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Effect of interaction of the various modes on the radial velocity curves of the polytropic models of rotationally and tidally distorted pulsating variable stars

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Abstract In our previous work, we developed a model to study the effects of rotation and/or tidal distortions on anharmonic radial oscillations and hence on the radial velocity curves of the polytropic models of pulsating variable stars. We considered the first three modes (fundamental and the next two higher modes) for the polytropic models of index 1.5 and 3.0 in that work. In the present paper, we are further extending our previous work to study the effect of the interaction of various modes on anharmonic radial oscillations and hence on radial velocity curves of the rotationally and/or tidally distorted polytropic models of pulsating variable stars. For this purpose, we have considered the following cases: (i) fundamental mode (ii) fundamental and the first mode, (iii) fundamental and the next two modes and finally (iv) fundamental and the next three higher modes of pulsation in our study. The objective of this paper is also to investigate whether the interaction of various modes affects the results of our previous study or not. The results of this study show that the interaction of the fundamental mode with higher modes appreciably changes the shape of the radial velocity curve of rotationally distorted and rotationally and tidally distorted polytropic models of pulsating variable stars.

Key words: stars: rotation — stars: oscillations — stars: binaries — technique: radial velocities

1 INTRODUCTION

Several types of pulsating variables stars have been observed in the sky. Many of these are single rotating stars whereas some of the pulsating variable stars have also been found in binary systems. The radial velocity curves of such pulsating variable stars have also been plotted through observations and it has been found that they are not exactly sinusoidal. Rosseland (1943) proposed the theory of anharmonic oscillations to explain such variation in the shapes of radial velocity curves of variable stars. According to this theory, the normal sinusoidal wave gets distorted on account of the interaction between various modes that are concurrently excited in the case of pulsating variable stars. Rosseland (1949) extended the theory of anharmonic pulsations to include the effect of higher modes and higher order on the shape of the radial velocity curve of a pulsating stellar model. This theory has been subsequently used by different investigators such as Schwarzschild & Savedoff (1949), Prasad (1949a,b), Bhatnagar & Kushwaha (1951), Chatterji (1952), Gurm (1963) and Kumar et al. (2018) for studying various problems associated with pulsating variable stars.

van der Borght & Murphy (1966) investigated the anharmonic adiabatic radial pulsations of an early main sequence nonrotating star of 10 solar masses. They found that the inclusion of higher modes such as fifth and sixth together with the third order terms appreciably changes the shape of the radial velocity curves of nonrotating stars. However, they also concluded that the inclusion of only higher modes does not improve agreement between the theoretical and observed radial velocity curves. Prasad & Mohan (1969) used the theory of anharmonic pulsation to study the shapes of the radial velocity curve for a 15.6 solar mass star in the helium-burning phase of its evolution. They also investigated the effect of the first four modes of pulsation on the shapes of radial velocity curves for non-rotating stars. They found that with the inclusion of second, third and fourth modes the skewness ratio decreases, bringing it nearer to the observed values. Mohan (1972) considered the anharmonic pulsation of a 15.6 solar mass star in the helium burning phase taking into account the fifth and sixth modes in the anharmonic pulsation equation of nonrotating stars. He found that with the inclusion of the fifth mode the skewness coefficient again increases and the inclusion of the sixth mode creates humps in the radial velocity curve of a nonrotating star. However, in both cases – with the inclusion of modes higher than fourth mode – the radial velocity curves start deviating from the observed radial velocity curves of Cepheids. So, they concluded that modes higher than the fourth are not active in most Cepheid type pulsating variable stars.

Using simultaneous photometric and spectroscopic observations, Gieren (1980) showed that the δ-Cephei star AH Vel is pulsating in the first and second harmonic. Kiss et al. (1999) presented the photometric measurement of variable V2109 Cyg and also obtained its radial velocity curve. They showed that V2109 Cyg is a pulsating variable star that pulsates in the second overtone. Arentoft et al. (2001) found multiperiodicity and cyclic amplitude variability in the light curve of the delta Scuti star V1162 Ori. They also concluded that the observed linear period changes are not caused by evolutionary effects, but rather by long-period binarity or nonlinear mode interaction. Kjurkchieva et al. (2017) conducted intensive photometric and spectral observations of the variable star V2551 Cyg. They also studied its radial velocity curves and found it to be a pulsating star that pulsates with the fundamental mode.

In our previous work (Kumar et al. 2018, hereafter Paper 1), we developed a model to study the effect of rotation and/or tidal distortions on the anharmonic radial oscillations and hence on the radial velocity curves of the polytropic models of pulsating variable stars. Extending this work further in the present paper, we are trying to study the effect of interaction of various modes on anharmonic radial oscillations and hence on the radial velocity curves of rotationally and/or tidally distorted (hereafter RTD) polytropic models of pulsating variable stars. One of the objectives of this paper is also to check whether interaction of the various modes affects the results obtained in our previous work where we have only taken into account the case of the first three modes.

In Paper 1 we considered the first three modes (fundamental and the next two higher modes) for polytropic models of index 1.5 and 3.0. However, in the present work we are considering the following cases: (i) fundamental mode (hereafter f mode); (ii) fundamental and the first mode (hereafter $f + 1$ mode); (iii) fundamental and the next two modes (hereafter $f + 2$ modes) and (iv) fundamental and the next three higher modes (hereafter $f + 3$ modes) of pulsation. We have chosen polytropic models of index $N = 1.5$, 3.0 and 4.0 (to represent stars in the different stages of evolution: $N = 1.5$ is for pre-main sequence stars, $N = 3.0$ for main sequence stars and $N = 4.0$ for post-main sequence stars). Also for the present work, we have considered the polytropic models for an undistorted star (no rotational or tidal distortions), a single rotating star (only rotational distortions) and a rotating star which is a primary component of a synchronous, circular and aligned binary system (or rotationally and tidally distorted star). As discussed in Paper 1, the present analysis is applicable to highly centrally condensed pulsating variable stars with small oscillations that are (i) single and rotating slowly and (ii) slowly rotating primary components of the synchronous, circular and aligned binary systems in which the mass of the secondary is much less than the mass of the primary star (for more details, the reader can see Paper 1).

This paper is organized as follows:

In Section 2 we present the equation that governs the anharmonic pulsation of RTD distorted polytropic models of the stars. A successive approximation method is discussed in Section 3 to solve the anharmonic pulsation equation. In Section 4, numerical computations are performed to obtain the solution of the anharmonic pulsation equation for certain RTD polytropic models of the stars. Numerical results thus obtained are analyzed and certain conclusions are discussed in Section 5.

2 ANHARMONIC RADIAL PULSATION EQUATION OF A RTD POLYTROPIC MODEL OF A STAR

The equation that governs the anharmonic radial pulsations of the RTD polytropic model of a star – as obtained in Paper 1 – can be written as

$$\frac{d^2 q_k}{d\tau^2} + \beta_k q_k = \frac{\gamma}{I_k^* \omega_1^2}\left[\sum_i A_{ii,k}^* q_i^2 + 2\sum_{\substack{i,j\\ i\neq j}} A_{ij,k}^* q_i q_j\right] ,$$

$$(j \geq i) \text{ for } k = 1,2,3,4\ldots , \tag{1}$$

where

$$\tau = \omega_1 t , \quad \beta_k = \frac{\omega_k^2}{\omega_1^2} , \quad A_{ii,k}^* = \frac{D_{ii,k}^*}{\omega_1^2 I_k^*} , \quad A_{ij,k}^* = \frac{D_{ij,k}^*}{\omega_1^2 I_k^*} , \quad I_k^* = \int_0^1 \theta^N x^4 \eta_k^2 dx ,$$

$$\begin{aligned} D_{ij,k}^* &= -\frac{(N+1)}{2}(3-\frac{4}{\gamma})(3\gamma+1)\int_0^1 \theta^N \theta' x^3 \eta_i \eta_j \eta_k dx \\ &+\frac{1}{2}(3\gamma-1)\int_0^1 x^4 \theta^{(N+1)}(\eta_i \eta_j' \eta_k' + \eta_i' \eta_j \eta_k' + \eta_i' \eta_j' \eta_k)dx \\ &+\frac{1}{2}(\gamma+1)\int_0^1 x^5 \theta^{(N+1)} \eta_i' \eta_j' \eta_k' dx . \end{aligned}$$

Here γ is the ratio of the specific heat, τ is the time variable, θ $(0 \leq \theta \leq 1)$ is a polytropic parameter, N is the polytropic index, ω_1^2 is the eigenfreqency of the fundamental radial mode, x is a nondimensional variable of the displacement varying from the center to the surface for a given polytropic model of a star and η_1,η_2,η_3,η_4, ... are the eigenfuctions of the various modes of the radial oscillation of the RTD polytropic models for the stars, as discussed in Mohan & Saxena (1985). The coefficients $D_{ij,k}^*$ and I_k^* are constants that can be computed for a given RTD polytropic model of a star. Also, the displacement q_b at the surface is given by

$$q_b = q_1 + q_2 + q_3 + q_4 + \ldots . \tag{2}$$

The eigenfunctions and eigenfrequencies have been obtained using the methodology discussed by Mohan & Saxena (1985). The equations governing radial oscillations of the RTD polytropic model of a star, as obtained by Mohan & Saxena (1985), in nondimensional form are given by

$$H_1 \frac{d^2\eta}{dr_0^2} + H_2 \frac{d\eta}{dr_0^2} + (H_3 \omega^2 - H_4)\eta = 0 , \tag{3}$$

where

$$\begin{aligned} H_1 &= 1 - \frac{16}{3} n r_0^3 - \Big(\frac{56}{5}q^2 + \frac{112}{15}nq + \frac{104}{45}n^2\Big)r_0^6 - \frac{90}{7}q^2 r_0^8 - \frac{44}{3}q^2 r_0^{10} + \ldots , \\ H_2 &= \frac{1}{r_0}\Big[4 - \frac{64}{3}nr_0^3 - (\frac{296}{5}q^2 + \frac{592}{15}nq + \frac{1064}{45}n^2)r_0^6 - \frac{560}{7}q^2 r_0^8 - \frac{316}{3}q^2 r_0^{10} + \ldots \\ &\quad +(N+1)\Big(\frac{1}{\theta_\psi}\frac{d\theta_\psi}{dx}\Big)r_0 H_1\Big] , \\ H_3 &= \frac{(N+1)}{3\gamma r_{0s}^2}\xi_u^2 K \frac{\bar{\rho}}{\rho_c}\frac{1}{\theta_\psi} , \\ H_4 &= -\Big(3 - \frac{4}{\gamma_1}\Big)(N+1)\Big(\frac{1}{\theta_\psi}\frac{d\theta_\psi}{dr_0}\Big)\frac{1}{r_0}\Big[1 - \frac{10n}{3}nr_0^3 - \Big(\frac{32}{5}q^2 + \frac{64}{15}nq + \frac{188}{45}n^2\Big)r_0^6 \\ &\quad - \frac{50}{7}q^2 r_0^8 - 8q^2 r_0^{10} + \ldots\Big] , \\ \omega^2 &= \frac{D^3 r_{0s}^2 \sigma^2}{GM_0}, \quad r_0 = x r_{0s} . \end{aligned}$$

Here n is a rotation parameter ($2n = \Omega^2$, Ω is the normalized angular velocity of rotation), q is the tidal parameter that represents the ratio of mass of the secondary to mass of the primary star, D is separation between the two components of a binary system, G is the universal gravitational constant, M_0 is the total mass of the star, ω^2 is the nondimensional form of the actual eigenfrequencies of oscillation σ, $r_{\rm os}$ is the value of r_0 on the outermost surface, N is the polytropic index, H_1, H_2, H_3 and H_4 are nonlinear functions of the distortion parameters n and q, $\rho_{\rm c}$ represents the density at the center, $\bar{\rho}$ is the average density of the undistorted polytropic model of a star and ξ_u is the value of ξ (where ξ is the Lane-Emden variable: specifically we have the values of $\xi_u = 3.65375$, 6.89685 and 14.97715 corresponding to the polytropic indexes $N = 1.5$, 3.0 and 4.0 respectively) at the outermost surface of the polytropic model.

3 METHOD FOR SOLVING THE EQUATION OF THE ANHARMONIC RADIAL OSCILLATION OF ROTATIONALLY AND TIDALLY DISTORTED POLYTROPIC MODEL OF A STAR

To solve Equation (1) we have followed the approach that was used in Paper 1. We consider the equations for q_1, q_2, q_3 and q_4

$$\begin{aligned}
\frac{d^2q_1}{d\tau^2} + q_1 &= A_{11,1}q_1^2 + 2A_{12,1}q_1q_2 + 2A_{13,1}q_1q_3 + 2A_{14,1}q_1q_4 \ , \\
\frac{d^2q_2}{d\tau^2} + \beta_2 q_2 &= A_{11,2}q_1^2 + 2A_{12,2}q_1q_2 + 2A_{13,2}q_1q_3 + 2A_{14,2}q_1q_4 \ , \\
\frac{d^2q_3}{d\tau^2} + \beta_3 q_3 &= A_{11,3}q_1^2 + 2A_{12,3}q_1q_2 + 2A_{13,3}q_1q_3 + 2A_{14,3}q_1q_4 \ , \\
\frac{d^2q_4}{d\tau^2} + \beta_4 q_4 &= A_{11,4}q_1^2 + 2A_{12,4}q_1q_2 + 2A_{13,4}q_1q_3 + 2A_{14,4}q_1q_4 \ ,
\end{aligned} \tag{4}$$

where $A_{11,1}$, $A_{12,1}$, $A_{13,1}$, $A_{14,1}$... are the constants that are to be determined. Following Prasad (1949a,b), we assume the solution of these equations is in the form

$$\begin{aligned}
q_1 &= a_{0,1} + a_{1,1}\cos n_1\tau + a_{2,1}\cos 2n_1\tau + a_{3,1}\cos 3n_1\tau + a_{4,1}\cos 4n_1\tau + \dots \ , \\
q_2 &= a_{0,2} + a_{1,2}\cos n_1\tau + a_{2,2}\cos 2n_1\tau + a_{3,2}\cos 3n_1\tau + a_{4,2}\cos 4n_1\tau + \dots \ , \\
q_3 &= a_{0,3} + a_{1,3}\cos n_1\tau + a_{2,3}\cos 2n_1\tau + a_{3,3}\cos 3n_1\tau + a_{4,3}\cos 4n_1\tau + \dots \ , \\
q_4 &= a_{0,4} + a_{1,4}\cos n_1\tau + a_{2,4}\cos 2n_1\tau + a_{3,4}\cos 3n_1\tau + a_{4,4}\cos 4n_1\tau + \dots \ ,
\end{aligned} \tag{5}$$

where

$$a_{0,1}, a_{1,1}, a_{2,1}, \dots,$$

$$a_{0,2}, a_{1,2}, a_{2,2}, \dots,$$

$$a_{0,3}, a_{1,3}, a_{2,3}, \dots,$$

$$a_{0,4}, a_{1,4}, a_{2,4}, \dots$$

and n_1 are constants that are to be determined. The values of q_1, q_2, q_3 and q_4 are substituted in Equation (4) and on equating the constant terms and the coefficients of $\cos kn_1\tau$ (for different k) to zero we get

$$\begin{aligned}
a_{0,1} =& A_{11,1}\left[a_{0,1}^2 + \frac{1}{2}a_{1,1}^2 + \frac{1}{2}a_{2,1}^2 + \frac{1}{2}a_{3,1}^2 + \dots\right] \\
&+ 2A_{12,1}\left[a_{0,1}a_{0,2} + \frac{1}{2}a_{1,1}a_{1,2} + \frac{1}{2}a_{2,1}a_{2,2} + \frac{1}{2}a_{3,1}a_{3,2} + \dots\right] \\
&+ 2A_{13,1}\left[a_{0,1}a_{0,3} + \frac{1}{2}a_{1,1}a_{1,3} + \frac{1}{2}a_{2,1}a_{2,3} + \frac{1}{2}a_{3,1}a_{3,3} + \dots\right] \\
&+ 2A_{14,1}\left[a_{0,1}a_{0,4} + \frac{1}{2}a_{1,1}a_{1,4} + \frac{1}{2}a_{2,1}a_{2,4} + \frac{1}{2}a_{3,1}a_{3,4} + \dots\right] \ ,
\end{aligned} \tag{6a}$$

$$\begin{aligned}
\left(1-n_1^2\right)a_{1,1} =& A_{11,1}\left[2a_{0,1}a_{1,1}+a_{1,1}a_{2,1}+a_{2,1}a_{3,1}+\ldots\right]\\
&+2A_{12,1}[a_{1,1}a_{0,2}+a_{0,1}a_{1,2}+a_{2,1}a_{0,2}+\frac{1}{2}\left(a_{1,1}a_{2,2}+a_{2,1}a_{1,2}\right)\\
&+\frac{1}{2}(a_{2,1}a_{3,2}+a_{3,1}a_{3,2})+\ldots]\\
&+2A_{13,1}\left[2a_{0,1}a_{1,3}+a_{1,1}a_{0,3}+\frac{1}{2}a_{1,1}a_{2,3}+\frac{1}{2}a_{2,1}a_{1,3}+\frac{1}{2}a_{3,1}a_{2,3}+\ldots\right]\\
&+2A_{14,1}\left[2a_{0,1}a_{1,4}+a_{1,1}a_{0,4}+\frac{1}{2}a_{1,1}a_{2,4}+\frac{1}{2}a_{2,1}a_{1,4}+\frac{1}{2}a_{3,1}a_{2,4}+\ldots\right]\ , \qquad (6b)
\end{aligned}$$

$$\begin{aligned}
\left(1-n_1^2k^2\right)a_{k,1} =& A_{11,1}\left[\frac{1}{2}\sum_{i=0}^{k}a_{i,1}a_{k-i,1}+\sum_{i=0}^{\infty}a_{i,1}a_{k+i,1}\right]\\
&+A_{12,1}\left[\frac{1}{2}\sum_{i=0}^{k}a_{i,1}a_{k-i,2}+\frac{1}{2}\sum_{i=0}^{\infty}(a_{i,1}a_{k+i,2}+a_{k+i,1}a_{i,2})\right]\\
&+A_{13,1}\left[\frac{1}{2}\sum_{i=0}^{k}a_{i,1}a_{k-i,3}+\frac{1}{2}\sum_{i=0}^{\infty}(a_{i,1}a_{k+i,3}+a_{k+i,1}a_{i,3})\right]\\
&+A_{14,1}\left[\frac{1}{2}\sum_{i=0}^{k}a_{i,1}a_{k-i,4}+\frac{1}{2}\sum_{i=0}^{\infty}(a_{i,1}a_{k+i,4}+a_{k+i,1}a_{i,4})\right]\ . \qquad (6c)
\end{aligned}$$

We get a similar equation from the second equation in Equation (4) which is the same as above but with $(\beta_2-k^2n_1^2)a_{k,2}$ in place of $(1-k^2n_1^2)a_{k,1}$ and $A_{11,2}$, $A_{12,2}$, $A_{13,2}$ and $A_{14,2}$ in place of $A_{11,1}$, $A_{12,1}$, $A_{13,1}$ and $A_{14,1}$ respectively.

In order to solve these algebraic (Eqs. (6a)–(6c)) equations, we suppose that $a_{1,1}$ is a known small quantity and we determine the other a's in terms of $a_{1,1}$. Considering Equation (6a) we see that $a_{0,1}$ is a small quantity of the second order, all other terms are square or products and they contain the term $a_{1,1}^2$. Similarly, we find that $a_{2,1}$ is a quantity of the second order, $a_{3,1}$ is of the third order and in general $a_{k,1}$ $(k>1)$ is of the k^{th} order. We consider the equation for $a_{1,2}$ of the third order and $a_{2,2}$ of the second order and in general $a_{k,2}$ $(k>1)$ of the k^{th} order. Therefore, a first approximation to the solution of Equations (6a)–(6c) can be given as:

$$\begin{aligned}
a_{0,1} &= \frac{1}{2}A_{11,1}a_{1,1}^2\ ,\\
a_{2,1} &= -\frac{1}{6}A_{11,1}a_{1,1}^2\ ,\\
a_{3,1} &= \frac{1}{16}\left[\frac{1}{3}A_{11,1}^2+\frac{A_{12,1}}{(4-\beta_2)}A_{11,2}\right]a_{1,1}^3\ ,\\
&\ldots\ldots\ldots\ldots\ldots\ldots\\
a_{0,2} &= \frac{1}{2}A_{11,2}a_{1,1}^2/\beta_2\ ,\\
a_{1,2} &= \left[\frac{5}{6}A_{11,1}+\frac{(8-3\beta_2)}{2\beta_2(4-\beta_2)}A_{12,2}\right]\frac{A_{11,2}a_{1,1}^3}{(\beta_2-1)}\ ,\\
a_{2,2} &= -\frac{1}{2}\frac{A_{11,2}a_{1,1}^2}{(4-\beta_2)}\ ,\\
a_{3,2} &= \frac{1}{2}\left[\frac{1}{3}A_{11,1}+\frac{A_{12,2}}{(4-\beta_2)}\right]\frac{A_{11,2}a_{1,1}^3}{(9-\beta_2)}\ ,\\
&\ldots\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

$$a_{0,3} = \frac{1}{2} A_{11,3} a_{1,1}^2 / \beta_3 \ ,$$

$$a_{1,3} = \left[\frac{5}{6} A_{11,1} + \frac{(8 - 3\beta_3)}{2\beta_3 (4 - \beta_3)} A_{13,3} \right] \frac{A_{11,3} a_{1,1}^3}{(\beta_3 - 1)} \ ,$$

$$a_{2,3} = -\frac{1}{2} \frac{A_{11,3} a_{1,1}^2}{(4 - \beta_3)} \ ,$$

$$a_{3,3} = \frac{1}{2} \left[\frac{1}{3} A_{11,1} + \frac{A_{13,3}}{(4 - \beta_3)} \right] \frac{A_{11,3} a_{1,1}^3}{(9 - \beta_3)} \ ,$$

....

$$a_{0,4} = \frac{1}{2} A_{11,4} a_{1,1}^2 / \beta_4 \ ,$$

$$a_{1,4} = \left[\frac{5}{6} A_{11,1} + \frac{(8 - 3\beta_4)}{2\beta_4 (4 - \beta_4)} A_{14,4} \right] \frac{A_{11,4} a_{1,1}^3}{(\beta_4 - 1)} \ ,$$

$$a_{2,4} = -\frac{1}{2} \frac{A_{11,4} a_{1,1}^2}{(4 - \beta_4)} \ ,$$

$$a_{3,4} = \frac{1}{2} \left[\frac{1}{3} A_{11,1} + \frac{A_{14,4}}{(4 - \beta_4)} \right] \frac{A_{11,4} a_{1,1}^3}{(9 - \beta_4)} \ ,$$

....

and

$$n_1{}^2 = 1 - \left[\frac{5}{6} A_{11,1}^2 + (8 - 3\beta_2)\, A_{12,1} A_{11,2} / \left(2\beta_2 \left(4 - \beta_2 \right) \right) \right] a_{1,1}^2 \ .$$

These values of $a_{i,1}$, $a_{i,2}$, $a_{i,3}$, $a_{i,4}$ and n_1 are substituted in Equations (6a)–(6c) and a better approximation is obtained. The new values are resubstituted in Equations (6a)–(6c) and the process is repeated a number of times till we attain the desired accuracy.

4 NUMERICAL COMPUTATIONS

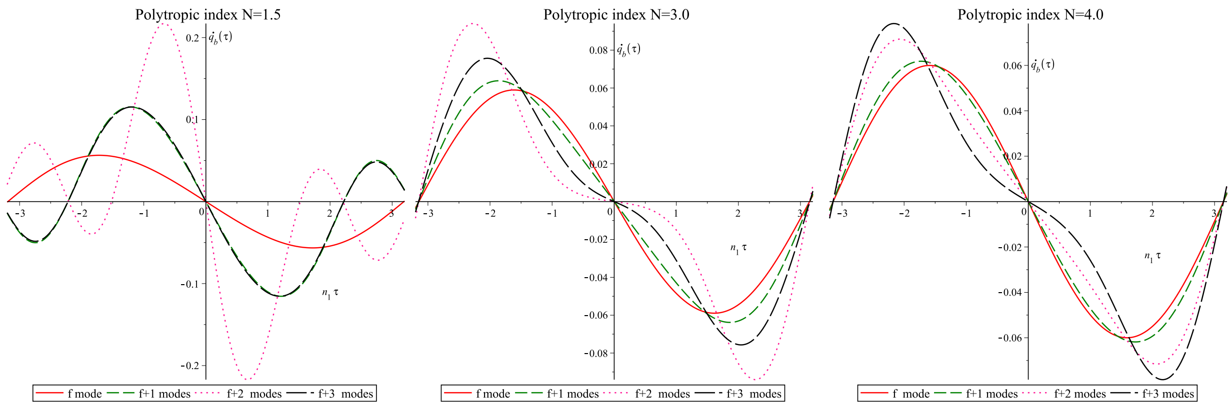
We have solved the equation of the anharmonic radial oscillation up to the first four modes for certain RTD polytropic models of the stars. We have considered the polytropic models with indices 1.5, 3.0 and 4.0 for different values of the rotational distortion parameter (n) and the tidal distortion parameter (q) with $\gamma = \frac{5}{3}$. Simpson's rule has been used to numerically evaluate the coefficients I_k^* and $D_{ij,k}^*$ in the anharmonic pulsation Equation (1). The numerical technique discussed in Section 3 has been used to solve the anharmonic pulsation equation, Equation (1).

In Table 1, we describe the important parameters that have been used in the manuscript. The eigenfrequencies of the pseudo-radial modes of oscillations in the RTD polytropic models have been obtained using the approach of Mohan & Saxena (1985) and are presented in Table 2 for certain polytropic models of the stars. The radial velocity curves for each model of polytropic index $N = 1.5,\ 3.0$ and 4.0 have been obtained and are shown in Figures 1–13. The value of the skewness coefficient K has also been computed in each case and these values are listed in Table 3.

In the present work, to study the effect of the interaction of various modes on the radial velocity curves of RTD polytropic models for the stars, we have considered the case of (i) fundamental mode, (ii) fundamental and the first mode, (iii) fundamental and the next two modes and finally (iv) fundamental and the next three higher modes of the pulsation. For computational and numerical purposes, the following polytropic models of the pulsating variable stars have been considered: (i) undistorted star (no rotational or tidal distortion), (ii) single rotating star (only rotational distortion) and (iii) rotating star which is a primary component of the synchronous, circular and aligned binary system (or rotationally and tidally distorted star) in which the mass of the secondary is assumed to be much less than the mass of the primary star.

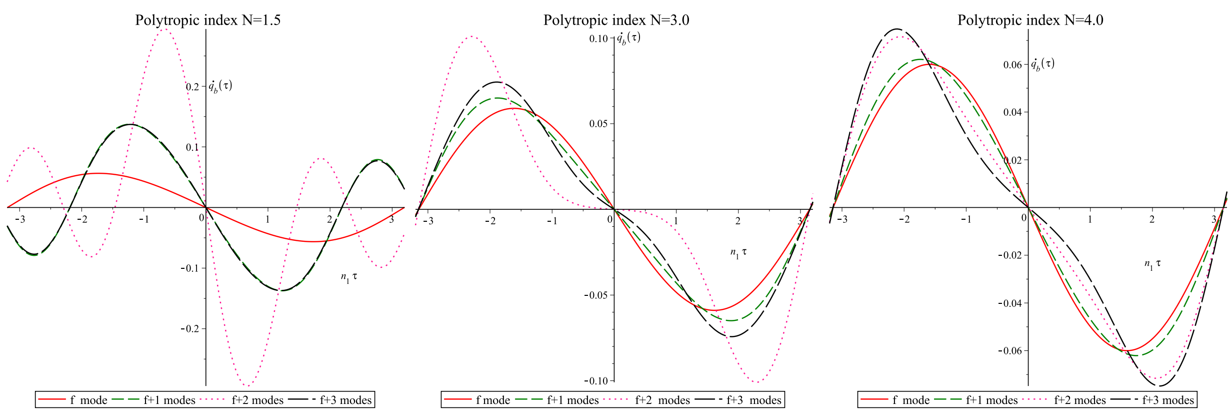
5 CONCLUDING OBSERVATIONS

The results shown in Table 2 represent the eigenfrequencies of the fundamental, first, second and third pseudo-



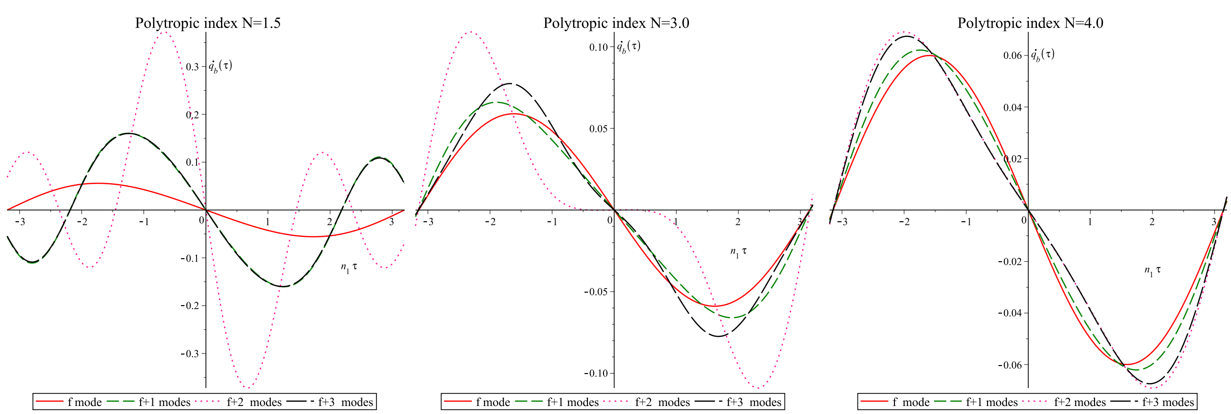
(a) (b) (c)

Fig. 1 Radial velocity curves of the undistorted polytropic models $n = 0.0$, $q = 0.0$).



(a) (b) (c)

Fig. 2 Radial velocity curves of the rotationally distorted polytropic models ($n = 0.03$, $q = 0.0$).



(a) (b) (c)

Fig. 3 Radial velocity curves of the rotationally distorted polytropic models ($n = 0.05$, $q = 0.0$).

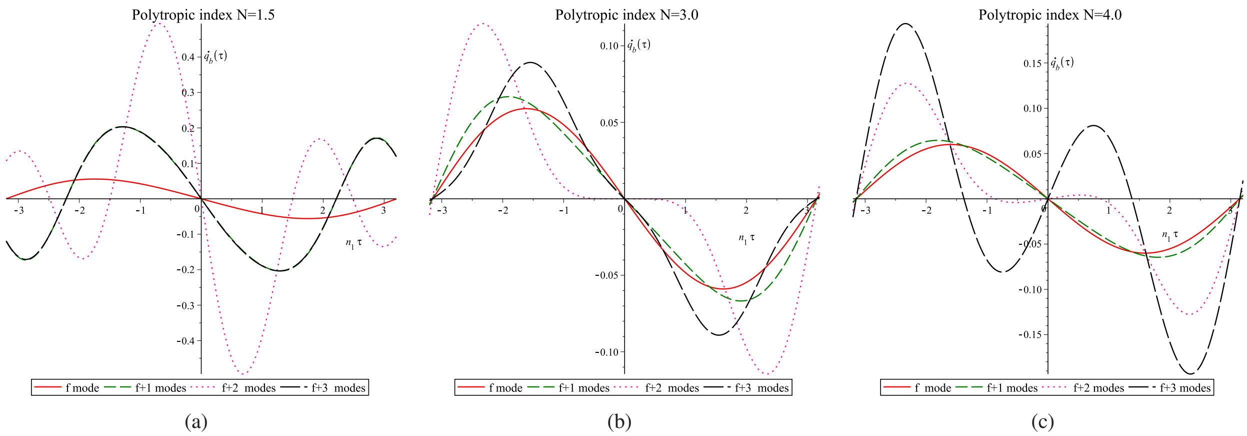


Fig. 4 Radial velocity curves of the rotationally distorted polytropic models ($n = 0.07$, $q = 0.0$).

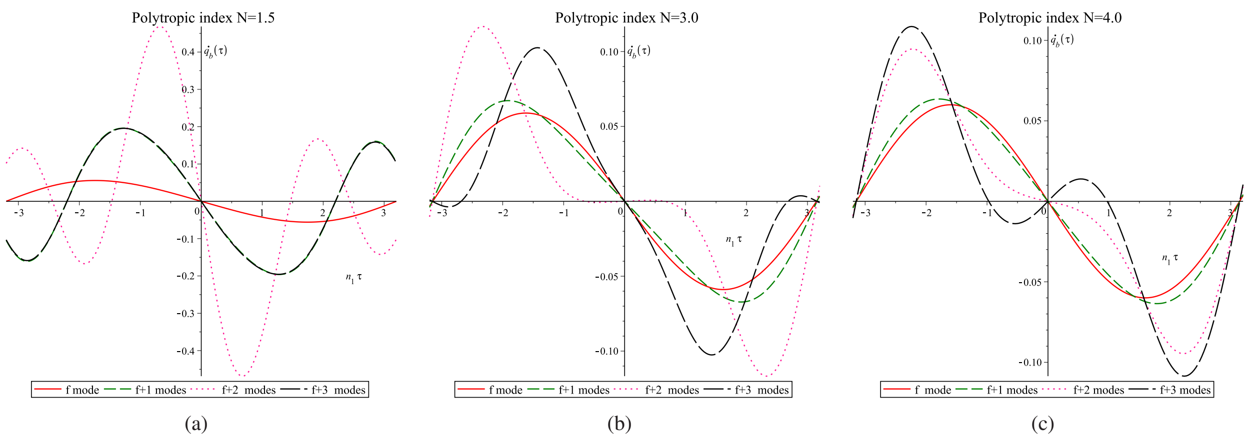


Fig. 5 Radial velocity curves of the rotationally and tidally distorted polytropic models ($n = 0.525$, $q = 0.05$).

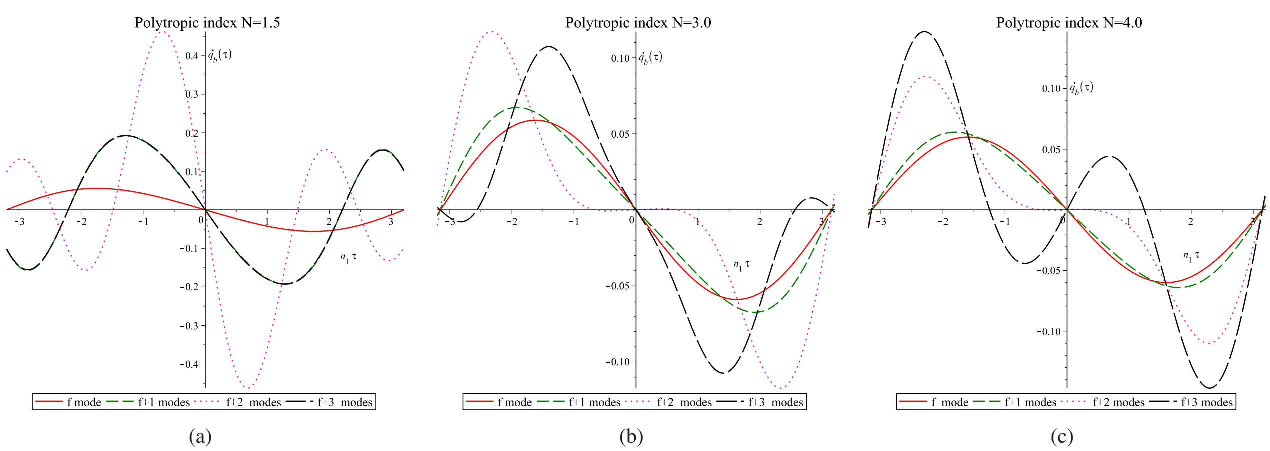
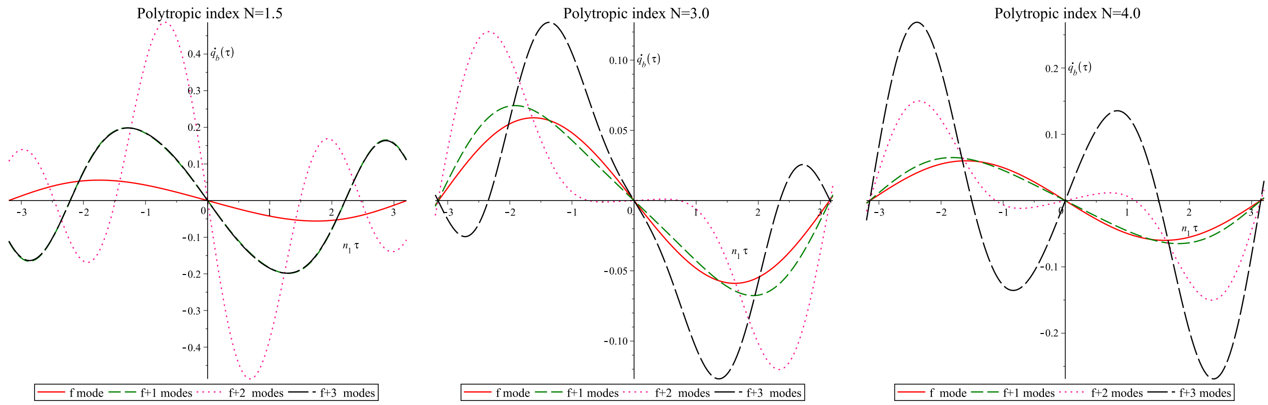
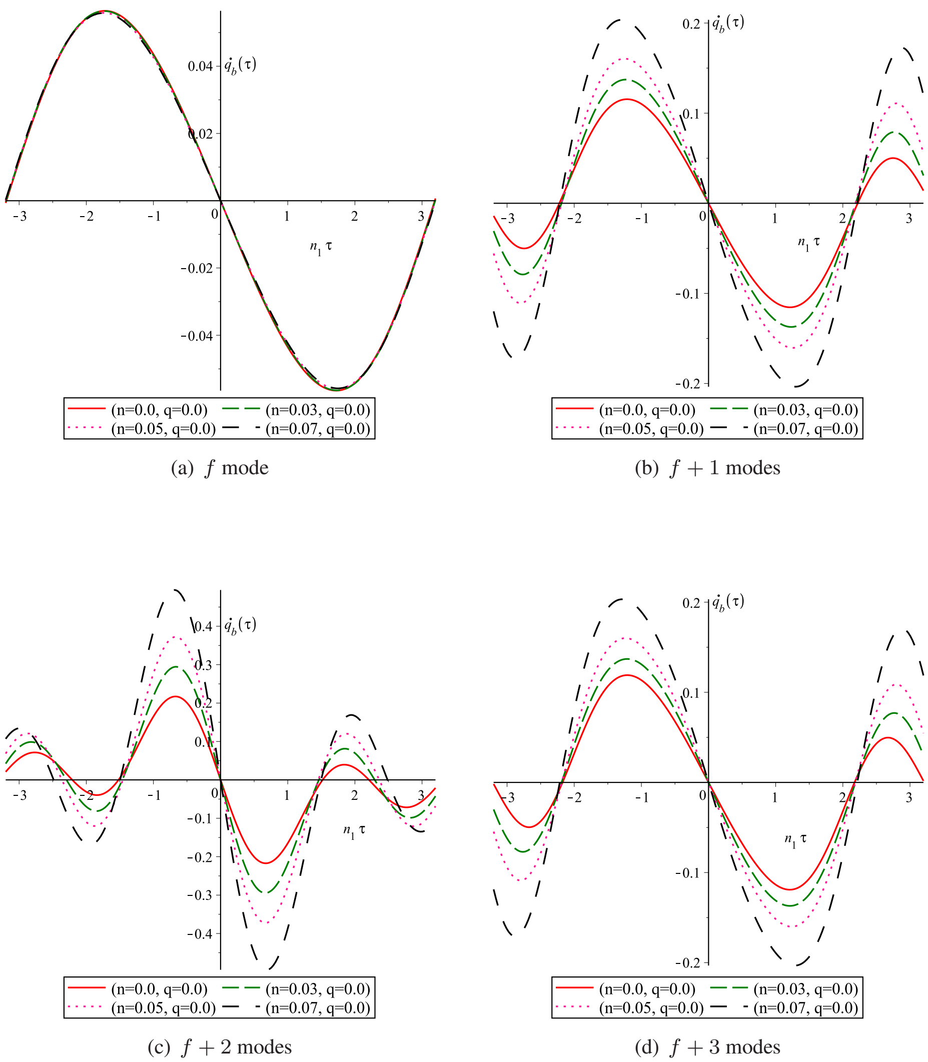


Fig. 6 Radial velocity curves of the rotationally and tidally distorted polytropic models ($n = 0.535$, $q = 0.07$).



(a) (b) (c)

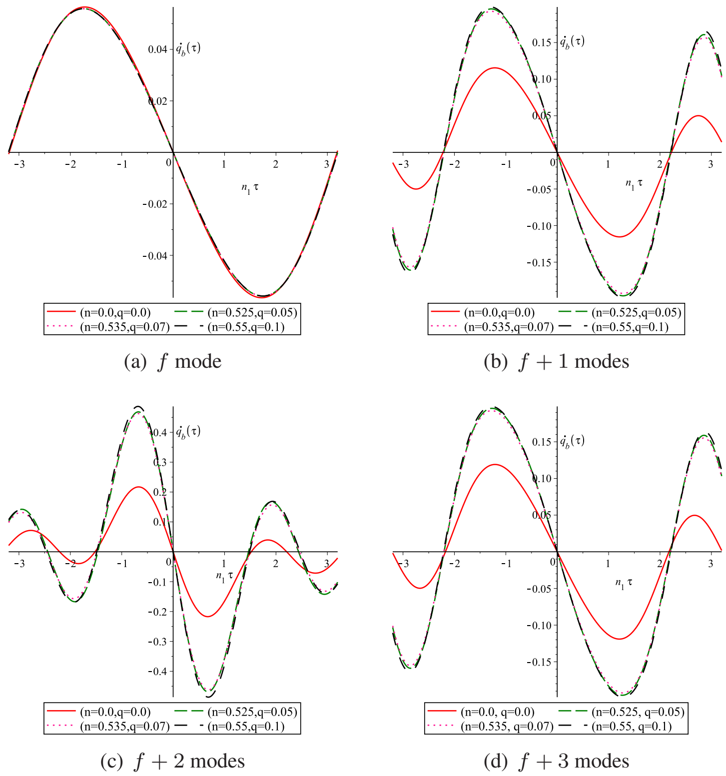
Fig. 7 Radial velocity curves of the rotationally and tidally distorted polytropic models ($n = 0.55$, $q = 0.1$).



(a) f mode (b) $f+1$ modes

(c) $f+2$ modes (d) $f+3$ modes

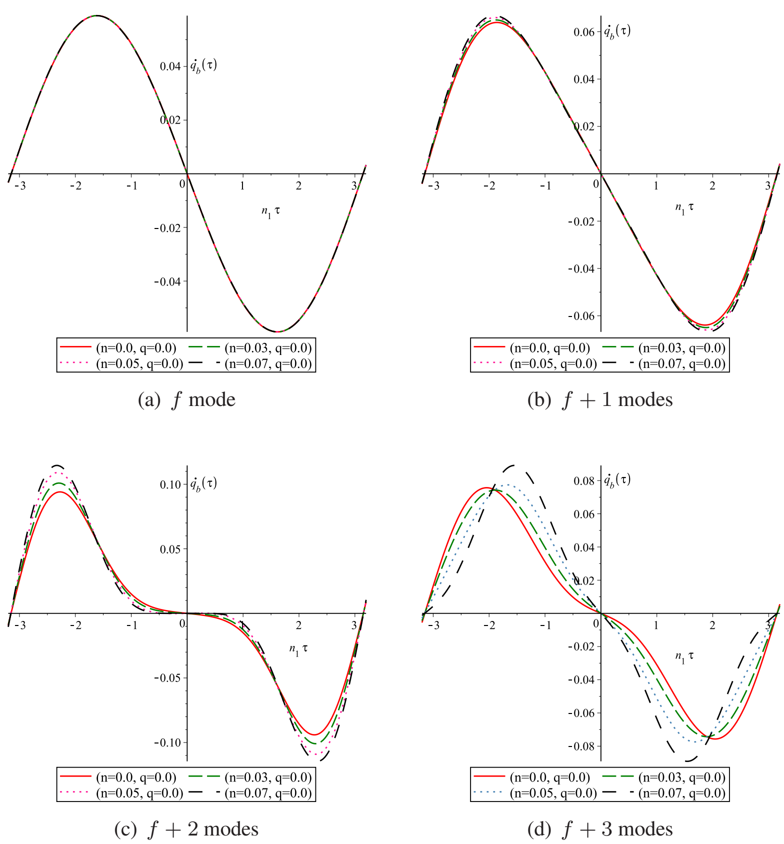
Fig. 8 Radial velocity curves of the rotationally distorted polytropic models (n varies, $q = 0$).



(a) f mode (b) $f+1$ modes

(c) $f+2$ modes (d) $f+3$ modes

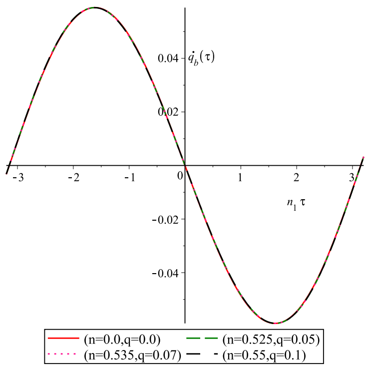
Fig. 9 Radial velocity curves of the rotationally and tidally distorted polytropic models with $N = 1.5$ (both n and q vary).



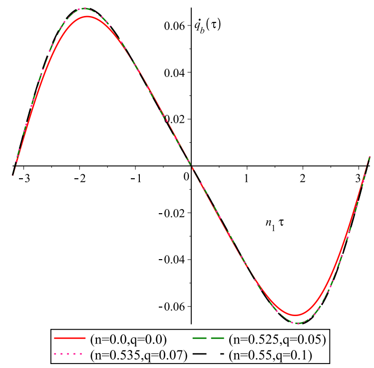
(a) f mode (b) $f+1$ modes

(c) $f+2$ modes (d) $f+3$ modes

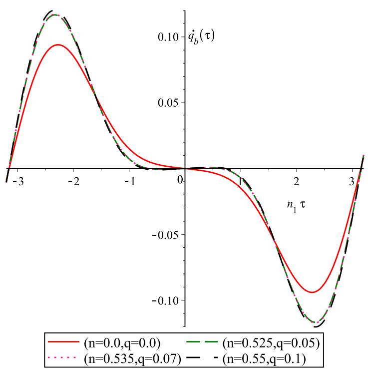
Fig. 10 Radial velocity curves of the rotationally distorted polytropic models with $N = 3.0$ (n varies and $q = 0$).



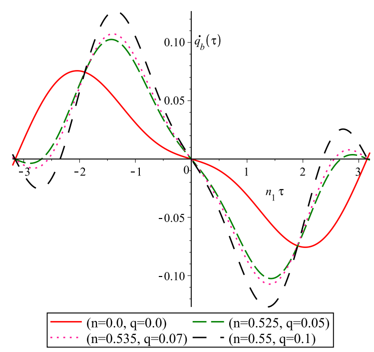
(a) f mode



(b) $f+1$ modes

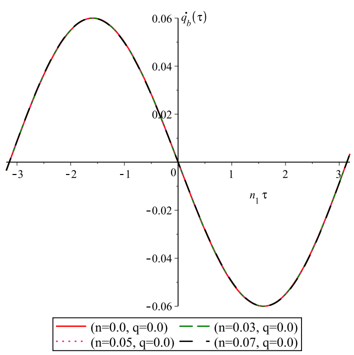


(c) $f+2$ modes

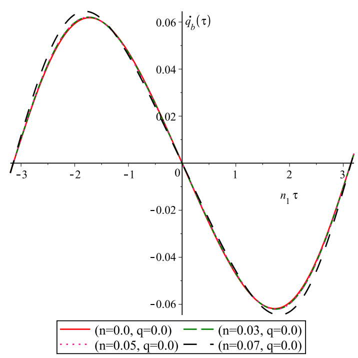


(d) $f+3$ modes

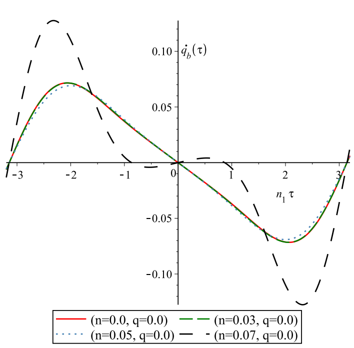
Fig. 11 Radial velocity curves of the rotationally and tidally distorted polytropic models with $N = 3.0$ (both n and q vary).



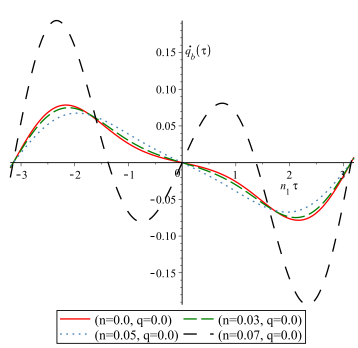
(a) f mode



(b) $f+1$ modes



(c) $f+2$ modes



(d) $f+3$ modes

Fig. 12 Radial velocity curves of the rotationally distorted polytropic models with $N = 4.0$ (n varies and $q = 0$).

Table 1 Various Parameters used in the Manuscript

S. No	Parameter	Definition
1	n	The rotation parameter that represents the distortions due to rotation ($2n = \Omega^2, \Omega$ is the normalized angular velocity of rotation of the star)
2	q	The tidal parameter that represents the distortions due to tidal effects ($q = \frac{m_2}{m_1}$, mass of the secondary component to the primary component in a binary system)
3	N	Polytropic index
4	$q_b(t)$	The displacement at the surface of a star at time t
5	τ	A variable for time ($\tau = \omega_1 t$)
6	ω_1	Eigenfrequency of the fundamental mode
7	η	Relative amplitude of the displacement of an element at a distance r from the center of the star
8	K	The skewness coefficient

Table 2 Eigenfrequencies of the Various Modes of the Polytropic Models for the Stars

	$N = 1.5$				$N = 3.0$				$N = 4.0$			
	ω_1^2	ω_2^2	ω_3^2	ω_4^2	ω_1^2	ω_2^2	ω_3^2	ω_4^2	ω_1^2	ω_2^2	ω_3^2	ω_4^2
Undistorted												
($n = 0, q = 0$)	2.705	12.533	26.533	44.509	9.254	16.983	28.554	44.749	15.149	24.834	35.477	48.859
Rotationally distorted												
($n = 0.03, q = 0.0$)	2.574	11.647	24.598	41.247	8.823	15.869	26.427	41.452	14.157	23.029	33.112	45.684
($n = 0.05, q = 0.0$)	2.486	11.042	23.297	39.138	8.519	15.099	24.598	38.495	13.463	21.827	31.618	43.568
($n = 0.07, q = 0.0$)	2.397	10.387	21.981	37.074	8.306	14.642	23.948	35.977	12.996	20.707	30.933	41.514
Rotationally and Tidally distorted												
($n = 0.525, q = 0.05$)	2.418	10.512	22.322	37.645	8.265	14.451	23.506	35.449	12.887	20.938	31.120	42.012
($n = 0.535, q = 0.07$)	2.409	10.499	22.144	37.337	8.241	14.391	23.352	35.110	12.835	20.850	31.058	41.701
($n = 0.55, q = 0.1$)	2.399	10.428	21.995	37.113	8.206	14.300	23.150	34.599	12.697	20.726	31.011	40.456

radial modes of the oscillations of the RTD polytropic models with indexes $N = 1.5$, 3.0 and 4.0. From this table, it is clear that the eigenfrequencies of the radial modes of the rotationally distorted and the rotationally and tidally distorted polytropic models are small as compared to the corresponding values of the undistorted polytropic model. Also, from this table it can be observed that with the increase in the rotational distortions (the value of n or angular velocity of the rotation of the star) and the rotational and tidal distortions (the values of both n and q where q represents tidal distortions due to the secondary component), the value of the eigenfrequencies decreases. These results are in accordance with previous results of Mohan & Saxena (1985).

From Figures 1–7, it can be observed that as we consider the interaction of the fundamental mode with the higher modes there is appreciable deviation in the shape of the radial velocity curve of the rotationally distorted and the rotationally and tidally distorted polytropic models of the pulsating variable stars. These deviations in the shapes of the radial velocity curves (as compared with the shapes of the radial velocity curves obtained for the f mode) are more in the case of the polytropes with index 1.5 than for the polytropes with indexes 3.0 and 4.0. However, for the polytropes with index 1.5 the $f + 3$ modes do not seem to change the radial velocity curve further, in fact the radial velocity curve coincides with the radial velocity curve of the $f + 1$ modes in all the considered cases.

From Figures 8–13, it can be observed that with an increase in the value of n (rotational distortions) and both n and q (rotational and tidal distortions) there is no change in the shape of the radial velocity curves (as compared to the shape of the radial velocity curve of the undistorted model) when only f mode is considered. However, considering the interaction of the f mode with the higher modes there is an appreciable change in the shape of the radial velocity curves for the various polytropic models when increasing the value of n and both n and q. Also, the radial velocity curve shows more de-

Table 3 Values of n_1^2 and the Skewness Coefficient K

Model	f mode		$f+1$ modes		$f+2$ modes		$f+3$ modes	
	n_1^2	K	n_1^2	K	n_1^2	K	n_1^2	K
Polytropic index ($N = 1.5$)								
($n = 0.0$, $q = 0.0$)	0.97029	0.4632	0.90616	0.6207	0.90616	0.7926	0.90616	0.6219
($n = 0.03$, $q = 0.0$)	0.96658	0.4588	0.88041	0.6152	0.88040	0.7920	0.88040	0.6213
($n = 0.05$, $q = 0.0$)	0.96434	0.4569	0.85341	0.6082	0.85341	0.7830	0.85341	0.6131
($n = 0.07$, $q = 0.0$)	0.96114	0.4541	0.80336	0.6082	0.80337	0.7821	0.80335	0.5983
($n = 0.525$, $q = 0.05$)	0.96243	0.4551	0.82527	0.5944	0.82170	0.7822	0.82526	0.6035
($n = 0.535$, $q = 0.07$)	0.96160	0.4544	0.81557	0.5975	0.81556	0.7813	0.81556	0.6013
($n = 0.55$, $q = 0.1$)	0.96123	0.4541	0.80885	0.5956	0.80884	0.7812	0.80882	0.5993
Polytropic index ($N = 3.0$)								
($n = 0.0$, $q = 0.0$)	0.99724	0.4965	0.99467	0.4181	0.99468	0.2891	0.99464	0.3615
($n = 0.03$, $q = 0.0$)	0.99682	0.4955	0.99349	0.4126	0.99353	0.2830	0.99349	0.4038
($n = 0.05$, $q = 0.0$)	0.99646	0.4947	0.99252	0.4074	0.99252	0.2770	0.99252	0.4754
($n = 0.07$, $q = 0.0$)	0.99649	0.4948	0.99235	0.4038	0.99235	0.2716	0.99235	0.5106
($n = 0.525$, $q = 0.05$)	0.99609	0.4939	0.99143	0.4026	0.99143	0.2718	0.99143	0.5541
($n = 0.535$, $q = 0.07$)	0.99605	0.4938	0.99133	0.4022	0.99133	0.2717	0.99133	0.5620
($n = 0.55$, $q = 0.1$)	0.99599	0.4937	0.99113	0.4021	0.99111	0.2698	0.99113	0.5735
Polytropic index ($N = 4.0$)								
($n = 0.0$, $q = 0.0$)	0.99939	0.5033	0.99855	0.4608	0.99855	0.3764	0.99855	0.3433
($n = 0.03$, $q = 0.0$)	0.99928	0.5028	0.99845	0.4567	0.99833	0.3764	0.99833	0.3388
($n = 0.05$, $q = 0.0$)	0.99919	0.5024	0.99819	0.4555	0.99819	0.3763	0.99819	0.3385
($n = 0.07$, $q = 0.0$)	0.99876	0.5003	0.99661	0.4443	0.99661	0.2760	0.99661	0.2701
($n = 0.525$, $q = 0.05$)	0.99891	0.5013	0.99723	0.4414	0.99723	0.3021	0.99722	0.3008
($n = 0.535$, $q = 0.07$)	0.99883	0.5010	0.99691	0.4377	0.99691	0.2882	0.99691	0.2818
($n = 0.55$, $q = 0.1$)	0.99872	0.5006	0.99642	0.4318	0.99642	0.2643	0.99642	0.2501

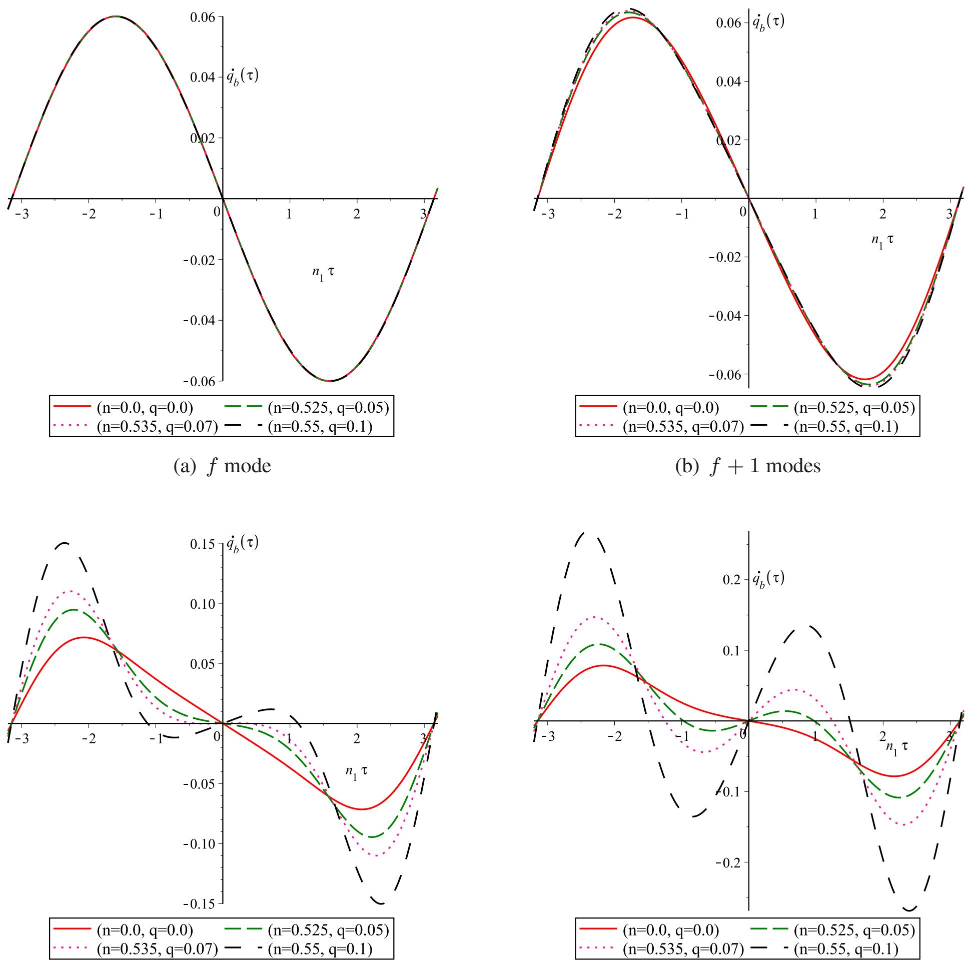
viation in the case of the rotational distortion (n) than the rotational and tidal distortion (both n and q). These results are in accordance with the results of our Paper 1 except for the case when we considered only the f mode.

From Table 3, it is clear that when only the f mode is considered and when the interaction of the f mode with higher modes is taken into account, in general, there is a slight decrease in the value of K with increasing value of n and both n and q. The value of K decreases more in the case of rotational distortions than rotational and tidal distortions. Again, these results are in accordance with the results of our Paper 1.

However, in Table 3, we also found some results that are contrary to the usual trend and also contrary to the trend we obtained in Paper 1. Firstly for polytropes of index $N = 3.0$, when $f + 3$ modes are considered then the value of K shows the opposite behavior. It tends to increase with an increase in the value of n and both n and q. Secondly, there is an appreciable decrease in the value of K with increase in the value of n and both n and q for polytropes of index $N = 4.0$ when higher modes $f + 2$ and $f + 3$ are considered. Whereas, in general, a very slight decrease is observed in the values of K with increasing values of n and both n and q in all other cases.

Again from Table 3, it can be observed that for polytropes of index $N = 1.5$ with an increase in the number of modes (up to $f + 2$ modes) the value of K increases and then it decreases for $f+3$ modes. However, for polytropes of index $N = 3.0$ and $N = 4.0$ with an increase in the number of modes the value of K decreases, except for polytropes with index $N = 3.0$ where, after $f + 2$ modes, the value starts increasing.

So, from the present study we can conclude that the interaction of the fundamental mode with the higher modes appreciably changes the shape of the radial velocity curves of the rotationally distorted and rotationally and tidally distorted polytropic models of the pulsating variable stars.



(a) f mode (b) $f+1$ modes (c) $f+2$ modes (d) $f+3$ modes

Fig. 13 Radial velocity curves of the rotationally and tidally distorted polytropic models with $N = 4.0$ (both n and q vary).

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