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Second post-Minkowskian order harmonic metric for a moving Kerr-Newman black hole

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Abstract We apply the Lorentz boosting method to the Kerr-Newman metric in harmonic coordinates, and obtain the second post-Minkowskian order harmonic metric for a moving Kerr-Newman black hole with an arbitrary constant speed. This metric may be useful for investigating observable relativistic effects due to the motion of the moving source. As an application, the post-Newtonian equations of motion for a particle and a photon in the far field of this black hole are calculated.

Key words: black hole physics — gravitation

1 INTRODUCTION

The deflection of test particles by a moving gravitational source is attracting more and more attention. Previous works have mainly concentrated on the effect of velocity in first-order deflection (Pyne & Birkinshaw 1993; Kopeikin & Schäfer 1999; Sereno 2002, 2005; Wucknitz & Sperhake 2004; Heyrovský 2005; Kopeikin & Makarov 2007), since the magnitude of the bending angle is rather small in astronomy measurements. Kopeikin & Schäfer (1999) investigated the relativistic perturbation on light propagation by the motion of an arbitrarily moving N-body system, which was later extended to study light deflection due to gravitomagnetism to leading order by sources with angular momentum (Kopeikin & Mashhoon 2002). Based on the Lorentz boost technique, Wucknitz & Sperhake (2004) tackled the effect of relativistic velocity of a uniformly moving deflector on gravitational deflection of light and a particle. However, with progress associated with μarcsec resolution in astronomical observations, such as the proposed *Space Interferometry Mission* (Laskin 2006; Turyshev 2009) and the *Gaia* mission (Lindegren et al. 2008), detection of the effect of velocity to higher-order (especially second order) gravitational deflection may be feasible in the near future. In order to consider the effect of motion in second order deflection, one must obtain the time-dependent metric of the background. Recently, harmonic metrics of arbitrarily and constantly moving Schwarzschild and Kerr black holes have been derived in He & Lin (2014a,b).

In this paper we take a step further, and extend the method of Lorentz transformation to calculating the harmonic metric to the second post-Minkowskian order (2PM) for a moving Kerr-Newman black hole with an arbitrary and constant speed. The post-Newtonian equations of motion for a particle and a photon in this field are also given. We work in geometrized units where $G = c = 1$.

2 2PM HARMONIC METRIC FOR A MOVING KERR-NEWMAN BLACK HOLE

Let us begin with the harmonic metric of a Kerr-Newman black hole. We assume $\boldsymbol{e}_i$ $(i = 1, 2, 3)$ to be the unit vector in 3-dimensional Cartesian coordinates. As shown in Lin & Jiang (2014), the exact harmonic metric of a Kerr-Newman black hole in the barycenter's rest frame $(X_0,\ X_1,\ X_2,\ X_3)$ takes the form

$$
\begin{aligned}
ds^2 \;=\; & -dX_0^2 + \frac{R^2(R+m)^2 + a^2X_3^2}{\left(R^2 + \frac{a^2}{R^2}X_3^2\right)^2}\left[\frac{\left(\mathbf{X}\cdot d\mathbf{X} + \frac{a^2}{R^2}X_3dX_3\right)^2}{R^2+a^2-m^2+Q^2} + \frac{X_3^2}{R^2}\frac{\left(\mathbf{X}\cdot d\mathbf{X} - \frac{R^2}{X_3}dX_3\right)^2}{R^2-X_3^2}\right] \\
& + \frac{(R+m)^2+a^2}{R^2-X_3^2}\left[\frac{aR^2(m^2-Q^2)(R^2-X_3^2)\left(\mathbf{X}\cdot d\mathbf{X} + \frac{a^2}{R^2}X_3dX_3\right)}{(R^2+a^2-m^2+Q^2)(R^2+a^2)(R^4+a^2X_3^2)} + \frac{R\,(X_2dX_1 - X_1dX_2)}{R^2+a^2}\right]^2 \\
& + \frac{2m(R+m)-Q^2}{(R+m)^2 + \frac{a^2}{R^2}X_3^2}\left[\frac{a^2R(m^2-Q^2)\,(R^2-X_3^2)\left(\mathbf{X}\cdot d\mathbf{X} + \frac{a^2}{R^2}X_3dX_3\right)}{(R^2+a^2-m^2+Q^2)(R^2+a^2)\,(R^4+a^2X_3^2)}\right. \\
& \left. + \frac{a\,(X_2dX_1 - X_1dX_2)}{R^2+a^2} + dX_0\right]^2 , \qquad (1)
\end{aligned}
$$

where m, a and Q stand for the rest mass, angular momentum per mass and electric charge of the black hole, respectively. The angular momentum $\boldsymbol{J}$ of the Kerr-Newman black hole is expressed as

$$
J\boldsymbol{e}_3 (= am\boldsymbol{e}_3)\,.
$$

$$
\mathbf{X}\cdot d\mathbf{X} \equiv X_1dX_1 + X_2dX_2 + X_3dX_3\,,
$$

$$
\frac{X_1^2+X_2^2}{R^2+a^2} + \frac{X_3^2}{R^2} = 1
$$

and the relation $m^2 \geq a^2 + Q^2$ has been used to avoid a naked singularity for the black hole. Notice that X_μ denotes the contravariant vector $x'^\mu = (t',\ x',\ y',\ z')$ for notational convenience.

Here we only consider the approximate metric to second post-Minkowskian order within which the leading effects of electric charge and intrinsic angular momentum of the black hole emerge. According to Equation (1), the harmonic metric up to the order of $1/R^2$ for the Kerr-Newman black hole is simplified to

$$
g_{00} = -1 - 2\Phi - 2\Phi^2 - \frac{Q^2}{R^2} + O(1/R^3)\,, \qquad (2)
$$

$$
g_{0i} = \zeta_i + O(1/R^3)\,, \qquad (3)
$$

$$
g_{ij} = (1-\Phi)^2\delta_{ij} + \frac{m^2-Q^2}{R^2}\frac{X_iX_j}{R^2} + O(1/R^3)\,, \qquad (4)
$$

where $i,\ j = 1,\ 2,\ 3$, and δ_{ij} denotes Kronecker delta. $\boldsymbol{\zeta} \equiv \frac{2am}{R^3}(\boldsymbol{X}\times\boldsymbol{e_3})$ and $\Phi \equiv -m/R$ represents the Newtonian gravitational potential.

According to the general covariance of field equations, we can apply a Lorentz boost to Equations (2)–(4) to get the metric of a constantly moving Kerr-Newman black hole. We denote the coordinate frame of the background as $(t,\ x,\ y,\ z)$, and the translational velocity in an arbitrary direction of the moving Kerr-Newman black hole is generally expressed as $\boldsymbol{v} = v_1\boldsymbol{e}_1 + v_2\boldsymbol{e}_2 + v_3\boldsymbol{e}_3$. The Lorentz transformation between $(t,\ x,\ y,\ z)$ and comoving frame $(t',\ x',\ y',\ z')$ of the moving black hole is

$$
x'^\alpha = \Lambda^\alpha_\beta x^\beta\,, \qquad (5)
$$

and

$$\Lambda^\alpha_\beta = \begin{pmatrix} \gamma & -v_1\gamma & -v_2\gamma & -v_3\gamma \\ -v_1\gamma & 1+\frac{v_1^2(\gamma-1)}{v^2} & \frac{v_1v_2(\gamma-1)}{v^2} & \frac{v_1v_3(\gamma-1)}{v^2} \\ -v_2\gamma & \frac{v_1v_2(\gamma-1)}{v^2} & 1+\frac{v_2^2(\gamma-1)}{v^2} & \frac{v_2v_3(\gamma-1)}{v^2} \\ -v_3\gamma & \frac{v_1v_3(\gamma-1)}{v^2} & \frac{v_2v_3(\gamma-1)}{v^2} & 1+\frac{v_3^2(\gamma-1)}{v^2} \end{pmatrix} , \tag{6}$$

where $\gamma = (1-v^2)^{-\frac{1}{2}}$ is the Lorentz factor and $v^2 = v_1^2 + v_2^2 + v_3^2$. The 2PM harmonic metric for the arbitrarily constantly moving Kerr-Newman black hole can be obtained as follows:

$$\begin{aligned} g_{00} &= -1-2(1+v^2)\gamma^2\Phi - (1+\gamma^2)\Phi^2 - \frac{\gamma^2 Q^2}{R^2} - 2\gamma^2\,(\boldsymbol{v}\cdot\boldsymbol{\zeta}) \\ &\quad + \frac{\gamma^2(m^2-Q^2)}{R^2}\frac{(\boldsymbol{v}\cdot\boldsymbol{X})^2}{R^2} , \qquad (7) \\ g_{0i} &= v_i\gamma^2\left(4\Phi+\Phi^2+\frac{Q^2}{R^2}\right) - \frac{\gamma(m^2-Q^2)}{R^2}\frac{(\boldsymbol{v}\cdot\boldsymbol{X})}{R^2}\left[X_i + \frac{v_i(\gamma-1)\,(\boldsymbol{v}\cdot\boldsymbol{X})}{\boldsymbol{v}^2}\right] \\ &\quad + \gamma\left[\zeta_i + v_i\left(\gamma+\frac{\gamma-1}{\boldsymbol{v}^2}\right)(\boldsymbol{v}\cdot\boldsymbol{\zeta})\right] , \qquad (8) \\ g_{ij} &= (1-\Phi)^2\delta_{ij} - v_iv_j\gamma^2\left(4\Phi+\Phi^2+\frac{Q^2}{R^2}\right) + \frac{m^2-Q^2}{R^4} \\ &\quad \times \left[X_i + \frac{v_i(\gamma-1)(\boldsymbol{v}\cdot\boldsymbol{X})}{\boldsymbol{v}^2}\right]\left[X_j + \frac{v_j(\gamma-1)(\boldsymbol{v}\cdot\boldsymbol{X})}{\boldsymbol{v}^2}\right] \\ &\quad - \gamma\left[v_i\zeta_j + v_j\zeta_i + 2\,(\boldsymbol{v}\cdot\boldsymbol{\zeta})\,\frac{v_iv_j(\gamma-1)}{\boldsymbol{v}^2}\right] . \qquad (9) \end{aligned}$$

Equations (7)–(9) reduce to the 2PM metric of a moving Kerr black hole (He & Lin 2014b) when there is no electric charge ($Q = 0$). When both the charge and the angular momentum are zero ($Q = a = 0$), these equations reduce to the 2PM harmonic metric of a moving Schwarzschild black hole with an arbitrary constant speed, which reads

$$\begin{aligned} g_{00} &= -1-2(1+v^2)\gamma^2\Phi - (1+\gamma^2)\Phi^2 + \frac{\gamma^2\Phi^2\,(\boldsymbol{v}\cdot\boldsymbol{X})^2}{R^2} , \qquad (10) \\ g_{0i} &= v_i\gamma^2\left(4\Phi+\Phi^2\right) - \frac{\gamma\Phi^2\,(\boldsymbol{v}\cdot\boldsymbol{X})}{R^2}\left[X_i + \frac{v_i(\gamma-1)\,(\boldsymbol{v}\cdot\boldsymbol{X})}{\boldsymbol{v}^2}\right] , \qquad (11) \\ g_{ij} &= (1-\Phi)^2\delta_{ij} - v_iv_j\gamma^2\left(4\Phi+\Phi^2\right) + \frac{\Phi^2}{R^2}\left[X_i + \frac{v_i(\gamma-1)(\boldsymbol{v}\cdot\boldsymbol{X})}{\boldsymbol{v}^2}\right] \\ &\quad \times \left[X_j + \frac{v_j(\gamma-1)(\boldsymbol{v}\cdot\boldsymbol{X})}{\boldsymbol{v}^2}\right] . \qquad (12) \end{aligned}$$

3 POST-NEWTONIAN DYNAMICS OF A PARTICLE AND A PHOTON

The resulting metric can be used to calculate the post-Newtonian equations of motion for a test particle. We consider a particle or a photon in the far field of a moving Kerr-Newman black hole. The geometrical relations of the gravitational source and the test particle are shown in Figure 1.

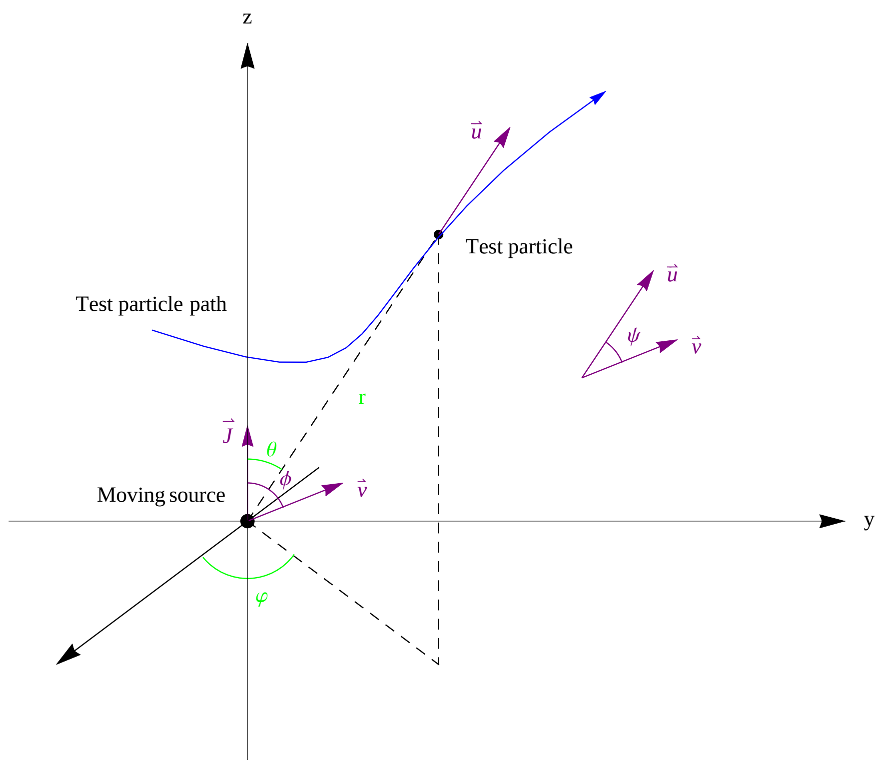

Fig. 1 Schematic geometrical relations of the moving Kerr-Newman black hole at $(0,\ 0\ ,0)$ and test particle at $(r,\ \theta\ ,\varphi)$ for some moment t. $\boldsymbol{v}$ and $\boldsymbol{u}$ are the instantaneous velocity vectors of the source and a test particle, respectively. The blue line is the motion trajectory of a particle or null path in the time-dependent gravitational field. $\phi \in [0,\ \pi]$ denotes the angle between the constant velocity $\boldsymbol{v}$ and the angular momentum $\boldsymbol{J}$ of the black hole and ψ is the angle between $\boldsymbol{v}$ and $\boldsymbol{u}$.

When the source's velocity is non-relativistic, Equations (7)–(9) can be simplified as

$$g_{00} = -1 - 2(1+2v^2)\Phi - 2\Phi^2 - 2\boldsymbol{v}\cdot\boldsymbol{\zeta} - \frac{Q^2}{R^2} + O(\overline{v}^6)\ , \tag{13}$$

$$g_{0i} = 4v_i\Phi + \zeta_i + O(\overline{v}^5)\ , \tag{14}$$

$$g_{ij} = (1-2\Phi)\delta_{ij} + O(\overline{v}^4)\ , \tag{15}$$

where $\overline{v}$ denotes the typical velocity of a non-relativistic system in the post-Newtonian approximation (Weinberg 1972). Notice that $\overline{v}$ is different from the magnitude v of the gravitational source's velocity.

For a particle, up to the order of $\overline{v}^5/\overline{r}$ with $\overline{r}$ being the typical separation of a non-relativistic system, we obtain the equation of motion as follows

$$\begin{aligned}\frac{d\boldsymbol{u}}{dt} =& -\nabla\left(\Phi+2v^2\Phi+2\Phi^2+\boldsymbol{v}\cdot\boldsymbol{\zeta}+\frac{Q^2}{2R^2}\right)-\frac{\partial\boldsymbol{\xi}}{\partial t}+\boldsymbol{u}\times(\nabla\times\boldsymbol{\xi})+3\boldsymbol{u}\frac{\partial\Phi}{\partial t}\\ &+4\boldsymbol{u}\,(\boldsymbol{u}\cdot\nabla)\,\Phi-\boldsymbol{u}^2\nabla\Phi\ ,\end{aligned} \tag{16}$$

where $\boldsymbol{\xi} = 4\boldsymbol{v}\Phi + \boldsymbol{\zeta}$. The equation of motion for a photon up to the order $\overline{v}^3$ reads

$$\frac{d\boldsymbol{u}}{dt} = -(1+\boldsymbol{u}^2)\nabla\Phi + 4(1-\boldsymbol{v}\cdot\boldsymbol{u})\boldsymbol{u}\,(\boldsymbol{u}\cdot\nabla)\,\Phi + \boldsymbol{u}\times[\nabla\times(4\boldsymbol{v}\Phi)] + (3-\boldsymbol{u}^2)\boldsymbol{u}\frac{\partial\Phi}{\partial t}\,. \tag{17}$$

It is worth pointing out that Equations (16)–(17) are consistent with the results for the case of the uncharged source, see equations (9.2.1) and (9.2.6) in the textbook Weinberg (1972).

4 CONCLUSIONS

In this paper, we have derived the 2PM harmonic metric of a uniformly moving Kerr-Newman black hole. We also apply this metric to obtain the post-Newtonian equations of motion for a particle and a photon in this field. The resulting metric can also be used to study relativistic effects, e.g., in the second post-Minkowskian deflection or the time delay of light, etc.

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