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The secular effect of gravitational radiation damping on the periastron advance of binary stars in second order perturbation theory

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Abstract The second order perturbation effect of gravitational radiation damping on the periastron advance of binary stars is studied. The second order analytic solution is obtained based on the first order theory in the 2014 article by Li. Theoretical results show that secular variation exists in the periastron advance of binary stars in the second order theory, but secular variation does not exist in the first order perturbation theory. Numerical results for two compact binary stars (PSR J0737–3039 and M33 X-7) are given, demonstrating the theoretical significance even though the effect is very small.

Key words: binaries: close — gravitational wave — celestial mechanics

1 INTRODUCTION

It is important and significant to study the secular effect of gravitational radiation damping on the time and space variation of the periastron of binary stars. The time variation refers to the time variation of periastron passage. The space variation of periastron refers to the advance of periastron. Lincoln & Will (1990) deduced from the equation for time variation of periastron passage that there is no secular variation of longitude of periastron ω, but only relativistic periastron advance in 5/2 post-Newtonian (PN) (gravitational-radiation emission) of first order perturbation theory. Li (2011) obtained the secular and periodic solutions of the time variation of periastron passage for binary stars. Moreover, Li (2009) studied gravitational radiation damping and the orbital evolution of compact binary pulsars by using the first perturbation method presented in Walker & Will (1979). In that paper, the author only obtained the periodic variation of longitude of periastron and not secular variation, that is, there is no periastron advance in first order perturbation theory for 5/2 PN. Li (2014) also studied gravitational radiation damping and the orbital evolution of compact binary stars by using the first perturbation method presented in Lincoln & Will (1990). However, in that paper the author obtained periodic variation and cases without secular variation of the longitude of periastron, that is, there is no periastron advance in first order perturbation theory for 5/2 PN.

It is necessary to examine the existence of secular variation of the orbit or periastron advance of binary stars in second order perturbation theory. The associated results demonstrate the existence of secular variation in the orbital elements of a binary star system in second order perturbation theory.

2 A METHOD FOR SOLVING THE PERTURBATION EQUATIONS IN HIGH ORDER PERTURBATION THEORY

It is necessary to research and explore how high order perturbation theory is related to the stability of a binary star system or the solar system. The most important consideration is the secular variation of orbital elements in high order perturbation theory. Some authors investigate this topic, such as the book Brouwer & Clemence (1961).

Perturbation equations describing time as an independent variable may be written as

$$\frac{d\sigma}{dt} = F(a, e, \ldots), \tag{1}$$

where σ denotes one of the orbital elements, a is the semi-major and e is the eccentricity.

Equation (1) can be transformed into an equation with true anomaly f as the independent variable from the case of time being an independent variable

$$\frac{d\sigma}{df} = F_0(a_0, e_0, \ldots). \quad (2)$$

The perturbation order of the orbital elements may be written as

$$\sigma = \sigma^{(0)} + \sigma^{(1)} + \sigma^{(2)} + \sigma^{(3)} + \ldots, \quad (3)$$

$$\therefore \frac{d\sigma}{df} = \frac{d\sigma^{(0)}}{df} + \frac{d\sigma^{(1)}}{df} + \frac{d\sigma^{(2)}}{df} + \frac{d\sigma^{(3)}}{df} + \ldots, \quad (4)$$

$$\begin{aligned} \frac{d\sigma}{df} &= F_0 + \frac{\partial F_0}{\partial a_0}(\delta a^{(1)} + \delta a^{(2)} + \ldots) \\ &+ \frac{\partial F_0}{\partial e_0}(\delta e^{(1)} + \delta e^{(2)} + \ldots) \\ &+ \ldots + \frac{1}{2}\frac{\partial^2 F_0}{\partial a_0^2}(\delta a^{(1)} + \delta a^{(2)} + \ldots)^2. \end{aligned} \quad (5)$$

Comparing Equation (4) with Equation (5), we find that

$$\frac{d\sigma^{(0)}}{dt} = 0, \quad (6)$$

$$\frac{d\sigma^{(1)}}{dt} = F_0(a_0, e_0, \ldots), \quad (7)$$

$$\frac{d\sigma^{(2)}}{dt} = \frac{\partial F_0}{\partial a_0}\delta a^{(1)} + \frac{\partial F_0}{\partial e_0}\delta e^{(1)} + \ldots, \quad (8)$$

$$\begin{aligned} \frac{d\sigma^{(3)}}{dt} &= \frac{\partial F_0}{\partial a_0}\delta a^{(2)} + \frac{\partial F_0}{\partial e_0}\delta e^{(2)} \\ &+ \ldots + \frac{1}{2}\frac{\partial^2 F_0}{\partial a_0^2}\delta a^{(2)} + \frac{1}{2}\frac{\partial^2 F_0}{\partial e_0^2} + \cdots. \end{aligned} \quad (9)$$

The purpose of this paper is to investigate the secular effect of second order perturbation on the longitude of periastron, i.e. letting $\sigma = \omega$

$$\frac{d\omega^{(1)}}{df} = F_0\{a_0, e_0, \ldots\}, \quad (10)$$

$$\frac{d\omega^{(2)}}{df} = \frac{\partial F_0}{\partial a_0}\delta a^{(1)} + \frac{\partial F_0}{\partial e_0}\delta e^{(1)} + \ldots, \quad (11)$$

$$\frac{d\varpi^{(2)}}{df} = \frac{d\omega^{(2)}}{df} + \frac{d\Omega^{(2)}}{df}. \quad (12)$$

Integrating Equation (11), we yield

$$\Delta\omega^{(2)} = \int_{f_0}^{f} \left(\frac{d\omega^{(2)}}{df}\right) df, \quad \varpi = \omega + \Omega, \quad (13)$$

where ω is the argument of periastron and ϖ is the longitude of periastron.

3 RESULTS FOR THE FIRST ORDER PERTURBATION EFFECT OF GRAVITATIONAL RADIATION DAMPING ON THE ORBIT OF BINARY STARS

Results for the first order perturbation effects have been given by Li (2014) and related results are listed as follows.

The formula for relative acceleration with 5/2 PN was provided by Lincoln & Will (1990)

$$\boldsymbol{a} = (m/r^2)\Big[(-1 + A)\boldsymbol{n} + B\boldsymbol{V}\Big]. \quad (14)$$

For gravitational emission, we take

$$\boldsymbol{a}_{5/2} = (m/r^2)\Big[(-1 + A_{5/2})\bar{n} + B_{5/2}\boldsymbol{V}\Big], \quad (15)$$

where $m = m_1 + m_2$ and r denotes the distance between the two binary stars. $\boldsymbol{n}$ and $\boldsymbol{V}$ denote the unit vectors of the radial direction and the relative velocity vector respectively.

Resolving the perturbation acceleration $\boldsymbol{a}$ into the radial component $R_{5/2}$, the transverse component $S_{5/2}$ perpendicular to $R_{5/2}$ in the orbital plane and the component $W_{5/2}$ normal to the orbital plane, we obtain (Li 2014)

$$\begin{aligned} R_{5/2} &= \frac{8}{15}\eta\left(\frac{m}{r}\right)^3\left(\frac{m}{p}\right)^{\frac{1}{2}} p^{-1} \\ &\times e\sin f\left(14 + 6e^2 + 20e\cos f\right), \end{aligned} \quad (16)$$

$$S_{5/2} = -\frac{8}{15}\eta\frac{m^3}{r^4}\left(\frac{m}{p}\right)^{\frac{1}{2}}(12 + 3e^2 + 15e\cos f), \quad (17)$$

$$W_{5/2} = 0. \quad (18)$$

Here $\eta = \mu/m$, $\mu = \frac{m_1 m_2}{m}$ and $G = c = 1$. These parameters have been defined in Lincoln & Will (1990). $p = a(1 - e^2)$, f denotes the true anomaly, and a and e denote the semi-major axis and eccentricity respectively.

We substitute $R_{5/2}$, $S_{5/2}$ and $W_{5/2}$ from Equations (16)–(18) into Gaussian equations (16)–(22) in Li (2014), which were derived from Lincoln & Will (1990) and Brouwer & Clemence (1961). We also change independent variable time t to an independent true anomaly f to transform the Gaussian equations by using $r^2\frac{df}{dt} = (mp)^{1/2}$. Then, by integrating Gaussian equations (16)–(22), we obtain the results for the first order perturbation effects in 5/2 PN as follows (Li 2014).

$$\delta a^{(1)} = A_0(f - f_0) + \sum_{i=1}^{3} A_i(\sin if - \sin if_0), \quad (19)$$

$$\delta e^{(1)} = E_0(f - f_0) + \sum_{i=1}^{3} E_i(\sin if - \sin if_0), \quad (20)$$

$$\delta\omega^{(1)} = \sum_{i=1}^{4} W_i(\cos if - \cos if_0), \quad (21)$$

$$\delta\varepsilon_0^{(1)} = \sum_{i=1}^{4} H_i(\cos if - \cos if_0), \quad (22)$$

$$\delta\lambda^{(1)} = n(t - t_0) + \delta\varepsilon_0^{(1)}, \quad (23)$$

$$\delta i^{(1)} = \delta\Omega^{(1)} = 0. \quad (24)$$

The coefficients of the secular terms are

$$A_0 = -\frac{8}{15}\eta m^{5/2} p_0^{-3/2}(1 - e_0^2)^{-2}(24 + 73e_0^2), \quad (25)$$

$$E_0 = -\frac{8}{15}\eta\left(\frac{m}{p_0}\right)^{5/2}\left(38e_0 + \frac{121}{8}e_0^2\right). \quad (26)$$

The amplitudes of the periodic terms are

$$\left.\begin{aligned} A_1 &= -\tfrac{8}{15}\eta m^{5/2} p_0^{-3/2}(1 - e_0^2)^{-2}(102e_0 + 18e_0^2),\\ A_2 &= -\tfrac{76}{3}\eta m^{5/2} p_0^{-3/2}\ e_0^2(1 - e_0)^{-2},\\ A_3 &= -\tfrac{8}{15}\eta m^{5/2} p_0^{-3/2}\ \tfrac{16}{3}e_0^2(1 - e_0)^{-2},\\ A_4 &= 0. \end{aligned}\right\} \quad (27)$$

$$\left.\begin{aligned} E_1 &= -\tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}(24e_0 + \tfrac{91}{2}e_0^2),\\ E_2 &= -\tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}(7e_0 + 12e_0^2),\\ E_3 &= -\tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}\tfrac{65}{12}e_0^2,\\ E_4 &= 0. \end{aligned}\right\} \quad (28)$$

$$\left.\begin{aligned} W_1 &= \tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}(24 + \tfrac{81}{4}e_0^2 + \tfrac{17}{2}e_0^3)/e_0,\\ W_2 &= \tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}\ \tfrac{1}{2}(33 + 7e_0 + \tfrac{33}{4}e^2),\\ W_3 &= \tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}\ \tfrac{1}{3}\left(\tfrac{57}{4}e_0 + \tfrac{17}{2}\ e_0^3\right),\\ W_4 &= \tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}\ \tfrac{15}{32}\ e_0^2. \end{aligned}\right\} \quad (29)$$

$$\left.\begin{aligned} H_1 &= \tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}\\ &\quad\left[\tfrac{24e_0 + \frac{81}{4}e_0^3}{1 - e_0^2 + \sqrt{1 - e_0^2}} + (1 - e_0^2)^{1/2}(28e_0 + 12e_0^3)\right],\\ H_2 &= \tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}\\ &\quad\left[\tfrac{33e_0^2 + 7e_0^3}{2[(i - e_0^2) + \sqrt{1 - e_0^2}]} + 10e_0^2(1 - e_0^2)^{1/2}\right].\\ H_3 &= \tfrac{8}{15}\eta(\tfrac{m}{p_0})^{5/2}\tfrac{1}{6}\left(\tfrac{\frac{57}{2}e_0^3 + 17e_0^4}{1 - e_0^2 + \sqrt{1 - e_0^2}}\right)\\ H_4 &= \tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}\left(\tfrac{15}{32}\right)\tfrac{e_0^4}{(1 - e_0^2 + \sqrt{1 - e_0^2})} \end{aligned}\right\} \quad (30)$$

where $p_0 = a_0(1 - e_0^2)$.

Expressions (19)–(30) indicate that there are both secular and periodic variations for the semi-major axis and the eccentricity, but there is only periodic variation for the argument of periastron ω in first order perturbation theory; that is, there is neither secular variation of the longitude of periastron, ϖ nor periastron advance for 5/2 PN in first order perturbation theory.

4 SECULAR SOLUTION FOR LONGITUDE OF PERIASTRON IN SECOND ORDER THEORY

Because there is no secular variation (shift) for the longitude of periastron in the first order theory, it is necessary to examine the secular variation (shift) for the longitude of periastron in second order theory.

First, one must write down the first order Gaussian equations for the longitude of argument ω with the anomaly as an independent variable by using Gaussian equations (Lincoln & Will 1990)

$$e\frac{d\omega}{dt}^{(1)} = (p/m)^{1/2}\Big\{-R_{5/2}\cos f + \left(1 + \frac{r}{p}\right) \sin f S_{5/2}\Big\} + e(r/p)\cos i \sin f W_{5/2}. \quad (31)$$

By substituting Equations (16)–(18) for $R_{5/2}$, $S_{5/2}$ and $W_{5/2}$ respectively into the above Gaussian equation and using $\frac{df}{dt} = (mp)^{1/2}/r^2$, the first order Gaussian equation with true anomaly as an independent variable for ω has been derived by Li (2014)

$$\begin{aligned}\frac{d\omega^{(1)}}{df} &= \frac{d\omega^{(1)}}{dt}\frac{dt}{df}\\ &= -\frac{8}{15}\eta(m/p)^{5/2}\frac{1}{e}\Big\{\left(24 + \frac{81}{4}e^2 + \frac{17}{2}e^3\right)\\ &\quad\times\sin f + e\left(33 + 7e + \frac{33}{4}e^2\right)\sin 2f\\ &\quad+e\left(\frac{57}{4}e + \frac{17}{2}e^2\right)\sin 3f\\ &\quad+\frac{15}{8}e^3\sin 4f\Big\}. \end{aligned} \quad (32)$$

For brevity, we write down the first order equation above in the following form ($\sum$ form)

$$\begin{aligned}\frac{d\omega^{(1)}}{df} &= \sum_{i=1}^{3} R_i \sin if\\ &= F_0(a_0, e_0, \omega_0, \ldots, \varepsilon_0, f), \end{aligned} \quad (33)$$

where

$$\left.\begin{aligned} R_1 &= -\tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}(24+\tfrac{81}{4}e_0^2+\tfrac{17}{2}e_0^3)/e_0,\\ R_2 &= -\tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}(33+7e_0+\tfrac{33}{4}e_0^2),\\ R_3 &= -\tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}(\tfrac{57}{4}e_0+\tfrac{17}{2}e_0^2),\\ R_4 &= -\tfrac{8}{15}\eta\left(\tfrac{m}{p_0}\right)^{5/2}\tfrac{15}{8}e_0^2 = -\eta\left(\tfrac{m}{p}\right)^{5/2}e_0^2. \end{aligned}\right\} \tag{34}$$

The second order equation for the longitude of periastron can be written according to Equation (7).

$$\begin{aligned}\frac{d\omega^{(2)}}{df} &= \frac{\partial F_0}{\partial a_0}\delta a^{(1)} + \frac{\partial F_0}{\partial e_0}\delta e^{(1)} + \frac{\partial F_0}{\partial i_0}\delta i^{(1)}\\ &+\frac{\partial F_0}{\partial \omega_0}\delta\omega^{(1)} + \frac{\partial F_0}{\partial \Omega_0}\delta\Omega^{(1)}\\ &+\frac{\partial F_0}{\partial \varepsilon_0}\delta\varepsilon^{(1)}. \end{aligned}\tag{35}$$

$$\frac{\partial F_0}{\partial a_0} = \sum_{i=1}^{3} Q_i \sin if, \tag{36}$$

where

$$\left.\begin{aligned} Q_1 &= \tfrac{8}{15}\eta m^{5/2}p_0^{-7/2}\tfrac{5}{2}(1-e_0^2)(24+\tfrac{81}{4}e_0^2+\tfrac{17}{2}e_0^3)/e^2,\\ Q_2 &= \tfrac{8}{15}\eta m^{5/2}p_0^{-7/2}\tfrac{5}{2}(1-e_0^2)\left(33+7e_0+\tfrac{33}{4}e_0^2\right),\\ Q_3 &= \tfrac{8}{15}\eta m^{5/2}p_0^{-7/2}(1-e_0^2)\left(\tfrac{51}{4}e_0+\tfrac{17}{2}e_0^2\right),\\ Q_4 &= \eta m^{5/2}p_0^{-7/2}(1-e_0^2)e_0^2. \end{aligned}\right\} \tag{37}$$

$$\frac{\partial F_0}{\partial e_0} = \sum_{i=1}^{3} N_i \sin if. \tag{38}$$

Here

$$\left\{\begin{aligned} N_1 &= -\tfrac{8}{15}\eta(\tfrac{m}{p_0})^{5/2}\left[\tfrac{299}{4}+33e_0+\tfrac{365}{4}e_0^2+O(e^3)\right],\\ N_2 &= -\tfrac{8}{15}\eta(\tfrac{m}{p_0})^{5/2}\left[7+\tfrac{363}{2}e_0+35e_0^2+O(e^3)\right],\\ N_3 &= -\tfrac{8}{15}\eta(\tfrac{m}{p_0})^{5/2}\left[\tfrac{57}{4}+17e_0+\tfrac{285}{4}e_0^2+O(e^3)\right],\\ N_4 &= -7\eta(\tfrac{m}{p_0})^{5/2}e_0, \end{aligned}\right. \tag{39}$$

where $p_0 = a_0(1-e_0^2)$.

$$\begin{aligned}\frac{\partial F_0}{\partial \omega_0} &= \frac{\partial F_0}{\partial i_0} = \frac{\partial F_0}{\partial \Omega_0} = \frac{\partial F_0}{\partial \varepsilon_0} = 0,\\ &(\delta\omega = \delta i = \delta\Omega = \delta\varepsilon_0 = 0). \end{aligned}\tag{40}$$

The expressions in Equation (19) with (36) yield

$$\begin{aligned}\frac{\partial F_0}{\partial a_0}\delta a^{(1)} &= \sum_{i=1}^{4} Q_i \sin if\Big\{A_0(f-f_0)\\ &+\sum_{i=1}^{4} A_i(\sin if - \sin if_0)\Big\}, \end{aligned}\tag{41}$$

$$\sum_{i=1}^{4} Q_iA_i(f-f_0)\sin if =$$

$$\sum_{i=1}^{4} Q_iA_i f\sin if - \sum_{i01}^{4} Q_iA_i f_0 \sin if.$$

Here

$$\sum_{i=1}^{4} Q_iA_i f_0 \sin if = \text{Periodic terms},$$

$$\sum_{i=1}^{4} Q_iA_i \sin if_0 \sin if = \text{Periodic terms},$$

$$\begin{aligned}\sum_{i=1}^{4} Q_iA_i \sin^2 if &= \frac{1}{2}\sum_{i=1}^{4} Q_iA_i(1-\cos 2f)\\ &= \frac{1}{2}\sum_{i=1}^{4} Q_iA_i - \frac{1}{2}\sum_{i=1}^{4} Q_iA_i\cos 2if. \end{aligned}\tag{42}$$

We take the secular term

$$\frac{1}{2}\sum_{i=1}^{4} Q_iA_i$$

and the term

$$\sum_{i=1}^{4} A_iQ_i f\sin if,$$

$$\frac{\partial F_0}{\partial a}\delta a^{(1)} = \frac{1}{2}\sum_{i=1}^{4} A_iQ_i + \sum_{i=1}^{4} A_iQ_i f\sin if. \tag{43}$$

Similarly the expressions in Equation (20) with Equation (38) give

$$\begin{aligned}\frac{\partial F_0}{\partial e_0}\delta e^{(1)} &= \sum_{i=1}^{4} N\sin if\\ &\Big[E_0(f-f_0) + \sum_{i=1}^{4} E_i(\sin if - \sin f_0)\Big], \end{aligned}$$

$$\frac{\partial F_0}{\partial e_0}\delta e^{(1)} = \frac{1}{2}\sum_{i=1}^{i} N_iE_i + \sum_{i=11}^{4} N_iE_i f\sin if. \tag{44}$$

In addition, Equations (21)–(24) with Equation (40) generate

$$\frac{\partial F_0}{\partial \omega_0}\delta\omega^{(1)} = \frac{\partial F_0}{\partial i_0}\delta i^{(1)} = \frac{\partial F_0}{\partial \Omega_0}\delta\Omega^{(1)} = \frac{\partial F_0}{\partial \varepsilon_0}\delta\varepsilon^{(1)} = 0. \quad (45)$$

Substituting Equations (43), (44) and (45) into Equation (35), one obtains

$$\frac{d\omega^{(2)}}{df} = \frac{1}{2}\Big(\sum_{i=1}^{4} A_iQ_i + \sum_{i=1}^{4} N_iE_i\Big) + \Big(\sum_{i=1}^{4} A_iQ_i f \sin if + \sum_{i=1}^{4} N_iE_i f \sin if\Big). \quad (46)$$

Integrating the above equation from 0 to 2π

$$\delta\omega^{(2)} = \frac{1}{2}\Big(\sum_{i=1}^{4} A_iQ_i + \sum_{i=1}^{4} N_iE_i\Big)\int_0^{2\pi} df + \Big(\int_0^{2\pi}\sum_{i=1}^{4} A_iQ_i f \sin f df + \int_0^{2\pi}\sum_{i=1}^{4} N_iE_i f \sin f df\Big). \quad (47)$$

We use the formula for integration

$$\int x \sin x dx = \sin x - x\cos x.$$

The integration

$$\int_0^{2\pi}\sum_{i=1}^{4} A_iQ_i f \sin if df = \sum_{i=1}^{4} AQ\frac{1}{i^2}\Big[\sin if - if\cos if\Big]_0^{2\pi} = -2\pi\sum_{i=1}^{4}\frac{A_iQ_i}{i}. \quad (48)$$

Similarly

$$\int_0^{2\pi}\sum_{1=1}^{4} N_iE_i f \sin i = -2\pi\sum_{i=1}^{4}\frac{N_iE_i}{i}. \quad (49)$$

The result of the integration (47) is

$$\delta\omega^{(2)} = \pi\Big(\sum_{i=1}^{4} A_iQ_i + \sum_{i=1}^{4} N_iE_i\Big) - 2\pi\Big(\sum_{i=1}^{4}\frac{A_i}{Q_i}i + \sum_{i=1}^{4}\frac{N_iE_i}{i}\Big),$$

or

$$\delta\omega^{(2)} = \Big[\pi\sum_{i=1}^{4} A_iQ_i\Big(1-\frac{2}{i}\Big) + \pi\sum_{i=1}^{4} N_iE_i \Big(1-\frac{2}{i}\Big)\Big]\,(\mathrm{rad\,cycle^{-1}}), \quad (50)$$

$$\dot\omega^{(2)} = \Big[\pi\sum_{i=1}^{4} A_iQ_i\Big(1-\frac{2}{i}\Big)/P + \pi\sum_{i=1}^{4} N_iE_i (1-\frac{2}{i})/P\Big](\mathrm{rad\,yr^{-1}}), \quad (51)$$

where P is the orbital period.

$$\delta\varpi = \delta\omega + \delta\Omega = \delta\omega, \dot\varpi = \dot\omega + \dot\Omega = \dot\omega. \quad \therefore \delta\Omega = \dot\Omega = 0, \quad (52)$$

where ϖ and ω are the longitudes of periastron and argument respectively.

5 NUMERICAL RESULTS FOR THE SECULAR EFFECT OF GRAVITATIONAL RADIATION DAMPING ON PERIASTRON ADVANCE IN SECOND ORDER PERTURBATION THEORY

In this paper we choose two compact binary star systems. One is PSR J0737–3039 and the other is black hole binary star M33 X-7. Their data are listed in Table 1.

The right hand side of Equations (16) – (18) needs to be multiplied by c^5 (light speed) and m should be multiplied by G (Gravitational constant).

$$\begin{aligned} \eta m^{5/2} &= (\tfrac{m_1m_2}{m^2})m^{5/2} = m_1m_2. \\ m^{1/2} &= \tfrac{G^{5/2}}{c^5}m_1m_2(m_1+m_2)^{1/2} \\ &= 2.2835\times10^{13}(\text{For PSR J0737}-3039) \\ &= 8.5883\times10^{16}(\text{For M33 X}-7). \end{aligned} \quad (53)$$

$$\begin{aligned} p_0 &= a_0(1-e_0^2) \\ &= 8.7019\times10^{10}(\text{For PSR J0737}-3039) \\ &= 2.9405\times10^{12}(\text{For M33 X}-7). \end{aligned} \quad (54)$$

Table 1 Table for data on PSR J0737–3039 and M33 X-7

Pulsar	P (d)	A ($R_\odot$)	M_1 ($M_\odot$)	M_2 ($M_\odot$)	e	Reference
PSR J0737–3039	0.10225	1.26	1.34	1.25	0.0878	Willems et al. 2004, Burgay et al. 2003
M33 X–7	3.4500	42.4	15.65	70.00	0.0185	Orosz et al. 2007

Table 2 Numerical values for the amplitudes of periodic terms for PSR J0373–3039

Amplitude	$A_i \times 10^{-4}$	$E_i \times 10^{-15}$	$Q_i \times 10^{-23}$	$N_i \times 10^{-13}$
$i = 1$	−23.8290	−1.3401	+47.5940	−9.1810
$i = 2$	−1.7468	−0.3855	+3.4493	−1.2651
$i = 3$	−0.1986	−0.0227	+0.0485	−0.8924
$i = 4$	0	0	+0.0003	−0.0628

Table 3 Numerical Values for the Amplitudes of Periodic Terms for M33 X-7

Amplitude	$A_i \times 10^{-1}$	$E_i \times 10^{-15}$	$Q_i \times 10^{-27}$	$N_i \times 10^{-15}$
$i = 1$	–1.7178	–1.4119	+73621	–231.7082
$i = 2$	–0.0146	–4.1082	+33.1253	–31.8699
$i = 3$	–0.0002	–0.0056	+4.9418	–44.4767
$i = 4$	0	0	+0.0003	–0.7462

Here m_1, m_2 and a_0 are expressed in the units of solar mass $M(M_\odot)$, $M_\odot = 1.989 \times 10^{33}$ g, and solar radius $a(R_\odot)$. $R_\odot = 6.9599 \times 10^{10}$ cm, $G = 6.67 \times 10^{-8}$ (cgs) and $c = 2.9979 \times 10^{10}\,\mathrm{cm\,s^{-1}}$.

Substituting data for the values of Equations (53)–(54) and M_1 ($M_\odot$), M_2 ($M_\odot$) and e_0 into Equations (27), (28), (37) and (39), we obtain Tables 2 and 3.

Expanding the terms in Equations (50) and (51) for

$$\sum_{i=1}^{4} A_i Q_i \Big(1 - \frac{2}{i}\Big)$$

and

$$\sum_{i=1}^{4} N_i E_i \Big(1 - \frac{2}{i}\Big)$$

yields

$$\begin{aligned}\sum_{i=1}^{4} A_i Q_i (1 - \tfrac{2}{i}) &= A_1 Q_1 (1 - \tfrac{2}{1}) + A_2 Q_2 (1 - \tfrac{2}{2}) \\ &\quad + A_3 Q_3 (1 - \tfrac{2}{3}) + A_4 Q_4 (1 - \tfrac{2}{4}) \\ &= \sum_{i=1}^{4} A_i Q_i (1 - \tfrac{2}{i}) \\ &= -A_1 Q_1 + \tfrac{1}{3} A_3 Q_3 + \tfrac{1}{2} A_4 Q_4. \end{aligned} \quad (55)$$

The next expression is similar to the above expression

$$\sum_{i-1}^{4} N_i E_i (1 - \frac{2}{i}) = -N_1 E_1 + \frac{1}{3} N_3 E_3 + \frac{1}{2} N_4 E_4. \quad (56)$$

Substituting Equations (55) and (56) into Equations (50) and (51), we obtain that

$$\begin{aligned}\delta\varpi^{(2)} &= \Big[\pi(-A_1 Q_1 + \frac{1}{3} A_3 Q_3 + \frac{1}{2} A_4 Q_4) + \pi(-N_1 E_1 \\ &\quad + \frac{1}{3} N_3 E_3 + \frac{1}{2} N_4 E_4)\Big] (\mathrm{rad\ cycle^{-1}}). \end{aligned} \quad (57)$$

$$\begin{aligned}\dot{\varpi}^{(2)} &= \Big[\pi(-A_1 Q_1 + \frac{1}{3} A_3 Q_3 + \frac{1}{2} A_4 Q_4) \\ &\quad + \pi(-N_1 E_1 + \frac{1}{3} N_3 E_3 + \frac{1}{2} N_4 E_4)\Big] \\ &\quad / P (\mathrm{rad\ yr^{-1}}). \end{aligned} \quad (58)$$

We can substitute the values of A_1, A_3, A_4, Q_1, Q_3, Q_4, E_1, E_3, E_4, N_1, N_3 and N_4 into Table 2, Table 3 and P (d) in Table 1.

Then, we can apply these results for PSR J0737–3039 and M33 X–7 in Equations (57) and (58) to obtain the second order secular solutions for periastron advance of pulsars or black holes that are part of binary systems as shown in Table 4.

6 DISCUSSION

(1) It is important to study the secular advance of periastron in binary star systems, because the period of

Table 4 The second order secular solutions for the periastron advance of PSR J0737–3–039 and M33 X–7 compared with the first order solution of the previous paper (Li 2014).

Binary Star	$\delta\varpi^{(1)}$ (rad cycle^{-1})	$\dot{\varpi}^{(1)}$ (rad yr^{-1})	$\delta\varpi^{(2)}$ (rad cycle $^{-1}$)	$\dot{\varpi}^{(2)}$ (rad yr^{-1})
PSR J0737–3039	0	0	6.55×10^{-24}	2.34×10^{-20}
M33 X–7	0	0	$3.97{\times}10^{-23}$	$4.21{\times}10^{-21}$

Notes: The first order values of $\delta\varpi^{(1)}$ and $\dot{\varpi}^{(1)}$ have been given in a previous article (Li 2014).

advance of the apsidal line is determined by the secular advance of periastron in binary stars. As is well known, there is not a secular advance of periastron in first order perturbation theory (Li 2009, 2014). It is necessary for us to study and explore whether or not there is a secular advance of periastron in second order perturbation theory; the work highlighted in this paper is such an attempt. We obtain the associated theoretical and numerical results. According to Burgay et al. (2005), pulsars A and B in the system PSR J0737–3039 will coalesce due to the emission of gravitational waves in a merger time of 85 Myr.

In Table 4 for PSR J0737–3039, $2.34 \times 10^{-20}(\mathrm{rad\,yr^{-1}}) \times 85 \times 10^6\mathrm{yr} = 1.99 \times 10^{-12}\,\mathrm{rad}$ is increased within the time 85 Myr. This effect is so small that we cannot measure it. Even though this effect is very small, it still has theoretical significance, especially as this paper demonstrates that there is secular advance of periastron in second order perturbation theory.

(2) The result obtained by Lincoln & Will (1990) for the gravitational radiation reaction in the first order perturbation theory was used to derive equation 2.11b for 1 PN (relativistic term), 2 PN and 5/2 PN for ω. We write the differential equation (2.11b) for ω in the following form.

$$\begin{aligned} e\frac{d\omega}{dt} &= \frac{m}{p}\left[3e + \sum_{i=1}^{3} A_i \cos if\right] + \left(\frac{m}{p}\right)^2 \\ &\quad \left[\left[e(7+5\eta-7\eta^2) - \frac{1}{8}e^3(2-2\eta+48\eta^2)\right]\right. \\ &\quad \left.+ \sum_{i-1}^{5} B_i \cos if\right] + \left(\frac{m}{p}\right)^{5/2} \\ &\quad \left(\sum_{i=1}^{4} C_i \sin if\right). \end{aligned}$$

They integrate the equation above and take the secular terms, which yield

$$\begin{aligned} \Delta\omega &= O\Big(\frac{m}{p}\Big) + O\Big(\frac{m}{p}\Big)^2 + O\Big(\frac{m}{p}\Big)^{5/2} \\ &= 6m/p + O\left(\frac{m}{p}\right)^2 . \end{aligned}$$

Here the first term is the relativistic periastron advance (secular terms + periodic terms); the second term is the 2 PN periastron advance (secular terms + periodic terms); the third term is the 5/2 PN (gravitational radiation reaction), and

$$O\Big(\frac{m}{p}\Big)^{5/2} = \Big(\frac{m}{p}\Big)^{5/2}\Big(\sum_{i=1}^{4}\frac{C_i}{i}\cos if\Big)$$

for which these periodic terms disappear in $\Delta\omega$. Because these terms are periodic terms and not secular terms, there exist secular variable terms in 1 PN and 2 PN, but no secular terms exist and only periodic terms exist in 5/2 PN in first order theory.

This implies that there is no secular term or there is no periastron advance but there are only periodic terms in the first order perturbation theory for 5/2 PN.

(3) We can compare the results of Li (2009, 2014) with those from Lincoln & Will (1990) in terms of first order theory for 5/2 PN theory. Li (2009, 2014) concludes that there are not secular variable terms and there are only periodic variable terms for $\Delta\varpi$ in first order theory. This is consistent with the result obtained by Lincoln & Will (1990) in the first order theory. However, in the present paper, Li obtains the secular variable term for $\Delta\varpi$ in second order perturbation theory. Moreover, there is no secular variable term in the first order perturbation theory. It must appear in the second order perturbation theory.

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