solving the TGWE (14). We introduce the three key parameters characterizing the equatorially antisymmetric zonal flow of Saturn. First, $U_{\rm A}(r=R_{\rm S},\theta)$ for $0\le\theta\le\pi$ at the outer bounding surface of Saturn is given by the observed profile depicted in Figure 1. Second, the equatorially antisymmetric flow is confined in the molecular region $0.6641\,R_{\rm S}\le r\le R_{\rm S}$, i.e., $U_{\rm A}(r,\theta)=0$ for $r\le 0.6641\,R_{\rm S}$ and $0\le\theta\le\pi$. Note that the location of the molecular envelope, $0.6641\,R_{\rm S}\le r\le R_{\rm S}$, is determined by the hydrostatic model of Saturn (Kong et al. 2018a). Third, a special parameter h is introduced to measure the depth of the equatorially antisymmetric winds within the molecular envelope $0.6641\,R_{\rm S}\le r\le R_{\rm S}$. Since the speed of the flow gradually decreases with depth, we define the ratio q of the two kinetic energies,

$$q=\frac{\int_{(1-h)R_{\rm S}}^{R_{\rm S}}\int_0^{2\pi}\int_0^{\pi}|U_{\rm A}(r,\theta)|^2\,r^2\sin\theta\,{\rm d}\theta\,{\rm d}\phi\,{\rm d}r}{\int_0^{R_{\rm S}}\int_0^{2\pi}\int_0^{\pi}|U_{\rm A}(r,\theta)|^2\,r^2\sin\theta\,{\rm d}\theta\,{\rm d}\phi\,{\rm d}r},$$

in order to quantify the penetration depth of the cloud-level winds, where the depth $hR_{\rm S}$ is defined such that 90% of the total kinetic energy is confined in the outer layer $(1-h)R_{\rm S}\le r\le R_{\rm S}$. For example, when $h=0.22$, 90% of the total kinetic energy of the flow is confined in the layer $(1-0.22)R_{\rm S}\le r\le R_{\rm S}$; when $h=0.14$, 90% of the energy is confined in $(1-0.14)\,R_{\rm S}\le r\le R_{\rm S}$; and when $h=0.076$, 90% of the energy is confined in $(1-0.076)\,R_{\rm S}\le r\le R_{\rm S}$. The three profiles of the equatorially antisymmetric zonal flow $U_{\rm A}(r,\theta)$ are depicted in Figure 2 for three different cases: the deep case with $h=0.22$, the intermediate case $h=0.14$ and the shallow case $h=0.076$. All three profiles $U_{\rm A}(r,\theta)$ shown in Figure 2 satisfy the solvability condition required for the TGWE (14).

An important property of the problem is that the equatorially antisymmetric zonal winds $U_{\rm A}(r,\theta)$ only induce the density anomaly $\rho'(r,\theta)$ obeying the same equatorial parity,

$$\rho'(r,\theta)(r,\theta)=-\rho'(r,\pi-\theta),$$

which produces only the odd gravitational coefficients ΔJ_{2k+1} with $k\ge 1$. After obtaining the density anomaly $\rho'(r,\theta)$, a solution of the TGWE (14) for a given zonal flow $U_{\rm A}(r,\theta)$, we can compute the odd zonal gravitational coefficients ΔJ_{2k+1} by performing the two-dimensional integration

$$\begin{aligned}\Delta J_{2k+1}=&-\frac{4\pi}{M_{\rm S}R_{\rm S}^{2k+1}}\int_0^{\pi/2}\int_0^{R_{\rm S}}\rho'(r,\theta)\\&\times P_{2k+1}(\cos\theta)r^{2k+3}\sin\theta\,{\rm d}r\,{\rm d}\theta,\\&k=1,2,3,\ldots\end{aligned}\qquad(16)$$

where the value of ΔJ_{2k+1} does not change by replacing $\rho'(r,\theta)$ with $\rho'(r,\theta)+\mathcal{F}(r)$. The form of the integral

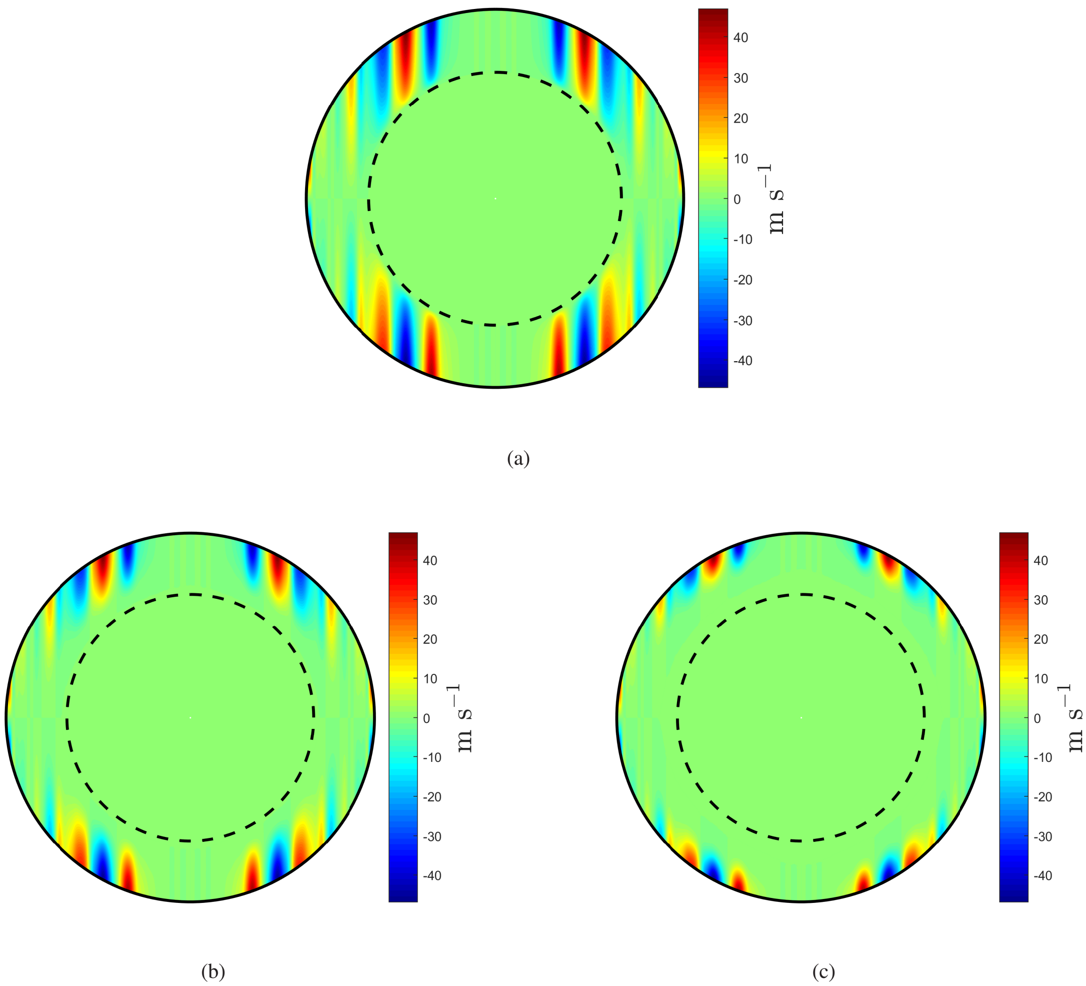

Fig. 2 The profiles of the parameterized equatorially antisymmetric zonal winds in a meridional plane of Saturn: (a) for the deep case $h = 0.22$, (b) for the intermediate case $h = 0.14$ and (c) for the shallow case $h = 0.076$. The *dashed circle* indicates the interface between the metallic dynamo region and the outer molecular envelope.

Table 1 The odd zonal gravitational coefficients $\Delta J_{2k+1}, k = 1, 2, 3, 4$ in the expansion (1) induced by the equatorially antisymmetric zonal winds of Saturn at different values of the depth parameter h.

The depth parameter	$h = 0.22$	$h = 0.14$	$h = 0.076$
$\Delta J_3 \times 10^8$	−4.197	−2.108	−0.765
$\Delta J_5 \times 10^8$	−0.075	−0.421	−0.333
$\Delta J_7 \times 10^8$	0.409	0.514	0.297
$\Delta J_9 \times 10^8$	−1.318	−0.714	−0.244

(16) suggests that the size of ΔJ_{2k+1} reflects the averaging effect of $\rho'(r, \theta)$ and, hence, depends on both the structure and amplitude of the density anomaly $\rho'(r, \theta)$.

Our analysis, based on solutions of the TGWE (13), has concentrated on the three typical profiles of the equatorially antisymmetric zonal flow $U_{\mathrm{A}}(r, \theta)$ shown in Figure 2. The results of the analysis, the lowermost odd coefficients $\Delta J_{2k+1}, k = 1, 2, 3, 4$ at different depth parameters, are presented in Table 1. In all the cases studied in this paper, the structure and amplitude of the zonal flow at the outer bounding surface $U_{\mathrm{A}}(r = R_{\mathrm{S}}, \theta)$ are prescribed while the penetration depth is marked by parameter h, which is inversely proportional to the exponentially decaying rate of the cloud-level winds with depth. Our attention is thus focused on the effect of depth h on the odd zonal gravitational coefficients ΔJ_{2k+1} with $k = 1, 2, 3, 4$. For $h = 0.22$ which corresponds to a substantial depth of about 13 000 km, our results show that the zonal winds yield the strongest gravitational signature in the lowermost-order coefficient J_3 with $\Delta J_3 = -4.197 \times 10^{-8}$ which is, however, of the same order as $\Delta J_9 = -1.318 \times 10^{-8}$. They perhaps represent an upper bound on the odd gravitational coefficients that can be induced by the equatorially antisymmetric zonal winds of Saturn. For $h = 0.14$, which corresponds to the wind depth of about $8500\,\mathrm{km}$, we found that the lowermost-order odd coefficient $\Delta J_3 = -2.108 \times 10^{-8}$ while $\Delta J_9 = -0.714 \times 10^{-8}$. For $h = 0.076$, which corresponds to the wind depth of about $4500\,\mathrm{km}$, the lowermost-order odd coefficient decreases to $\Delta J_3 = 0.765 \times 10^{-8}$. An interesting feature of the results is that the spectral structure of ΔJ_{2k+1}, the distribution of ΔJ_{2k+1} as a function of k, strongly depends on the size of the depth parameter h. For example, $\Delta J_3/\Delta J_5 \approx 60$ at $h = 0.22$ in the deep case while $\Delta J_3/\Delta J_5 \approx 2$ at $h = 0.076$ in the shallow case, indicating a complicated relationship between the profile of the equatorially antisymmetric zonal winds and the spectrum of the wind-induced odd coefficients ΔJ_{2k+1} with $k \geq 1$ for Saturn.

It is anticipated that the size of the odd coefficients $\Delta J_3, \Delta J_5, \Delta J_7$ and ΔJ_9 shown in Table 1 are within the accuracy of the high-precision gravity measurements carried out by the Cassini Grand Finale. If the measurements produce a similar spectrum of the odd coefficients ΔJ_{2k+1} to that in Table 1 at a particular h, we may conclude that the deep structure of the equatorially antisymmetric zonal winds in the Saturnian interior reflects what is observed at the cloud level, as suggested in Figure 2. If the measurements produce a spectrum of the odd coefficients ΔJ_{2k+1} which is profoundly different from that shown in Table 1 at any depth parameter h, we may conclude that the antisymmetric winds at the cloud level are confined to a very thin weather layer and, therefore, are not directly connected to the flow taking place in the Saturnian interior in which case an a priori unknown deep circulation accounts for the gravitational signal.

4 SUMMARY AND REMARKS

This paper is the third in a series on the gravitational field, shape and zonal winds of Saturn. By assuming that the cloud-level winds of the equatorially antisymmetric component on Saturn are sufficiently deep, we have computed the odd zonal gravitational coefficients ΔJ_{2k+1} with $k = 1, 2, 3, 4$ induced by the antisymmetric zonal winds. We have adopted an interior model of Saturn comprising an ice-rock core, a metallic region and an outer molecular envelope whose physical and geometrical parameters are determined by the hydrostatic problem (Kong et al. 2018a).

Our results presented in Table 1 are based on solutions of the TGWE (14) using the equatorially antisymmetric zonal flow $U_{\mathrm{A}}(r, \theta)$ depicted in Figure 2. The parameterized zonal flows used in our model are confined in the outer region of the molecular envelope and satisfy the solvability condition required for the TGWE (14). We have focused on the four lowermost-order odd gravitational coefficients, $\Delta J_3, \Delta J_5, \Delta J_7$, ΔJ_9, at the three different depth parameters: $h = 0.22$ corresponding to the wind depth of about $13\,000\,\mathrm{km}$, $h = 0.14$ for the depth of about $8500\,\mathrm{km}$ and $h = 0.076$ for the depth of about $4500\,\mathrm{km}$. We believe that our model forms a theoretical basis needed for interpreting the odd gravitational coefficients of Saturn when the high-precision measurements carried out by the Cassini Grand Finale become available.

Since the size of ΔJ_{2k+1} with $k \geq 1$ is related solely to the equatorially antisymmetric zonal flow $U_{\mathrm{A}}(r, \theta)$, the spectrum of the odd coefficients plays a vital role in understanding the structure and amplitude of large-scale fluid motion taking place in the deep interior of Saturn. If the cloud-level winds are confined to a very thin upper layer in which the winds are associated with horizontal temperature differences between belts and zones (Ingersoll & Cuzzi 1969), the spectrum ΔJ_{2k+1} of Saturn is then largely unrelated to the structure and amplitude of the cloud-level zonal winds. In this case, the profiles of the equatorially antisymmetric zonal flow $U_{\mathrm{A}}(r, \theta)$ depicted in Figure 2 are no longer physically relevant and the observed cloud-level winds are not directly associated with a deeper zonal flow that is responsible for the Saturnian gravitational signal. Although the governing equation, the TGWE (14), remains unchanged, the approach must be changed such that the antisymmetric gravitational field is computed without making a priori assumptions about the nature of the deeper zonal flow. We shall carry out further studies when the gravitational signal for the equatorially antisymmetric gravitational field of Saturn detected by the Cassini Grand Finale becomes available.

Finally, it should be stressed that there exist two significant differences between our model and that of Kaspi (2013). First, our model is based on Equation (14) that represents a two-dimensional inhomogeneous integral equation while his model is based on Equation (15) that represents a diagnostic relation. Second, the zonal flow $U_{\rm A}(r,\theta)$ used in our model respects the solvability condition required for Equation (14) while the equatorially antisymmetric zonal flow used by Kaspi (2013) does not satisfy the solvability condition and, hence, may lead to a spurious spectrum ΔJ_{2k+1}.

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