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Numerical simulation of the Moon's rotation in a rigorous relativistic framework

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Abstract This paper describes a numerical simulation of the rigid rotation of the Moon in a relativistic framework. Following a resolution passed by the International Astronomical Union (IAU) in 2000, we construct a kinematically non-rotating reference system named the Selenocentric Celestial Reference System (SCRS) and give the time transformation between the Selenocentric Coordinate Time (TCS) and Barycentric Coordinate Time (TCB). The post-Newtonian equations of the Moon's rotation are written in the SCRS, and they are integrated numerically. We calculate the correction to the rotation of the Moon due to total relativistic torque which includes post-Newtonian and gravitomagnetic torques as well as geodetic precession. We find two dominant periods associated with this correction: 18.6 yr and 80.1 yr. In addition, the precession of the rotating axes caused by fourth-degree and fifth-degree harmonics of the Moon is also analyzed, and we have found that the main periods of this precession are 27.3 d, 2.9 yr, 18.6 yr and 80.1 yr.

Key words: lunar libration — Selenocentric Celestial Reference System — geodetic precession — general relativity

1 INTRODUCTION

Studying the rotation of celestial bodies is an important research topic in astronomy and astrophysics. From the rotation, one can derive important information about dynamics, gravitational interaction, internal structure of a celestial body and so on. For example, highly accurate data on the Earth's rotation have been applied in geophysics, observation technologies like VLBI, and constructing astronomical reference systems. Therefore, researchers have investigated this issue in many works. The P03 precession (Capitaine et al. 2003) is a successful theory for the precession of the Earth's rotation axis and it is very accurate over a few centuries. The expressions of Earth's precession for a longer time span have also been derived by Vondrák et al. (2011) and Laskar et al. (2011). Recently, we solved the equations describing Earth's rotation numerically in a rigorous general relativistic framework and obtained an expression for the long-term precession of Earth (Tang et al. 2015). This relativistic theory of Earth's rotation was constructed by Klioner et al. (Klioner et al. 2010, Klioner et al. 2003).

The Moon, the nearest neighbor to our planet, is also very interesting for scientific research, especially following the start of a new upsurge of lunar exploration missions (for example China's Chang'e program). Accurately observing and calculating the rotation of the Moon are useful for studying the dynamical evolution of the Earth-Moon system, the internal composition of the Moon, and for building lunar based reference systems. It is well known that the period of the Moon's rotation is approximately equal to the period of the Moon's orbital motion. However, researchers have also observed a kind of oscillating motion in the Moon's visual surface as seen from Earth. This oscillating motion is called lunar libration. A part of libration is due to the rotation of the Moon. A comprehensive theory of the libration of the Moon was given by Eckhardt (1981), who solved the Euler equations in a Newtonian framework. Moons also studied the libration of the Moon (Moons 1982), in which he used a Hamiltonian equation and phase space to produce an analytical theory of the libration of the Moon. Bois et al. (1992) considered an effect of fifth-degree harmonics in the Moon's gravitational field. Williams used the method of numerical integration to construct the ephemerides of the Moon. The initial conditions of the Moon's motion, mass ratio and other parameters he used came from joint fits of lunar and planetary data (Williams & Dickey 2002).

However, none of these studies considered the effect of general relativity. Currently, lunar laser ranging experiments can measure distances with sub-centimeter accuracy (Rambaux & Williams 2011; Kopeikin et al. 2008). A new project proposed in China plans to send a telescope to the Moon and measure the Moon's rotation with high accuracy (C.-L. Huang, private communications). With the improvement in accuracy, the relativistic effects on the Moon's rotation may be observed in the near future. On the other hand, the Moon's rotation can also be used to test Einstein's theory of gravitation. The first investigation of the effect of general relativity on the Moon's rotation was given by Bois & Vokrouhlicky (1995). In their paper, they used the Damour-Soffel-Xu (DSX) theory to numerically calculate the libration of the Moon due to the gravitoelectric part. However, the local reference system for the Moon they used was not rigorous and they also did not include the effect of the gravitomagnetic part. Consequently, we need a more accurate calculation of the Moon's rotation in a rigorous relativistic theory that incorporates this important relativistic effect.

The relativistic theory of a rigid body's rotation that was developed by Klioner et al. (Klioner et al. 2010) was previously used to calculate the rotation of the Earth. In principle, because this theory is constructed from a DSX framework (Damour et al. 1991; Damour et al. 1992; Damour et al. 1993), it can be used for any rigid body in the solar system. Therefore, in the present paper, we use Klioner's theory to simulate the rotation of the Moon. To this end, we first construct a kinematically non-rotating reference system named the Selenocentric Celestial Reference System (SCRS) which is consistent with the International Celestial Reference System (ICRS) convention to write the evolution equations of the Moon. Second, we need to consider the relativistic scaling of quantities which appear in the Moon's rotation (Klioner 2008). Third, the torques on the Moon are described by mass multipole moments and tidal moments. Finally, the post-Newtonian equations of the Moon's rotation are integrated numerically in the range of 600 yr and the relativistic effects on the three Euler angles are analyzed.

This paper is organized as follows. In the next section, we construct the SCRS. In Section 3, we introduce our numerical method for solving the post-Newtonian Euler equations. The results of the calculation are presented and analyzed in Section 4. Finally, in the last section, conclusions and discussions are provided.

2 RELATIVISTIC EQUATIONS OF THE "RIGID" MOON

2.1 Selenocentric Celestial Reference System

In order to study the rotation of the Moon in a rigorous relativistic framework, we should first construct a local reference system in which we can establish post-Newtonian equations of the Moon's rotation. Xie and Kopeikin have already constructed a dynamically non-rotating Selenocentric Reference System (Xie & Kopeikin 2010; Kopeikin & Xie 2010). In the present paper, we establish a kinematically non-rotating system in order to be consistent with the ICRS convention. According to the DSX framework, we can define such a reference system (Fig. 1), which is called SCRS, as follows: its origin is assumed to agree with the post-Newtonian center of mass of the Moon and the orientation of its space axes coincides with the ICRS axes, just as in the Barycentric Celestial Reference System (BCRS) and Geocentric Celestial Reference System (GCRS) (Petit & Luzum 2010).

Kopeikin and Klioner have constructed the relation between BCRS and GCRS (Kopejkin 1988; Klioner & Voinov 1993), and we can use their method to establish a transformation between BCRS and SCRS. After this, we have the relation between the Barycentric Coordinate Time (TCB) and Selenocentric Coordinate Time (TCS)

$$\begin{aligned}\mathrm{TCS} \;=\; \mathrm{TCB} - \frac{1}{c^2}\Big\{ \int_0^{\mathrm{TCB}} \Big[\frac{v_{\mathrm{M}}^2}{2} + w_{\mathrm{ext}}(\boldsymbol{x}_{\mathrm{M}})\Big] d\mathrm{TCB} \\ + v_{\mathrm{M}}^i r_{\mathrm{M}}^i \Big\}\,, \qquad (1)\end{aligned}$$

where x_{M}^i and v_{M}^i are the barycentric coordinate position and velocity of the selenocenter, $r_{\mathrm{M}}^i = x^i - x_{\mathrm{M}}^i$ with x^i the barycentric position, and $w_{\mathrm{ext}}(\boldsymbol{x}_{\mathrm{M}})$ is the Newtonian potential of all solar system bodies apart from the Moon evaluated at the selenocenter. We define the origin of TCS in terms of International Atomic Time (TAI). The reading of TCS on 1997 January 1, $0^{\mathrm{h}}0^{\mathrm{m}}0^{\mathrm{s}}$ TAI must be 1997 January 1, $0^{\mathrm{h}}0^{\mathrm{m}}32.184^{\mathrm{s}}$, just like geocentric coordinate time (Soffel et al. 2003), so we also set this moment as the origin for the numerical integration. The unit we use in TCS is SI second.

2.2 The Equations describing the Moon's Rotation in a Relativistic Framework

According to the work of Damour et al. (Damour et al. 1993), for practical application in the solar system, we have the following equations

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}T}\Big(S^a\Big) \;=\; \sum_{l=1}^{\infty}\frac{1}{l!}\varepsilon_{abc}M_{bL}G_{cL} \\ + \sum_{l=0}^{\infty}\frac{1}{c^2}\frac{l+1}{l+2}\varepsilon_{abc}S_{bL}H_{cL}\,, \qquad (2)\end{aligned}$$

where S^a is the spin dipole and H_{ab} is the gravitomagnetic tidal moments of the external gravitational field experienced by the Moon in the SCRS.

Based on the work of Soffel et al. (2003), we can define a simple Newtonian model of a rigidly rotating Moon with $S_L = C_{Ld}\omega^d$ and the meaning of C_{Ld} is similar to the work of Soffel et al. (2003). We also find that if we consider the Moon to be a rigidly rotating homogeneous oblate spheroid with equatorial radius A and polar radius C, then the torque term resulting from the spin octuple of the Moon is more than 10^6 smaller than the one from the spin dipole, so finally the evolution equations of the Moon in the SCRS can be written as follows

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}T}\left(C^{ab}\omega^b\right) \;=\; \sum_{l=1}^{\infty}\frac{1}{l!}\varepsilon_{abc}M_{bL}G_{cL} \\ + \varepsilon_{abc}\boldsymbol{\Omega}_{\mathrm{iner}}^b C^{cd}\omega^d\,, \qquad (3)\end{aligned}$$

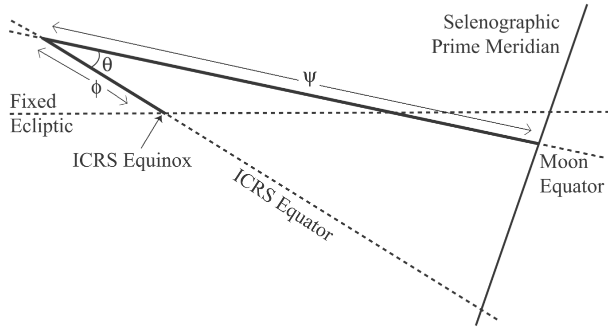

Fig. 1 Three Euler angles ϕ, θ, ψ which describe the orientation of the Moon relative to SCRS and the orientation of space axes of SCRS that coincide with the ICRS axes (Taylor et al. 2010).

where T = TCS is the selenocentric time and C^{ab} is the Newtonian tensor of inertia of the Moon. $\omega = \omega^a$ is the angular velocity of the post-Newtonian Tisserand axes. M_L ($L = i_1 i_2 \cdots i_l$) are the mass multipole moments of the lunar gravitational field defined in the SCRS, and G_L are the gravitoelectric tidal moments of the external gravitational field experienced by the Moon in the SCRS.

When comparing with the rotational equations (6.33–6.39) given by Kopeikin (Kopeikin et al. 2011), we find that there are some additional terms which we do not consider here.

Considering that our work is only in the framework of general relativity, all terms with β and γ in (6.33–6.39) are omitted. Terms with the external monopole moments Q and Y can also be ignored based on Kopeikin's analysis. We simply set the gauge multipoles $Z_L = 0$. After these approximations, we finally find that equation (6.33) in Kopeikin et al. (2011) reduces to Equation (3) in the present paper.

The inertial torque $\varepsilon_{abc}\boldsymbol{\Omega}^b_{\text{iner}}C^{cd}\omega^d$ depends on ω, $\mathbf{C}$ and the angular velocity $\boldsymbol{\Omega}_{\text{iner}}$ which only describes the rotation of the SCRS with respect to a dynamical non-rotating selenocentric reference system (Soffel et al. (2003)). In our model, this rotation mainly comes from the geodetic precession and gravitomagnetic effect. The effect caused by the spin of the Earth is too small, so we do not consider it in this work (about 100 μas/century). Therefore, the expression of $\boldsymbol{\Omega}_{\text{iner}}$ is (Soffel et al. 2003)

$$\boldsymbol{\Omega}_{\text{iner}} = \frac{1}{c^2}\sum_{\text{A}\neq\text{M}} \frac{Gm_{\text{A}}}{|\boldsymbol{x}_{\text{M}} - \boldsymbol{x}_{\text{A}}|^3}(\boldsymbol{x}_{\text{M}} - \boldsymbol{x}_{\text{A}}) \times \left(\frac{3}{2}\boldsymbol{v}_{\text{M}} - 2\boldsymbol{v}_{\text{A}}\right), \tag{4}$$

where $\boldsymbol{x}_{\text{M}}$ and $\boldsymbol{v}_{\text{M}}$ represent the position and velocity of the Moon in the BCRS respectively, and $\boldsymbol{x}_{\text{A}}$ and $\boldsymbol{v}_{\text{A}}$ are the position and velocity of a celestial body in the BCRS respectively.

We need to construct a rotating reference system fixed on the Moon to transform Equation (2) to the analogous expression from Newtonian equations that describe the Moon's rotation, just as Klioner et al. did for the Earth (Klioner et al. 2010):

– $\boldsymbol{C}^{ab} = P^{ac}P^{bd}\bar{C}^{cd}, \ \ \bar{C}^{cd} = \text{const.}$
– $M_{a_1 \ldots a_l} = P^{a_1 b_1} \ldots P^{a_1 b_1}\bar{M}_{b_1 \ldots b_l}, \ \ \bar{M}_{b_1 \ldots b_l} = \text{const.}$
– The orthogonal matrix $P^{ab}(T)$ is assumed to be

$$\omega^a = \frac{1}{2}\varepsilon_{abc}P^{db}(T)\frac{\text{d}}{\text{d}T}P^{dc}(T), \tag{5}$$

where ω^a is the component of angular velocity in the SCRS.
– Because our model of the Moon is rigid, we can regard the rotating reference frame which is associated with the Moon as a system of principal axes of inertia (PAI) and the matrix P^{ij} can be parameterized by the three Euler angles ϕ, θ, ψ (see Fig. 1). They define the orientation of the Moon in the SCRS.

In the PAI reference system, the time scale is the same as in SCRS, and the transformation between the two space axes is

$$\begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix} = R_z(\phi)R_x(\theta)\,R_z(\psi)\begin{bmatrix} X \\ Y \\ Z \end{bmatrix}, \tag{6}$$

where ξ, η, ζ are the spatial coordinates of the PAI system, X, Y, Z are the spatial coordinates of SCRS, R_x and R_z are the rotation matrix about the x axis and z axis respectively and orthogonal matrix P^{ij} can be written as

$$P^{ij} = R_z^{il}(\phi)R_x^{lm}(\theta)R_z^{mj}(\psi).$$

3 NUMERICAL INTEGRATION OF THE MOON'S ROTATION

In this section, we introduce our numerical method for solving the post-Newtonian Euler equations in detail.

3.1 The Torque on the Moon

To compute the torque on the Moon, we also need some assumptions to make the numerical integration more efficient and to achieve sufficient precision:

- All of the external bodies except the Earth are supposed to be mass-monopoles, that is point masses only characterized by their masses and BCRS positions.
- According to Eckhardt (1981), the figure-figure torques are important for the Moon

$$\begin{aligned}\boldsymbol{N} \;=\; & \frac{15GM_{\mathrm{E}}R_{\mathrm{E}}^2J_{2,\mathrm{E}}}{2r_{\mathrm{EM}}^5}\Big\{\left(1-7\sin^2\varphi_{\mathrm{M,E}}\right)[\hat{\boldsymbol{r}}_{\mathrm{EM}}\times\boldsymbol{I}_{\mathrm{M}}\hat{\boldsymbol{r}}_{\mathrm{EM}}] \\ & +2\sin\varphi_{\mathrm{M,E}}\left[\hat{\boldsymbol{r}}_{\mathrm{EM}}\times\boldsymbol{I}_{\mathrm{M}}\hat{\boldsymbol{p}}_{\mathrm{E}}+\hat{\boldsymbol{p}}_{\mathrm{E}}\times\boldsymbol{I}_{\mathrm{M}}\hat{\boldsymbol{r}}_{\mathrm{EM}}\right]-\frac{2}{5}\left[\hat{\boldsymbol{p}}_{\mathrm{E}}\times\boldsymbol{I}_{\mathrm{M}}\hat{\boldsymbol{p}}_{\mathrm{E}}\right]\Big\}\,,\end{aligned} \tag{7}$$

  where $\hat{\boldsymbol{p}}_{\mathrm{E}}$ is the direction vector of the Earth's pole and $\hat{\boldsymbol{r}}_{\mathrm{EM}}$ is the direction vector of the Earth from the Moon. $\boldsymbol{I}_{\mathrm{M}}$ is the Moon's moment of inertia tensor; R_{E} and M_{E} are the mass and equatorial radius of the Earth respectively; $\sin\varphi_{\mathrm{M,E}}=\hat{\boldsymbol{r}}_{\mathrm{EM}}\cdot\hat{\boldsymbol{p}}_{\mathrm{E}}$.
- The relativistic inertial torque is expressed in terms of symmetric and trace-free Cartesian tensors M_L and G_L, and the explicit formulas for the external tidal moments G_L influencing the Moon (M) can be written as

$$G_L=\sum_{\mathrm{A}\neq\mathrm{M}}GM_{\mathrm{A}}g_L^A\,, \tag{8}$$

  where G is the gravitational constant and M_A the mass of a celestial body like Earth, Sun, Venus or Jupiter. The expression of g_L^A (Klioner et al. 2003) is

$$\begin{aligned}g_L^A \;=\; & \frac{(-1)^l\,(2l-1)!!}{r_{\mathrm{M}A}^{l+1}}\Bigg[\hat{n}_{\mathrm{M}A}^L\Bigg\{1+\frac{1}{c^2}\Bigg(2v_{\mathrm{M}A}^2-\frac{1}{2}\boldsymbol{a}_A\cdot\boldsymbol{r}_{\mathrm{M}A}-l\bar{w}_{\mathrm{M}}(\boldsymbol{x}_{\mathrm{M}})-\bar{w}_A(\boldsymbol{x}_A) \\ & -\frac{1}{2}\left(2l+1\right)\left(\boldsymbol{v}_A\cdot\boldsymbol{n}_{\mathrm{M}A}\right)^2\Bigg)\Bigg\}-\frac{1}{c^2}\frac{(l-1)\,(l-8)}{2\,(2l-1)}v_{\mathrm{M}A}^{\langle i_l}v_{\mathrm{M}A}^{i_{l-1}}n_{\mathrm{M}A}^{L-2\rangle}+\frac{1}{c^2}\frac{l}{2l-1}r_{\mathrm{M}A}a^{\langle i_l}n_{\mathrm{M}A}^{L-1\rangle} \\ & +\frac{1}{c^2}\frac{l}{2}\left(\boldsymbol{v}_{\mathrm{M}}\cdot\boldsymbol{n}_{\mathrm{M}A}\right)v_{\mathrm{M}}^{\langle i_l}n_{\mathrm{M}A}^{L-1\rangle}-\frac{1}{c^2}\left(l\boldsymbol{v}_{\mathrm{M}}\cdot\boldsymbol{n}_{\mathrm{M}A}+4\boldsymbol{v}_{\mathrm{M}A}\cdot\boldsymbol{n}_{\mathrm{M}A}\right)v_{\mathrm{M}A}^{\langle i_l}n_{\mathrm{M}A}^{L-1\rangle}\Bigg]\,,\end{aligned} \tag{9}$$

  where

$$\boldsymbol{a}\;=\;\Big(l^2-l+4\Big)\boldsymbol{a}_{\mathrm{M}}+\frac{1}{2}\Big(l-8\Big)\boldsymbol{a}_A\,, \tag{10}$$

$$\bar{w}_A(\boldsymbol{x}_A)\;=\;\sum_{B\neq A}\frac{GM_B}{r_{AB}}+G\sum_{l=2}^{\infty}\frac{\Big(-1\Big)^l\Big(2l-1\Big)!!}{l!r_{\mathrm{M}A}^{l+1}}M_L\hat{n}_{\mathrm{M}A}^L\,, \tag{11}$$

$$\bar{w}_{\mathrm{M}}(\boldsymbol{x}_{\mathrm{M}})\;=\;\sum_{B\neq\mathrm{M}}\frac{GM_B}{r_{\mathrm{M}B}}\,. \tag{12}$$

  and $\boldsymbol{r}_{AB}=\boldsymbol{x}_A-\boldsymbol{x}_B$, $\boldsymbol{v}_{AB}=\boldsymbol{v}_A-\boldsymbol{v}_B$, $n_{AB}^L=\frac{r_{AB}^{a_{i_1}}\cdots r_{AB}^{a_{i_l}}}{r_{AB}^l}$ and $T^{\langle i_1 i_2\cdots i_l\rangle}$ denotes the symmetric and trace-free part of a tensor.

  All of the terms with the factor $1/c^2$ in Equation (6) are the corrections, and the corresponding torque is called post-Newtonian torque. The sum of the post-Newtonian torque and the inertial torque is named the total relativistic torque. For the Newtonian part, we consider up to $l=5$ but for the post-Newtonian part we only consider up to $l=2$.

3.2 Time Transformation

To calculate the Moon's rotation in the relativistic frame, we need to deal with three relativistic time scales: TCS, TCB and TDB. The motion of bodies in the solar system is described in the BCRS with TCB as a time scale, while the rotation of the Moon is described in the SCRS with TCS as the coordinate time scale. When computing the torque on the Moon, we also need to reference the positions and velocities of the bodies in the solar system from the JPL ephemeris DE405, and the time scale of DE405 is the TDB.

According to Equation (1), the transformation between TCS and TCB at the selenocenter is given by

$$\mathrm{TCB} = \mathrm{TCS} + \frac{1}{c^2}\Big(\int_0^{\mathrm{TCB}}\Big(\frac{v_{\mathrm{M}}^2}{2} + w_{\mathrm{ext}}(\boldsymbol{x}_{\mathrm{M}})\Big)d\mathrm{TCB}\Big)\,. \tag{13}$$

When we numerically integrate the equations of the Moon's rotation, there are two problems we need to deal with. Firstly, the time scale that we use is TCS, but we must know the corresponding time scale TDB (TCB) so that we can reference the positions and velocities of bodies from DE405. Another problem is that the algorithm we use to integrate the equations of the Moon's rotation is an 8th-order Runge-Kutta method with a step size of 0.1 d and the range of the numerical integration is about 600 yr, so we need to transfer TCS to TCB (or TDB) many times. One way to solve these two problems is given by Klioner et al. (2010), but if we use his method, we have to store significantly more values from TCS and TCB. In order to take the accuracy and efficiency into account, we use this approach:

– Let $\mathrm{TCS}_1 = 0.1$ d; we use the iterative method (Atkinson & Han 2009) with the initial iteration value $\mathrm{TCB} = 0.1$ d to give the value at this moment and denote this value by TCB_1.
– To compute the value of TCB_2 when $\mathrm{TCS}_2 = 0.2$ d, we rewrite Equation (13) as

$$\mathrm{TCB}_2 = \mathrm{TCS}_2 - \mathrm{TCS}_1 + \mathrm{TCS}_1 + \frac{1}{c^2}\Big(\int_0^{\mathrm{TCB}_2}\Big(\frac{v_{\mathrm{M}}^2}{2} + w_{\mathrm{ext}}(\boldsymbol{x}_{\mathrm{M}})\Big)d\mathrm{TCB}\Big)\,, \tag{14}$$

then we assume that

$$\mathrm{TCS}_1 \;=\; \mathrm{TCB}_1 - \frac{1}{c^2}\Big(\int_0^{\mathrm{TCB}_1}\Big(\frac{v_{\mathrm{M}}^2}{2} + w_{\mathrm{ext}}(\boldsymbol{x}_{\mathrm{M}})\Big)d\mathrm{TCB}\Big)\,. \tag{15}$$

After this, we put Equation (15) into Equation (14) and have

$$\mathrm{TCB}_2 \;=\; \mathrm{TCS}_2 - \mathrm{TCS}_1 + \mathrm{TCB}_1 + \frac{1}{c^2}\Big(\int_{\mathrm{TCB}_1}^{\mathrm{TCB}_2}\Big(\frac{v_{\mathrm{M}}^2}{2} + w_{\mathrm{ext}}(\boldsymbol{x}_{\mathrm{M}})\Big)d\mathrm{TCB}\Big)\,. \tag{16}$$

Finally, we use the iterative method with the initial iteration value $\mathrm{TCB}_2 = \mathrm{TCS}_2$ to generate the value of TCB_2.
– To compute the value of TCB at TCS=0.3 d, we let $\mathrm{TCS}_1 = 0.2$ d, $\mathrm{TCB}_2 \to \mathrm{TCB}_1$, $\mathrm{TCS}_2 = 0.3$ d and then use the iterative method to solve Equation (16) for TCB_2 with initial iteration value $\mathrm{TCB}_2 = \mathrm{TCL}_2$. By reusing the above method, we can get the value of TCB at any given value of TCS.
– To ensure the desired accuracy, the difference between TCS_2 and TCS_1 should be small. In our program, $\mathrm{TCS}_2 - \mathrm{TCS}_1 \leq 0.1$ d.

To test the accuracy of this method, we compare with Klioner's approach (Klioner et al. 2010) which solves the differential equations for TCB and TCS. The algorithm that we use to solve the differential equation is an 8th-order Runge-Kutta method with a step size of 0.01 d. Their relative error is about 10^{-11} as plotted in Figure 2. Therefore, our method for time transformation satisfies the requirements.

4 RELATIVISTIC EFFECTS INVOLVED IN THE ROTATION OF THE MOON

4.1 Relativistic Scaling of Parameters

In the model of the rotation of the Moon, we have three time scales of TCS, TCB and TDB. There are also two classes of quantities entering Equations (1)–(5) that are defined in the BCRS and SCRS and parametrized by TCB and TCS. The relativistic equations describing the rotation of the Moon are only valid if non-scaled time scales TCS and TCB are used. If we use TDB as the time variable, the equations will have a different form (Klioner 2008).

– The position $\boldsymbol{x}$, velocity $\boldsymbol{v}$, acceleration $\boldsymbol{v}_{\mathrm{A}}$, mass parameter GM_{A} of a massive solar system body and radius of the Moon are all from JPL ephemeris DE405 (Standish & Williams 2010). They are TDB-compatible.
– The coefficients of the Moon's gravitational field are independent of any time scale.
– According to the formulas for torques on the Moon, the TCS-compatible torque originating from Earth, Sun, Venus and Jupiter can be computed by using TDB-compatible values of all external bodies, which come from the JPL ephemerides. By denoting the torque by F^a_{TDB}, we can get the TCS-compatible value $F^a_{\mathrm{TCS}} = (1 - L_B)^{-1}F^a_{\mathrm{TDB}}$.
– The moment of inertia for the Moon also comes from the JPL ephemerides. Therefore, it is TDB-compatible

$$C_i^{\mathrm{TDB}} = (1 - L_B)^3 C_i^{\mathrm{TCB}}. \tag{17}$$

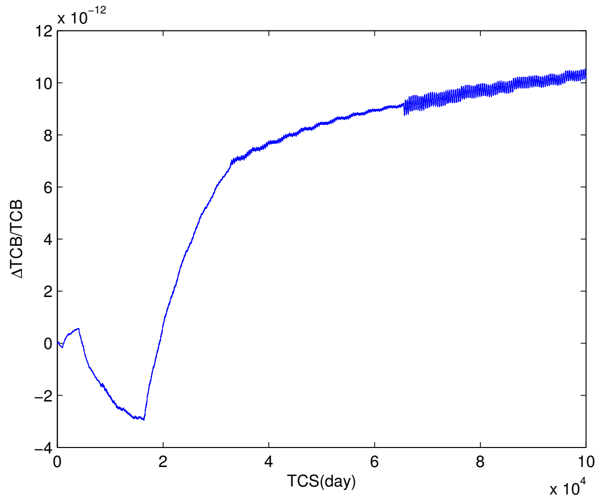

Fig. 2 Relative error of the numerical integration method and the iterative method.

4.2 Initial Value and a Change of Variable

The initial values ($\phi,\ \theta,\ \psi,\ \frac{d\phi}{dt},\ \frac{d\theta}{dt},\ \frac{d\psi}{dt}$) of the Moon's rotation come from JPL ephemerides (Standish & Williams 2010). The initial values of three Euler angles are independent of time scales, but their derivatives are TDB-compatible. Because Equation (3) is parametrized by TCS, we need to change our variables when we integrate numerically.

$$\frac{\mathrm{d}\Theta}{\mathrm{dTDB}} = (1 - L_B)^{-1} \left(\frac{\mathrm{dTCS}}{\mathrm{dTCB}}\right)_{\boldsymbol{x}_\mathrm{M}} \frac{\mathrm{d}\Theta}{\mathrm{dTCS}}\,, \qquad (18)$$

where Θ represents $\phi,\ \theta$ or ψ, and $\boldsymbol{x}_\mathrm{M}$ is the displacement vector from the center of mass of the solar system to the selenocenter.

Because the method that we have used is similar to the case of the rigid Earth (Bretagnon et al. 1997), we can use their results to test our codes.

Figures 3 and 4 show that the results of calculation using our codes are close to the results of SMART 97 (the codes used in Bretagnon et al. 1998) and the relative error of ϕ, θ, ψ are about $10^{-7}, 10^{-9}, 10^{-10}$, respectively. The reason why the relative error of ϕ is about 10^{-7} in the neighborhood of the year 2000 is that the value of ϕ is close to zero in this region.

The left panels of Figure 5 show effects on three Euler angles caused by the total relativistic torque including post-Newtonian torque and inertial torque $\varepsilon_{abc}\boldsymbol{\Omega}^b_{\mathrm{iner}}C^{cd}\omega^d$. We can find that in 20–30 yr, the relativistic effect will cause a variation of more than 10 mas on the three angles. The results suggest that we should consider post-Newtonian corrections when analyzing highly accurate observations of the Moon's rotation.

We also use a fast Fourier transform to identify frequencies arising from effects of the total relativistic torque. The dominant periods of three Euler angles are 18.6 yr and 80.1 yr. For the period of 18.6 yr, we know that longitude of the ascending node of the Moon is regressing by one revolution in 18.6 yr, and the dominant period of the Moon's rotation angles ϕ and θ is also 18.6 yr. For the case of 80.1 yr, we know from Yoder (1981) that there is retrograde precession of the spin axis in space. According to Bois (1995), the periods of 80 yr is the result of the dynamics of the Earth-Moon system.

Our codes can compute the rotation of the Moon for both the Newtonian and the post-Newtonian cases and we can consider the effect of each term in Equation (3). The approach to finding them is as follows: if we want to describe the effects of each term in the right hand of Equation (3), we will compute the difference between the result which has this term and the result which is free from this term.

4.3 Effects of Post-Newtonian Torque and Inertial Torque

The left panel of Figure 6 plots the effects of post-Newtonian torque on the three Euler angles ϕ, θ, ψ. The largest differences shown by the three Euler angles are about 1.5 mas in ϕ, ψ and 0.5 mas in θ. Their dominant periods are the same as the total relativistic torque. This result is similar to the conclusion of Bois & Vokrouhlicky (1995).

Because the inertial torque $\boldsymbol{\Omega}^b_{\mathrm{iner}}C^{cd}\omega^d$ arises from a purely geometrical reason, we may let the first term on the right hand side of Equation (3) be zero just as Damour et al. did (Damour et al. 1993) and give the difference between this solution and the solution that is free from all torques.

Figure 7 shows the effect of geodetic precession and the gravitomagnetic effect due to motion of the Earth and other bodies.

To discover how much influence the post-Newtonian torque and the inertial torque have, we compare them with Newtonian multipole moments.

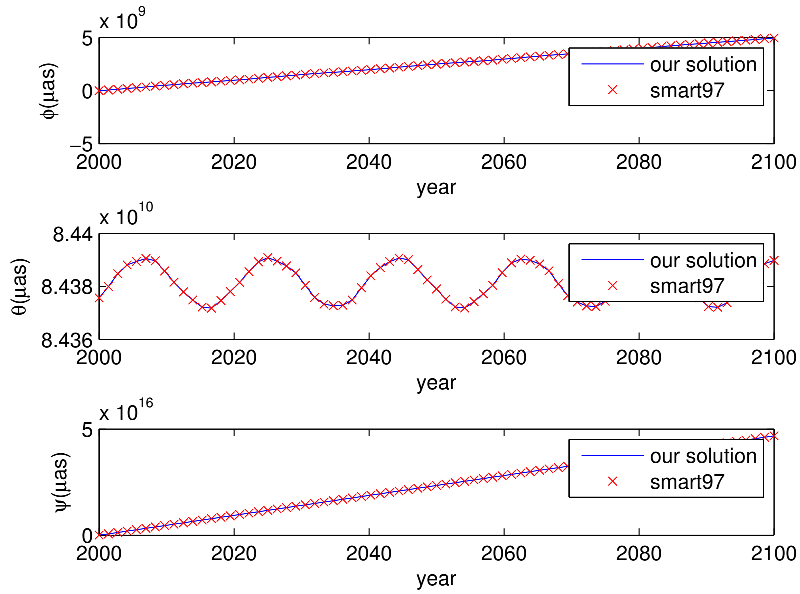

Fig. 3 Comparison of our numerical solution with that generated by SMART 97.

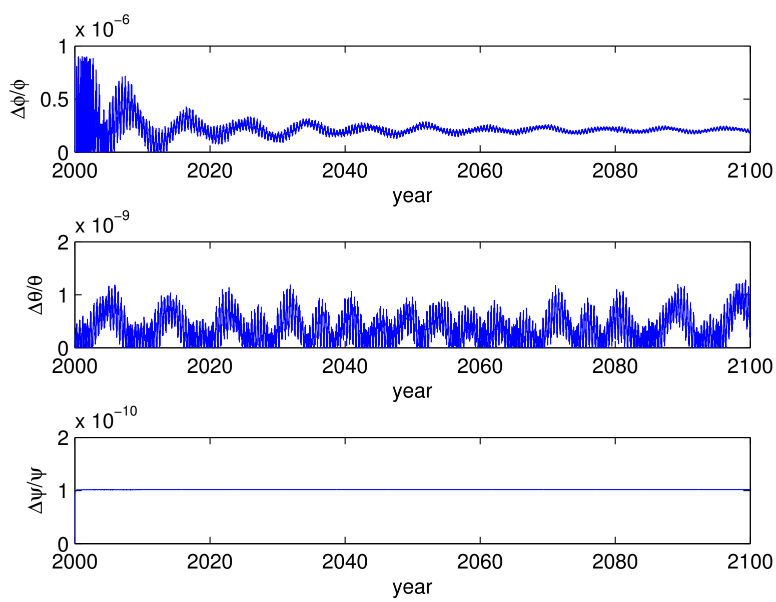

Fig. 4 Relative error of the three Euler angles between our solution and that generated by SMART 97.

Figures 7 and 8 display the effects of Newtonian multipole moments. The magnitudes of the effects of post-Newtonian torque are the same as the effects of fifth-degree harmonics and the main periods of fifth-degree harmonics are 27.3 d, 18.6, 80.1 and 2.9 yr (for ϕ). The effects of inertial torque in the range of 600 yr and fourth-degree harmonics in ϕ, ψ have the same orders of magnitude, but the global behaviors are completely different; the dominant periods of the effects of fourth-degree harmonics are the same as those of fifth-degree harmonics and the effects of inertial torque have no explicit periods in the range of 600 yr.

5 CONCLUSIONS

The goal of our work is to compute the Moon's rotation in the framework of general relativity. The motion of the Moon is decomposed into orbital motion and rotational motion, but we only focus on the rotational motion. The orbital motion of planets in the solar system comes from JPL ephemeris DE405. We construct the SCRS which is kine-

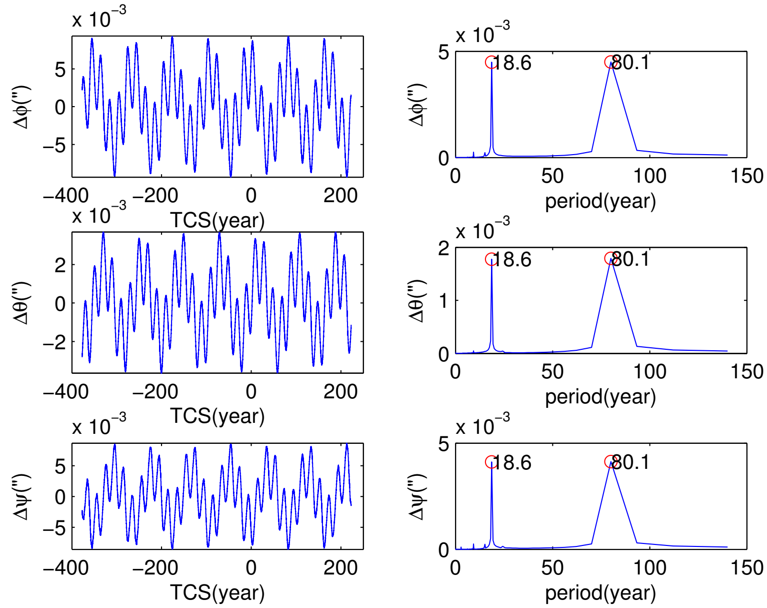

Fig. 5 Left panels: sum of the effects caused by the total relativistic torque. The largest differences shown by the three Euler angles are about 9 milliarcseconds (mas) in ϕ, ψ and 4 mas in θ. Right panels: spectra of the time series in the left panels.

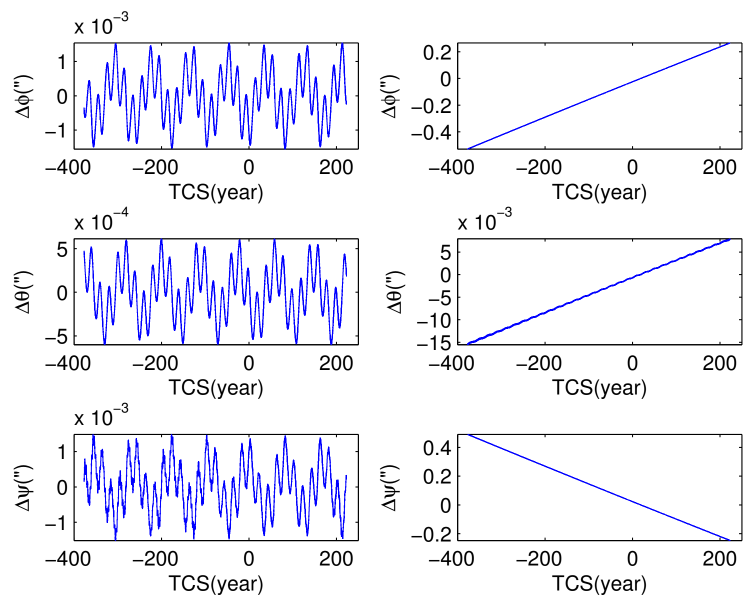

Fig. 6 *Left panels*: the variation of three Euler angles due to the post-Newtonian torque in the time domain. *Right panels*: effects of geodetic precession and the gravitomagnetic effect due to motion of the Earth and other bodies in the range of 600 yr. The change in ϕ and ψ is about 0.8 arcsec and in θ it is about 20 mas.

matically non-rotating in the post-Newtonian cases and provide the transformation between TCS and TCB. The post-Newtonian equations of the Moon's rotation are given in the SCRS. In our work, we consider two relativistic reference systems: SCRS and BCRS. Three time scales of TCS, TCB and TDB and relevant relativistic scaling of parameters are taken into account. The geodetic precession is treated as an additional torque on the right hand side of Equation (3). Afterwards, we numerically integrate the post-Newtonian equations to derive the time series of three Euler angles which describe the Moon's rotation and obtain the difference caused by total relativistic torque. After this, we give the effects of post-Newtonian torque and inertial torque. We find that the effects of total relativistic

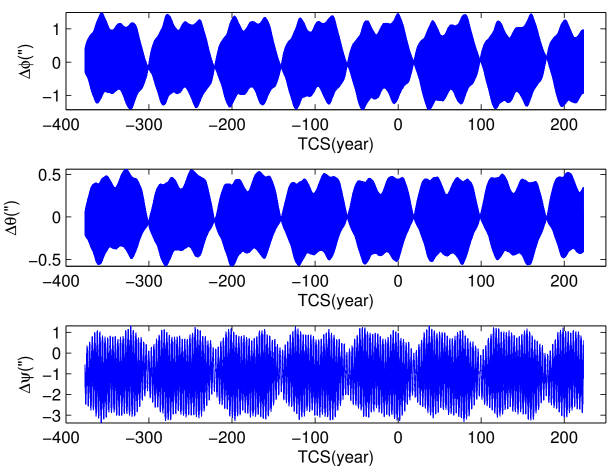

Fig. 7 Effects of fourth-degree harmonics on the Moon's rotation.

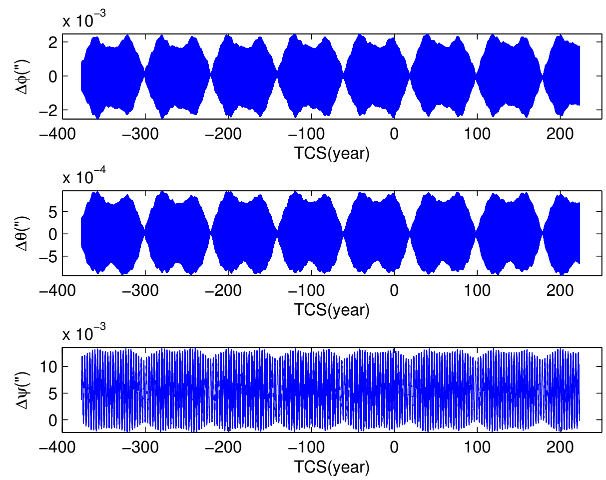

Fig. 8 Effects of fifth-degree harmonics on the Moon's rotation.

torque are larger than those from the Earth (Klioner et al. 2010). The amplitudes of the effects of total relativistic torque are about 9 mas in ϕ, ψ and 4 mas in θ, and their periods are 18.6 yr and 80.1 yr respectively which coincide with important periods in the Moon-Earth-Sun system.

In this work, we do not consider non-rigid aspects of the Moon such as elastic deformation and rotational deformation in terms of general relativity. This should be investigated in the near future. However, this is the first time that the rotation of the Moon has been calculated in a rigorous relativistic frame and these results should offer a reference for both theoretical research and observations of the Moon's rotation.

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