

Scientific Reminiscences

## Research career of an astronomer who has studied celestial mechanics

Yoshihide Kozai

National Astronomical Observatory of Japan, Mitaka, Tokyo 181-8588, Japan; [kozai@wave.plala.or.jp](mailto:kozai@wave.plala.or.jp)

Received 2016 May 1; accepted 2016 May 11

**Abstract** Celestial mechanics has been a classical field of astronomy. Only a few astronomers were in this field and not so many papers on this subject had been published during the first half of the 20<sup>th</sup> century. However, as the beauty of classical dynamics and celestial mechanics attracted me very much, I decided to take celestial mechanics as my research subject and entered university, where a very famous professor of celestial mechanics was a member of the faculty. Then as artificial satellites were launched starting from October 1958, new topics were investigated in the field of celestial mechanics. Moreover, planetary rings, asteroids with moderate values of eccentricity, inclination and so on have become new fields of celestial mechanics. In fact I have tried to solve such problems in an analytical way. Finally, to understand what gravitation is I joined the TAMA300 gravitational wave detector group.

**Key words:** celestial mechanics — occultation — minor planets — asteroids: general — planets and satellites: general — planets and satellites: rings — artificial satellite — gravitational waves

### 1 SCHOOL DAYS

I was born in 1928 and was educated in the Japanese old system, which was continued until roughly 1952. In fact the schools we attended were a primary school of six years, which was the only one obligatory school at that time, a middle school of five years, a high school of three years and a university of three years. At our primary school a total of 60 boys and girls were in one class with one teacher, at the middle school 50 boys were in one of five classes, and at the high school one class consisted of 40 boys. There were five science oriented classes and five non-science-oriented classes.

Since there were many pupils in one class and not so good experimental facilities were available, I do not have a good memory of being trained in science education.

In December 1941, when I was a second year middle school boy, the Japanese navy attacked Pearl Harbor. Nearly all middle schools including ours practically stopped their educational activities from May 1944 to August 1945. We were obliged to work at military factories for 11 hours a day by a two shift system till June 1945. Still I took the entrance examination for a high school and received permission in March 1945 to enter the high school which is now the first two year course at the University of Tokyo. In fact I could not enter the high school by July 1945 after having worked at the factory.

Actually, even after I entered the high school, I had to continue to work at the factory. Finally a few weeks after August 15, when Japan declared its surrender to the Allied

Forces, the high school resumed its operation as an educational organization.

However, another important condition that reduced our study hours occurred. In fact, we could not obtain enough food to survive, particularly, for some students including me, as my father passed away in June 1945 after he had stayed in bed for 10 years. Namely, my family lost all income during the period of severe inflation after the war. Many students had to work to obtain some income and, therefore, the school was only open in the morning. I myself had to work as a private home teacher in the afternoon and at night.

Even though I was studying at high school, I could not decide which field I would choose at a university. However, a half year before the end of high school I found a book just published on celestial mechanics by Yusuke Hagihara, who was a professor at the University of Tokyo (Hagihara 1947). This book mainly described fundamental principles of analytical dynamics, and the author intended to address many problems of celestial mechanics in later volumes. This book attracted me very much and I decided to study astronomy, particularly celestial mechanics.

The entrance examination permitted me to become a student at the Department of Astronomy, University of Tokyo and during the second year I could attend lectures on celestial mechanics by Prof. Hagihara. During the same period I read the four volumes of books by Tisserand (Tisserand 1889–1896) at my house. In these books the author described how to solve many problems in celestial mechanics.



**Fig. 1** The author demonstrates how to use a hand computer (Tiger calculator). The Japanese characters at the back of the computer say that this is property of Tokyo Astronomical Observatory. The photo was taken in around 1955.

As the third year started, Prof. Hagihara suggested a problem for the thesis I needed to graduate the university. That was on the motion of the asteroid (279) Thule, for which the mean motion is  $400''$  per day, being at the ratio of 4:3 with respect to that of Jupiter. According to this relation, the two bodies do not approach each other and I investigated why this is so. For this work I spent many hours to compute some coefficients in the disturbing function by using a hand computer (see Fig. 1). I subsequently submitted the paper to Prof. Hagihara and published it (Kozai 1952). Also, I read the three volumes of books by Poincaré, *Les Méthodes Nouvelles de la Mécanique Céleste*, which describe many fundamental ideas related to the three body problem (Poincaré 1892).

## 2 OCCULTATION OBSERVATIONS

In April 1951 I started to work in the group lead by Prof. H. Hirose at Tokyo Astronomical Observatory, a research institute at the University of Tokyo. Dr. Hirose's main interest had been in astrometric observations and motions of comets and asteroids. He had also been interested in using a lunar occultation to determine positions and the motion of the Moon. As he analyzed occultation observation data obtained in Japan and compared them with worldwide data, he found that there were some systematic differences between the results. He thought that the differences were

due to the fact that geodetic coordinates in Japan were shifted systematically with respect to most coordinate systems used in other countries. In fact it had been known already that there was some other evidence for such differences. Then Dr. Hirose proposed to connect the Japanese geodetic system to the US one across the Pacific Ocean by timing a lunar occultation very accurately.

For this purpose we had made test observations that could be used to relate measurements recorded at any remote location with those by the site of the 65 cm telescope at Tokyo Astronomical Observatory. This study was conducted by timing the same occultation of a star with an accuracy of 0.001 second by using a photomultiplier device. In addition, through analyzing such data we could relate any two sites inside of Japan with an accuracy of 20 m.

Later, an organization to relate Japanese and US data by observing a lunar occultation was set up. However, as the first Sputnik satellite was launched in October 1957, everybody believed it was better to use artificial satellites for this purpose, and the lunar occultation campaign was discontinued.

## 3 SATURNIAN SATELLITES

Also I spent time for my own research on celestial mechanics. In 1953, Hall et al. (1953) published astrometric observations of the inner satellites of Saturn during 1927–1947. The orbital elements of the satellites used at that time were published by G. Struve in 1930 (Struve 1930). Therefore, I tried to improve them by analyzing data by Hall et al. and derived their orbital elements almost every year between 1927–1947 (Kozai 1957a,b). For this purpose I also improved the theory of motion of some of the satellites and determined the mass of the satellites as well as the dynamical figure parameters of Saturn by using the secular motions of inner satellites. It should be mentioned that for the mean longitude of Mimas a secular acceleration term,  $2.8^\circ$  (per century)<sup>2</sup>, was detected.

These papers were submitted for a doctor's degree to the University of Tokyo and the degree was awarded to me in 1958.

## 4 MOTIONS OF ARTIFICIAL SATELLITES

The International Geophysical Organization proposed to organize an international campaign called the "International Geophysical Year (IGY)" in 1957–1958 and the US declared that during the IGY artificial satellites would be launched. Then Dr. F. L. Whipple at the Smithsonian Astrophysical Observatory (SAO) agreed to be responsible for optical tracking of satellites by distributing 12 Baker-Nunn cameras worldwide. Among other things, he wanted to publicize the predicted positions of satellites. The cameras were of Schmidt type with a 50 cm aperture and  $f = 1$ . To follow motions of satellites, the alt-azimuth mountings had a third axis and the field of view was particularly wide in the direction of satellite motion.

In October 1957 the USSR launched the first artificial satellite to orbit Earth and in January 1958 the US started to launch artificial satellites. Then Dr. Whipple wanted to have researchers on celestial mechanics and I was invited to work at SAO's satellite tracking program. I joined the group late in October 1958.

As soon as I arrived at SAO, I was told that the Minitrack radio-interferometer system detected strange variations in the orbital elements, particularly eccentricity. Later it was made clear that they were due to odd order harmonics (north-south asymmetry) of the geopotential.

For close artificial satellites, the periods of one revolution of the perigee and the ascending node of the orbital plane are generally of the order of 100 days. One must include second-order perturbation terms to derive the theory of motion that is valid for a 100 day time interval. As for artificial satellites one cannot assume that the eccentricity and the inclination have any small values, that is, we must derive the theory of motion for satellites under these conditions. Then as soon as I arrived at SAO, I started to formulate such a theory and I was able to succeed in deriving such a theory in three months (Kozai 1959b).

I also noticed the importance of luni-solar perturbations and included them in the equations (Kozai 1959a). By using luni-solar perturbation theory, Whitney and I concluded that the lifetime of a satellite with a large eccentricity, 1959  $\delta 2$ , for which NASA had announced the lifetime to be as long as 20 years, would be reduced to two years due to the solar effect, which would reduce the perigee height very much (Kozai & Whitney 1959).

For satellite motions I analyzed several effects like solar radiation pressure, which is very important for balloon-type satellites, effects by the deformed Earth and by luni-solar tides, effects caused by using a non-inertial coordinate system (namely referring to the moving equator of the Earth) and so on.

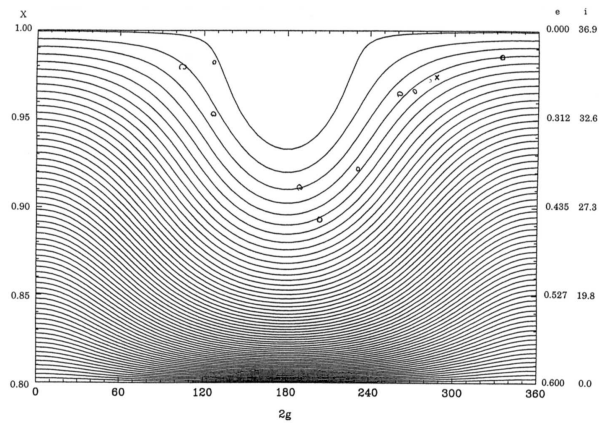
Finally I derived the geopotential by using satellite motions (Kozai 1961a,b). In fact I enjoyed staying at SAO as Dr. Whipple encouraged my work very much.

### 5 SECULAR PERTURBATIONS OF ASTEROIDS

As far as I know in textbooks for celestial mechanics published before 1960, the authors treated secular perturbations of asteroids under the assumption that both the eccentricity,  $e$ , and the inclination,  $i$ , are small enough that the solutions could be derived by solving linear differential equations. Therefore, the idea for proper eccentricity and proper inclination could be introduced.

K. Hirayama computed the proper eccentricities and inclinations for asteroids, when about 800 asteroids had been discovered, and found that there were some groups, in which the semi-major axes, proper eccentricities and inclinations have similar values. Later he named such groups families of asteroids (Hirayama 1918, 1922).

However, now we know that there are many asteroids with not so small eccentricities and/or inclinations. Therefore, I tried to derive secular perturbation theory for



**Fig. 2** The diagram expressing the solution for the asteroid (2) Pallas with  $a = 2.77$  AU and  $\Theta = 0.80$ .

asteroids with arbitrary values of the eccentricity and inclination. Still I had to assume that the disturbing planets move along circular orbits on the same plane, namely, the ecliptic plane.

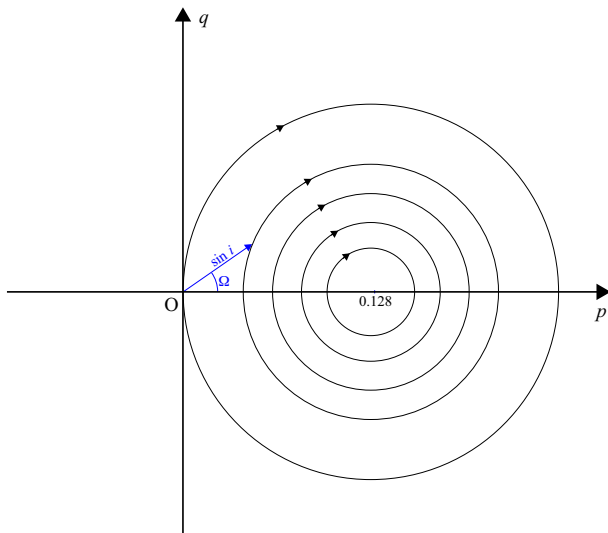
In secular perturbation theory, as the mean longitude is eliminated, the semi-major axis,  $a$ , is constant. Moreover, because in the present case for the disturbing function, the longitude of the ascending node does not appear, so  $\Theta = \sqrt{1 - e^2} \cos i$ , which represents the  $z$ -component of angular momentum of the asteroid, is constant. Then we have a dynamical system with one degree of freedom. Here,  $X = \sqrt{1 - e^2}$  and  $g$ , the argument of perihelion, are a set of canonical variables, with the constant Hamiltonian,  $F$ , in which time does not appear explicitly. In fact the Hamiltonian is the averaged disturbing function. Therefore, this system of equations can be solved by quadrature (Kozai 1962), and  $X$  as well as  $e$  and  $i$  are expressed as periodic functions of  $2g$ .

In Figure 2, the diagram to express the solution is shown for the asteroid (2) Pallas with  $a = 2.77$  AU and  $\Theta = 0.80$ . The vertical axis on the left side shows the value of  $X$ , the vertical axis on the right side shows  $e$  and  $i$ , while the horizontal axis shows  $2g$ . In the figure, the solution is along one of the equi- $F$ -value curves and the present value for (2) Pallas is denoted by  $\times$ . Notice that there are several other asteroids denoted by  $\circ$  with similar values of  $a$  and  $\Theta$ . This shows that there is a family containing (2) Pallas, although this family cannot be detected by the linear theory.

Note that there is a stationary point where  $X$  and  $2g = 180^\circ$  do not change. Around the stationary point there is a region where  $2g$  cannot make any complete revolution. In fact, if  $\Theta$  is less than roughly 0.8, a so-called libration region appears where  $2g$  changes within a limited region.

After I submitted this paper to the Astronomical Journal, I came back to Tokyo Astronomical Observatory in October 1962.





**Fig. 3** A schematic diagram illustrating how the inclination  $i$  and the longitude of the ascending node  $\Omega$  of a geostationary satellite with respect to the equator change with time. The definition of variables is  $p = \sin i \cos \Omega$  and  $q = \sin i \sin \Omega$ . As mentioned in the main text, there is a stationary point approximately at  $(p, q) = (0.128, 0)$ . The initial condition that I mentioned in the main text,  $(\Omega, i) = (270^\circ, 1^\circ)$ , is approximately located at  $(p, q) = (0, -0.01745)$  in this figure.

## 6 IMPROVEMENT IN SATELLITE TRACKING METHOD

Soon after I returned to Tokyo Astronomical Observatory, I assumed the position of Dr. H. Hirose, who was also responsible for the observational side of the observatory. In fact there was one of the 12 Baker-Nunn cameras. The Baker-Nunn camera had been a good satellite tracking device. However it had a disadvantage in precisely determining orbital elements of satellites, as the observed data cannot be well distributed along any orbit, since observations can only be made when any satellite is sunlit whereas the camera is on the night side of Earth.

As far as radio observations were concerned, the radio-transmitters onboard satellites before 1960 were not stable enough to be used to determine the orbits of satellites. However, as soon as the stability of the transmitters had been improved, radio Doppler observations became an important technique to determine the Earth rotation parameters including polar motions.

Later, the laser ranging technique was introduced and the ranging accuracy had been much increased after the Lageos 1 and Lageos 2 satellites, which had retro-reflectors covering their surface, were launched, respectively, in 1976 and 1992. In fact they have greatly increased the accuracy of determining geodetic parameters. It is also notable that the Apollo 11 spacecraft, launched in July 1969, brought a laser retroreflector to the surface of the Moon. We then tried to install laser ranging instruments at Tokyo Astronomical Observatory without much success.

Even after I came back to Japan, I tried to derive more geodetic data by using SAO's observations. In fact I derived Love's number, the ratio of the actual tides on the Earth with respect to the theoretical ones (Kozai 1967). I also tried to determine temporal variations of the geopotential by satellite motions, but the accuracy was not good (Kozai 1971). I understand that such data have been very accurately determined by using Lageos as well as other satellites used for laser ranging data.

## 7 ORBIT OF GEOSTATIONARY SATELLITE

After the 1960s, use of high-speed computers became very popular and I myself learned a computer language, FORTRAN, in 1959. However, I never tried to use such a method to solve any differential equation. I think that analytical methods are useful to have some idea of how to find any adequate orbit, even though the solutions derived are not so exact. In fact I believe that they give us some idea of their dynamical property. Here I would like to describe one example of this kind of research (Kozai 1979a).

As geostationary satellites are usually used for radio communication, weather forecast and so on, they should be kept at 37 000 km above a fixed point on the equator. However, as the geopotential is not symmetrical with respect to the rotation axis of the Earth, the longitude of any synchronous satellite changes between two fixed points with a period of 820 days as the semi-major axis changes on the order of 0.001. However, if the longitude of the satellite is  $70^\circ$  or  $250^\circ$  it does not move in the east-west direction. The luni-solar perturbation does not change the eccentricity by more than 0.0005.

Still the inclination of the orbital plane and its longitude of the ascending node with respect to the equatorial plane,  $\Omega$ , change widely due to the luni-solar perturbation. In fact the solution is easily expressed when we use the polar coordinate system with  $\sin i$  as the radius and  $\Omega$  as the angle (Fig. 3). Moreover, it can be shown that the point corresponding to  $\sin i = 0.128$  and  $\Omega = 0^\circ$  is the stationary point. Other solutions move along nearly circular curves around the stationary point in the clockwise direction with a period of 54 years. Among them there is a solution passing through the origin of the coordinate system, however it soon leaves the origin. In order to increase the time length, during which the inclination stays in the region with small inclination as long as possible, say  $1^\circ$ , the initial value of  $\Omega$  should be near  $270^\circ$  and  $i$  is nearly equal to  $1^\circ$ . For me it is a good initial orbit for any geostationary satellite.

## 8 DYNAMICAL PROPERTY OF ASTEROIDS, COMETS, AND PLANETARY RINGS

I continued to work on secular perturbations of asteroids and comets based on the paper Kozai (1979b) by plotting the values for observed asteroids and short-period comets in figures similar to Figure 2. Generally speaking if the value of  $\Theta = \sqrt{1 - e^2} \cos i$  is nearly equal to 1, both the eccentricity,  $e$ , and the inclination,  $i$ , are small enough and



**Fig. 4** The author's photo during the time he was general director of National Astronomical Observatory of Japan in 1988–1994.

do not change very much. However, if  $\Theta$  is small like 0.8,  $e$  and  $i$  show large changes.  $e$  generally takes its maximum value at  $2g = 180^\circ$ , where the major axis of the orbit is greatly inclined with respect to the reference plane;  $e$  takes its minimum value at  $2g = 0^\circ$ , where the major axis is placed on the reference plane.

As an example, I show variations of the orbital elements for (1922) Zula having  $a = 3.25$  AU and  $\Theta = 0.52$ . For this asteroid  $e$  changes between 0.36 and 0.66 and  $i$  changes between  $18^\circ$  and  $40^\circ$ . Therefore, the aphelion distance changes between 4.42 and 5.40 AU, which is larger than 5.20 AU, the mean heliocentric distance of Jupiter. However, when the aphelion distance takes the largest value,  $2g$  is equal to  $180^\circ$ . This means that the position of the aphelion is far from the orbital plane of Jupiter when  $e$  takes the largest value. A similar situation occurs for asteroids with small  $\Theta$ . That is why any asteroid can avoid very close approaches to Jupiter.

On the other hand, generally speaking, for a short-period comet,  $e$  takes a large value whereas  $i$  is small. This means that positions of short-period comets are found in the lower part of figures corresponding to Figure 2. For this case the eccentricity does not change so much with the argument of perihelion. Therefore, comets have a relatively large chance of closely approaching Jupiter.

Since there had been few papers treating the motion of particles with eccentric orbits, the paper addressing secular perturbations with high eccentricity and inclination (Kozai 1962), which essentially analyzes the motion of a test particle in a field with axial symmetry, had been applied to several problems. In fact, many exoplanets with highly eccentric orbits were found recently. The same method was

also applied to the problem of merging black holes, as any black-hole pairs with highly eccentric orbits merge easily since more intense gravitational waves are emitted near the area with short mutual distance. A post-Newtonian version of the paper was published (e.g. Hinder et al. 2010).

In fact the paper Kozai (1962) on secular perturbations was selected as one of about 30 influential papers published in the *Astronomical Journal* and the *Astrophysical Journal* in the 20<sup>th</sup> century when the American Astronomical Society celebrated its centennial anniversary. According to SAO/NASA ADS, the number of citations for this paper had reached 916 as of 2016 August 18.

I also studied the dynamics of Uranian ring particles together with shepherding satellites. The rings of Uranus were discovered in 1977 by observing a star being occulted by Uranus. They are mostly narrow rings, however, the  $\varepsilon$  ring has a unique structure (Elliot & Nicholson 1984). In fact, all the particles have the same value of eccentricity ( $e = 0.0079$ ) with very small inclination and this ring revolves around Uranus like a solid plate. This means that the major axes of all the particles in the ring move with the same angular velocity. This contradicts the idea that the axes rotate due to the oblateness of Uranus. This interested me very much and I thought that undiscovered shepherding satellites help maintain this structure, although this problem has not yet been solved (Kozai 1992, 1993).

## 9 GRAVITATIONAL DETECTOR TAMA300

The idea of gravitational waves was proposed by Einstein in 1916 and several attempts have been made to detect this phenomenon. Indirect evidence of them was identified by observing the orbital variations of a neutron star binary caused by emitting gravitational waves. Also in the 1980s, a new method of using a Fabry-Pérot interferometer was proposed. Then the US started to construct LIGO detectors with 4 km baselines, one in Hanford, Washington and one in Livingston, Louisiana. France and Italy had the idea to construct one with 3 km baselines in Pisa, Italy, and the UK and Germany together proposed building GEOS600 with 600 m baselines in Hanover, Germany.

I served as the general director of Tokyo Astronomical Observatory during 1981–1988. And, after Tokyo Astronomical Observatory was reorganized into the National Astronomical Observatory of Japan (NAOJ), one of the Inter-university Institutes, I served as its general director during 1988–1994 (see Fig. 4). Then officially I left NAOJ and joined the group managing a gravitational wave detector.

The group consisted of scientists in various fields and from various research and educational institutes. They decided to construct a gravitational wave detector with 300 m arms by making underground tunnels within the Mitaka campus of NAOJ. We named this instrument TAMA300. In 1995 the five year budget for this project was approved and the project started. In this project, we intended to detect gravitational waves caused by the merging of neutron

star binaries, so we tried to eliminate any noise in the instrument around a frequency of 1 kHz.

Also, in the summer of 1999 before the other instruments, we tested the operation of TAMA300 and in September 1999 we tried to detect gravitational waves. Then we worked to increase its sensitivity and in August 2000 we conducted more than 100 hours of observations. In March 2001 we could acquire continuous observations over 24 hours and in early spring 2004 we recorded simultaneous observations with LIGO.

In total we could obtain more than 3000 hours of data, although we did not detect any gravitational waves. Finally we concluded that when a neutron star binary merger takes place within a distance of 82 kpc, TAMA300 could detect any gravitational wave caused by such an event (Kuroda et al. 2002).

The work has been transformed into KAGRA, which has 3 km arms and was constructed at the Kamioka mine in Gifu, by trying to reduce any thermal or other noise. We expect that the sensitivity of KAGRA will be a few 100 times better than that of TAMA300. It is expected to begin operation soon.

In the meantime, the gravitational wave caused by the merging of a black hole binary was detected by the Advanced LIGO instrument in September 2015 (Abbott et al. 2016).

## 10 ADMINISTRATIVE WORKS

After I originally attended the General Assembly of the International Astronomical Union (IAU) in 1961, I often attended subsequent meetings of the IAU and the International Union of Geodesy and Geophysics (IUGG). I served as the chairman of the Committee of Satellite Geodesy in 1975–1979, the president of the Commission of Celestial Mechanics in 1979–1982 and the president of the Commission of Asteroids and Comets in 1985–1988. Later, I served as the President of IAU in 1988–1991. Through this way, I could make several friends from various countries. Therefore, not all the duties were very taxing. However, it is true that I lost my own research hours.

As I already wrote briefly, I served as the general director of the Tokyo Astronomical Observatory in 1981–1988 and that of NAOJ in 1988–1994. During these periods and with much cooperation from my colleagues, I was able to make facility changes of observatories and succeeded in constructing an 8 m optical-infrared telescope (the Subaru telescope) at the top of Mauna Kea, Hawaii, US. After it was completed I am very glad to say that many fruitful results were published.

**Acknowledgements** I must express great appreciation to Dr. Fumi Yoshida for her giving me much advice.

## References

- Abbott, B. P., Abbott, R., Abbott, T. D., et al. 2016, *Physical Review Letters*, 116, 061102
- Elliot, J. L., & Nicholson, P. D. 1984, in *IAU Colloq. 75: Planetary Rings*, ed. R. Greenberg & A. Brahic, 25
- Hagihara, Y. 1947, *The Foundation of Celestial Mechanics* (in Japanese) (Kawadeshobo)
- Hall, A. jr et al., 1953, *Publ. US Naval Obs. Ser.2*, 17
- Hinder, I., Herrmann, F., Laguna, P., & Shoemaker, D. 2010, *Phys. Rev. D*, 82, 024033
- Hirayama, K. 1918, *AJ*, 31, 185
- Hirayama, K. 1922, *Japanese Journal of Astronomy and Geophysics*, 1, 55
- Kozai, Y. 1952, *PASJ*, 3, 184
- Kozai, Y. 1957a, *Ann. Tokyo Astron. Obs.*, Ser. 2, 5, 73
- Kozai, Y. 1957b, *Ann. Tokyo Astron. Obs. Ser. 2*, 5, 107
- Kozai, Y. 1959a, *SAO Special Report*, 22
- Kozai, Y. 1959b, *AJ*, 64, 367
- Kozai, Y., & Whitney, C. A. 1959, *SAO Special Report*, 30
- Kozai, Y. 1961a, *AJ*, 66, 8
- Kozai, Y. 1961b, *AJ*, 66, 355
- Kozai, Y. 1962, *AJ*, 67, 591
- Kozai, Y. 1967, *Philosophical Transactions of the Royal Society of London Series A*, 262, 135
- Kozai, Y. 1971, in *Space Research XI*, ed. K. Y. Kondratyev, M. J. Rycroft, & C. Sagan, 469
- Kozai, Y. 1979a, *Orbit of Artificial Satellites*, National Space Development Agency of Japan (NASDA, published in Japanese)
- Kozai, Y. 1979b, in *IAU Symposium, 81, Dynamics of the Solar System*, ed. R. L. Duncombe, 231
- Kozai, Y. 1992, *PASJ*, 44, 135
- Kozai, Y. 1993, *PASJ*, 45, 263
- Kozai, Y. 2004, *Proceedings of the Japan Academy, Series B*, 80, 157
- Kuroda, K., Ohashi, M., Miyoki, S., et al. 2002, *Classical and Quantum Gravity*, 19, 1237
- Poincaré, H. 1892, *Les Méthodes Nouvelles de la Mécanique Céleste*
- Struve, G. 1930, *Veröff. Univ.-Sternwarte, Berlin-Babelsberg*, 4, Heft 4
- Tisserand, F. F. (1889–1896) *Traité de la Mécanique Céleste 1-4*, Gauthier Villars. Paris (<https://archive.org/details/traitemecanique00tissrich>)