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A long time span relativistic precession model of the Earth *

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Abstract A numerical solution to the Earth's precession in a relativistic framework for a long time span is presented here. We obtain the motion of the solar system in the Barycentric Celestial Reference System by numerical integration with a symplectic integrator. Special Newtonian corrections accounting for tidal dissipation are included in the force model. The part representing Earth's rotation is calculated in the Geocentric Celestial Reference System by integrating the post-Newtonian equations of motion published by Klioner et al. All the main relativistic effects are included following Klioner et al. In particular, we consider several relativistic reference systems with corresponding time scales, scaled constants and parameters. Approximate expressions for Earth's precession in the interval ± 1 Myr around J2000.0 are provided. In the interval ± 2000 years around J2000.0, the difference compared to the P03 precession theory is only several arcseconds and the results are consistent with other long-term precession theories.

Key words: astrometry — ephemeris — post-Newtonian

1 INTRODUCTION

The P03 precession theory (Capitaine et al. 2003) was adopted by IAU 2006 Resolution B1 as the IAU precession model to replace the precession component of the IAU 2000A precession-nutation model, beginning on 2009 January 1. The precession of the ecliptic in P03 is derived from the analytical theory VSOP87 (Bretagnon & Francou 1988) that was fitted to the JPL ephemeris DE406; it has taken advantage of VLBI observations and incorporated a dynamical theory of the Moon to develop expressions for the precession of the equator based on theoretical contributions to precession (Williams 1994, W94) and on MHB (Mathews et al. 2002) estimates of the precession rates. The P03 theory is given as polynomial expressions of various precession quantities, which are known to be very accurate over a few centuries, but to diverge rapidly from numerical integration for more distant epochs.

Precession expressions valid for long time intervals have been derived by Vondrák et al. (2011) to provide an extension of IAU 2006 to scales of several thousand centuries. They use the Mercury 6 package (Chambers 1999) for the ecliptic and the La93 (Laskar et al. 1993) solution to represent

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general precession p_A and the precession of obliquity ϵ_A. Some corrections are added due to the IAU 2006 solutions. This long-term precession is then expressed in the form of a cubic polynomial plus 8 to 14 periodic terms. It is consistent with the IAU 2006 precession in the vicinity of J2000.0, with differences being less than a few arcseconds throughout the historical period. The accuracy of this precession model reaches a few tenths of a degree at the extreme epochs ± 200 millennia from J2000.0. The interval was intentionally reduced to two thousand centuries, because of the limited predictive knowledge of the changes in the dynamical ellipticity of the Earth and the tidal dissipation in the Earth-Moon system. However, this work does not consider effects from General Relativity.

Some other authors also investigated the orbital motion of the Earth and Earth's rotation for longer time spans. A $3\,$Myr integration was made by Quinn et al. (1991). The model used in this work is based on classical mechanics and it replaced the Moon by a ring lying in the plane of the ecliptic. They estimated that their errors in the Earth's position and the direction of Earth's pole are within about $6\,200''$. In addition, Laskar et al. derived solutions for the general precession p_A and obliquity ϵ_A of the Earth in La93 (Laskar et al. 1993) and La2004 (Laskar et al. 2004) based on the rigid-Earth theory of Kinoshita (Kinoshita 1977), and the orbital motion of the Earth spanning the time from -250 to $0\,$Myr in La2010 (Laskar et al. 2011). Their dynamical models all include the dominant relativistic corrections: the 1PN corrections due to the Sun and the geodetic precession. Their standard way to account for geodetic precession is to solve the purely Newtonian equations of rotational motion and add the geodetic precession as a correction to the solution, which is not fully consistent with General Relativity.

Recently, Klioner et al. (2010) have constructed a relativistic theory of Earth's rotation. According to the post-Newtonian equations of rotational motion given by Klioner et al. (2003), they explain how to calculate relativistic torque, and discuss how to deal with different relativistic reference systems including time scales and relativistic scaling. Geodetic precession and nutation are also taken into account in a natural way. This theory of Earth's rotation is consistent with General Relativity. A numerical integration of Earth's rotation in the limit of rigidly rotating multipoles over several centuries is made and a comparison of the result with SMART97 shows that they have succeeded in repeating SMART97 within the full accuracy of the latter. This approach allows us to obtain the long-term precession of the Earth in a more rigorous relativistic framework.

Our work is to obtain the Earth's long-term precession in a relativistic framework. The precession of the ecliptic is obtained by numerical integration as in most previous works. However, the precession of the equator is calculated with a relativistic theory of Earth rotation which is mentioned above. This part of the work starts with a post-Newtonian rigid-multipole formalism that has been published by Klioner et al. (2003). Then, the equations are integrated numerically and the results are modified due to the effect from tidal dissipation, and an approximation for the precession is derived and expressed in the form of a linear polynomial plus 20-30 periodic terms. Finally the relativistic effects on the precession are obtained and analyzed.

In this paper, expressions for the relativistic long term precession of the Earth are given. In Section 2, we describe the way to calculate the precession in detail. An approximation for the precession is provided in Section 3. Finally in Section 4, the influences of relativistic effects on the precession are discussed.

2 NUMERICAL INTEGRATION FOR THE PRECESSION

2.1 Precession of the Ecliptic

The precession of the ecliptic represents the motion of the ecliptic pole, relative to a fixed ecliptic, due to planetary perturbations. The basic quantities are $P_A = \sin\pi_A \sin\Pi_A$ and $Q_A = \sin\pi_A \cos\Pi_A$, where π and Π are the osculating elements of the orbit of the Earth-Moon barycenter (π, the inclination and Π, the longitude of ascending node), and these angles refer to the fixed ecliptic at J2000.0. Here the motion of the solar system is integrated in the Barycentric Celestial

Reference System (BCRS), with the $x-y$ plane rotated into the J2000 ecliptic frame, that has its x-axis through the J2000 (inertial) mean equinox and its z-axis through the J2000 ecliptic pole. Barycentric Dynamical Time (TDB) is used as the coordinate time. The P03 theory gives the transformation from the DE406 frame to the J2000 ecliptic frame the J2000 in the form of the rotation matrix $R_z\,(0.03862'')\,R_x\,(84381.40889'')\,R_z\,(-0.05132'')$ in the DE406 heliocentric coordinates (Capitaine et al. 2003). Then the motion of the solar system will be obtained in the BCRS.

For the barycentric translational motion, we use a similar dynamical model as in La2010. The Sun, all eight planets of the solar system and Pluto are all taken into account. The Moon is treated as a separate object. The first post-Newtonian correction due to the Sun is considered by following Saha & Tremaine (1994). We also take into account the effects of the quadruple moment of the Sun and the Earth, the solar mass loss and tidal dissipation in the Earth-Moon System.

Our numerical integrator is based on the symplectic SABA4 scheme (Laskar & Robutel 2001; Wu et al. 2003). The results are modified due to tidal dissipation in the Earth-Moon system. To reduce round-off errors, we use compensated summation in our program. The integration is started at J2000.0 and goes to ± 1 Myr. With regard to the period of the Moon's orbit, the stepsize is set to one day. The initial conditions of the integration are taken from the JPL DE406 ephemeris. The main constants used here are listed in Appendix A.

After integrating the solar system motion, we directly calculate the orbit of the Earth-Moon System from their positions and velocities, and then smooth the orbit by the Vondrák method (Vondrák 1969). The precession parameters P and Q are finally obtained.

The motion of the Earth-Moon barycenter in the solar system over this time span is well known. Quinn et al. (1991) calculate $p = \sin\pi\sin\Pi$ and $q = \sin\pi\cos\Pi$ measured in ecliptic coordinates and give a solution (QTD) for the motion of this barycenter over 3.05 Myr with an accuracy better than $6200''$. Furthermore, orbital solutions for the long term motion of the Earth are given by Laskar et al. (2011). Our aim is to test our integrator and get a solution that is consistent with those solutions.

Figures 1 and 2 show comparisons of our solution with the solution QTD for p and q over the past 1 Myr from J2000.0; relative differences are smaller than 0.01.

2.2 Precession of the Equator

The precession of the equator describes the motion of the mean equator with respect to a fixed plane, due to the luni-solar and planetary torques acting on the oblate Earth. In this paper, we represent it by the general precession in longitude $p_{\rm A}$ and the mean obliquity of date $\epsilon_{\rm A}$, which is the orientation angle of the mean equatorial plane with respect to the ecliptic plane. These precession quantities are obtained from the motion of the Earth's spin axis, mostly determined by torques from the Sun and the Moon, with a small contribution from General Relativity.

Rotation of the Earth is modeled in the Geocentric Celestial Reference System (GCRS) as indicated in Klioner et al. (2010). IAU 2000 resolutions B give the transformation between the BCRS and the GCRS. Another important reference system is the terrestrial reference system in which the model of Earth's gravity field with potential coefficients C_{lm} and S_{lm} is defined. The spatial coordinates of this terrestrial reference system (ξ, η, ζ) are obtained by rotating the GCRS spatial coordinates (X, Y, Z) with a time-dependent matrix $P^{ab} = R_z\,(\phi)\,R_x\,(\omega)\,R_z\,(\psi)$. The meaning of the angles ϕ, ψ, ω and the terrestrial system (ξ, η, ζ) here is the same as in Bretagnon et al. (1997). This implies that the sign convention for ψ is not in agreement with the traditional astronomical one. These Euler angles are defined in the GCRS.

The post-Newtonian equation of Earth's rotation that we use reads

$$\frac{d}{dT}\left(C^{ab}\omega^b\right) = \sum_{l=1}^{\infty}\frac{1}{l!}\epsilon_{abc}M_{bL}G_{cL} + \epsilon_{abc}\Omega^b_{\rm iner}C^{cd}\omega^d + L^a_{\rm other}, \tag{1}$$

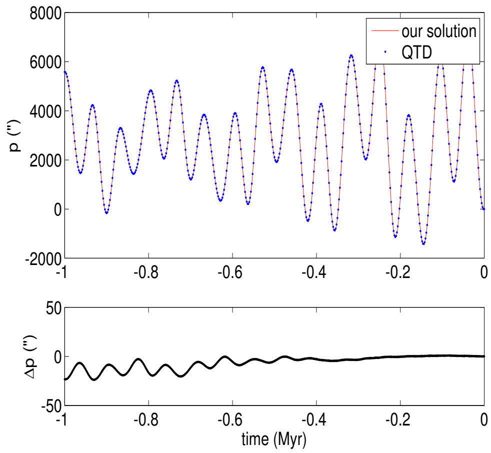

Fig. 1 *Top panel*: comparision between our solution (*solid line*) and the QTD solution (*dotted line*) (Quinn et al., 1991) for p over the past 1 Myr from J2000.0. *Bottom panel*: the difference between the two solutions (dotted line).

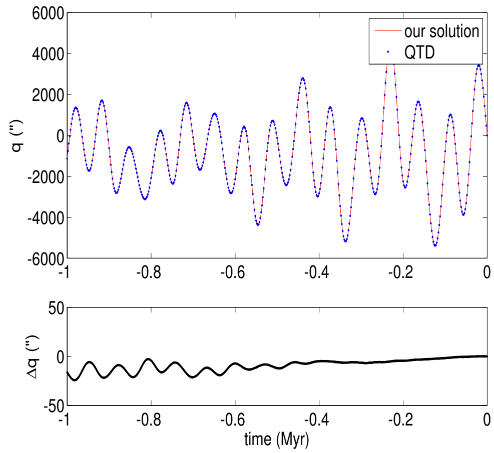

Fig. 2 *Top panel*: comparision between our solution (*solid line*) and the QTD solution (*dotted line*) (Quinn et al., 1991) for q over the past 1 Myr from J2000.0. *Bottom panel*: the difference between the two solutions (*dotted line*).

where $\boldsymbol{C} = C^{ab}$ is the post-Newtonian inertia tensor and $\boldsymbol{\omega} = \omega^a$ is the angular velocity of the post-Newtonian Tisserand axis (Klioner 1996) defined by the orthogonal matrix $P^{ab}(T)$. The first term on the right hand side is the relativistic inertial torque (without relativistic precessions). The second term is the additional torque due to the relativistic precessions (Geodetic, Lense-Thirring and Thomas precessions). The third term describes tidal dissipation. Details will be given below.

The motion of the solar system is described in the BCRS with TDB as time scales, while the rotation of the Earth is described in the GCRS using Geocentric Coordinate Time (TCG) as the coordinate time scale parameterizing the equations. So, the problem of how to treat these different relativistic time scales is inevitable. For different TDB times, we calculate the corresponding Barycentric Coordinate Times (TCB), Terrestrial Times (TT) and TCG times, as described in Irwin & Fukushima (1999) and Klioner (2008). The differential equations for these time scales are numerically integrated by Romberg's method using our solutions for motion in the solar system. Another important part of our program is to deal with the relativistic scaling of various parameters. The proper relativistic scaling of constants and parameters is chosen to be like that described in Klioner et al. (2010):

(a) The position $\boldsymbol{x}_{\mathrm{A}}$, velocity $\boldsymbol{v}_{\mathrm{A}}$, the acceleration $\boldsymbol{a}_{\mathrm{A}}$ which we calculate and the mass parameter GM_{A} of a massive solar system body A (given by IAU 2009 Resolution B2) are TDB-compatible.
(b) The radius of the Earth R_{E} (given by IAU 2009 Resolution B2) and the values of the Earth's moments of inertia $\boldsymbol{C}$ are TT-compatible.
(c) The post-Newtonian inertial torque $L^a = \sum_{l=1}^{\infty} \frac{1}{l!} \epsilon_{abc} M_{bL} G_{cL}$ is calculated by using TDB-compatible parameters (see Klioner et al. 2010).

To compute the torque, we only take into account the influence of the Sun and the Moon. It contains the following:

(a) The relativistic inertial torque is expressed with Symmetric and Trace-Free Cartesian (STF) tensors. M_L are the multipole moments of the Earth defined in the GCRS, while G_L are the multipole moments of the external tidal gravitoelectric field in the GCRS. The formulas for M_L are given by equations (5.1)–(5.6) of Klioner et al. (2010) and G_L are from equations (19)–(23) of Klioner et al. (2003). Here we only consider terms with $l = 2$.
(b) The additional torque $\epsilon_{abc}\Omega^b_{\mathrm{iner}}C^{cd}\omega^d$ depends on $\boldsymbol{C}$, $\boldsymbol{\omega}$ and the angular velocity $\boldsymbol{\Omega}_{\mathrm{iner}}$ describing the relativistic precessions. In our work, only geodetic precession and nutation are considered (Fukushima 1991).
(c) The torque from the tidal dissipation describes another important effect in the Earth-Moon system. It results from the tidal forces of the Sun (neglected here) and the Moon on the Earth, and induces small changes in the speed of rotation of the Earth and in the mean motion of the Moon. Here we use the tidal dissipation model given by Mignard (1979) and Touma & Wisdom (1994). It assumes that the torque resulting from tidal friction is proportional to the time lag Δt that the deformation takes to reach equilibrium. This time lag is supposed to be constant.

Explicitly, the tidal torque acting on Earth reads

$$\Delta \boldsymbol{L} = 3\frac{k_2 G M_{\mathrm{M}}^2 R_{\mathrm{E}}^5}{r_{\mathrm{EM}}^8}\Delta t\left[\left(\boldsymbol{r}_{\mathrm{EM}}\cdot\boldsymbol{\omega}_{\mathrm{E}}\right)\boldsymbol{r}_{\mathrm{EM}} - r_{\mathrm{EM}}^2\boldsymbol{\omega}_{\mathrm{E}} + \boldsymbol{r}_{\mathrm{EM}}\times\boldsymbol{v}_{\mathrm{EM}}\right], \tag{2}$$

where k_2 is the potential Love number of the Earth. There is also a force acting on the Moon due to a delayed tidal bulge on the Earth

$$\boldsymbol{F} = -3\frac{k_2 G M_{\mathrm{M}}^2 R_{\mathrm{E}}^5}{r_{\mathrm{EM}}^{10}}\left\{r_{\mathrm{EM}}^2\boldsymbol{r}_{\mathrm{EM}} + \Delta t\left[2\boldsymbol{r}_{\mathrm{EM}}\left(\boldsymbol{r}_{\mathrm{EM}}\cdot\boldsymbol{v}_{\mathrm{EM}}\right) + r_{\mathrm{EM}}^2\left(\boldsymbol{r}_{\mathrm{EM}}\times\boldsymbol{\omega}_{\mathrm{E}} + \boldsymbol{v}_{\mathrm{EM}}\right)\right]\right\}. \tag{3}$$

This force is added to the equation of motion for the Earth-Moon system. As a consequence of the decreasing angular rotation rate of the Earth, the moment J_2 of the Earth will also change in proportion to $\boldsymbol{\omega}^2$.

Klioner et al. (2010) try to get the most accurate results for precession/nutation for a relatively short interval of time including relativistic effects. In this paper, we focus on very long time scales and integrate the rotational equation of motion, Equation (1), using a 4th-order Runge-Kutta method with a stepsize of 0.1 day. The potential coefficients of the gravity field of the Earth in the terrestrial system are computed from the GEM2008 normalized coefficients:

$$\begin{aligned} C_{20} &= -1082.626173852223\times 10^{-6}\,, \\ C_{22} &= 1.574615325722917\times 10^{-6}\,, \\ S_{22} &= -0.9038727891965667\times 10^{-6}\,. \end{aligned}$$

Because of this and the different values of the constants we use, the moments of inertia are slightly different from the ones of SMART97 (Bretagnon et al. 1998):

$$\begin{aligned} A &= 1.799538227025858\times 10^{-15}\,M_{\mathrm{S}}\,\mathrm{au}^2\,, \\ B &= 1.799577876994722\times 10^{-15}\,M_{\mathrm{S}}\,\mathrm{au}^2\,, \\ C &= 1.805468786696834\times 10^{-15}\,M_{\mathrm{S}}\,\mathrm{au}^2\,. \end{aligned}$$

The initial conditions for $t_0 = J2000.0$ are from SMART97:

$$\begin{aligned} \psi\left(t_0\right) &= 0.00006789546085\,\mathrm{rad}\,, \\ \omega\left(t_0\right) &= -0.4090646190715125\,\mathrm{rad}\,, \\ \varphi\left(t_0\right) &= 4.8948989303002346\,\mathrm{rad}\,, \\ \dot{\psi}\left(t_0\right) &= -0.7010549586589918\times 10^{-6}\,\mathrm{rad\,d}^{-1}\,, \\ \dot{\omega}\left(t_0\right) &= 0.0960673662260632\times 10^{-6}\,\mathrm{rad\,d}^{-1}\,, \\ \dot{\varphi}\left(t_0\right) &= 6.30038813041313\,\mathrm{rad\,d}^{-1}\,. \end{aligned}$$

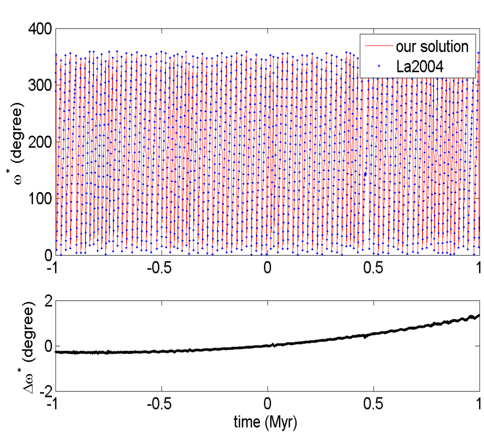

Fig. 3 *Top panel*: comparison of our solution (*solid line*) and La2004 (*dotted line*) (Laskar et al., 2004) for ω^* from -1 Myr to 1 Myr. *Bottom panel*: the difference between the two solutions (*dotted line*).

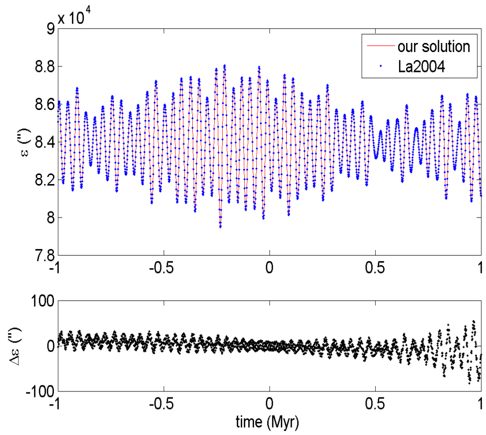

Fig. 4 *Top panel*: comparison of our solution (*solid line*) and La2004 (*dotted line*) (Laskar et al., 2004) for the obliquity of the Earth ϵ from -1 Myr to 1 Myr. *Bottom panel*: the difference between the two solutions (*dotted line*).

With these parameters and initial conditions, the rotational equation of motion will be numerically integrated. The parameters p_{A} and ϵ are computed from π, Π, ψ and ω.

The values for longitude of perihelion from moving equinox of the date ω^* and obliquity of the Earth from -50 to 20 Myr are provided by La2004 (Laskar et al., 2004) in which the precession quantities are integrated using the rigid-Earth theory of Kinoshita (Kinoshita 1977; Neron de Surgy & Laskar 1997).

The longitude of perihelion from the equinox of reference' is also a term. ω^* is defined as $\omega^* = \varpi + p_{\mathrm{A}}$, where ϖ is the longitude of perihelion derived from the equinox that is used as a reference. We made a comparison for ω^* and ϵ over ± 1 Myr with the results of Laskar et al. (2004). Figures 3 and 4 show that our numerical solution is close to the results of La2004, with relative differences being smaller than 0.01.

3 ANALYTICAL EXPRESSIONS FOR THE PRECESSION

3.1 Numerical Analysis

Some algorithms are applied to our data to get approximations for the precession parameters. To this end, we use a polynomial curve fitting with the least squares method. After removing the linear drift, a frequency analysis algorithm is used to search for periodic terms.

Frequency analysis has the goal of determining the fundamental frequencies for the numerical solution of a dynamical system and enables us to derive approximate analytic theories for the long-term behavior of the solar system. The method we use is a combination of the Numerical Analysis of Fundamental Frequency (NAFF) (Laskar et al. 1992) and Vaníček's method (Vaníček 1971):

(1) A Fast Fourier Transform (FFT) is applied to determine the frequency f of the largest amplitude.
(2) The least squares method is used to make a small change in f and derive good approximations for sine/cosine amplitudes of the corresponding term. A Hanning window is used to improve the frequency determination.
(3) The contribution with frequency f is removed from the data.

(4) The steps above are repeated to find other frequencies.

Precession represents the secular part of the motion. From Vondrák et al. (2011), we assume that precession covers all periods longer than 10^4 yr, while shorter ones are included in the nutation. In this paper, we use this frequency analysis method to extract the long periodic terms to get the precession of the Earth and remove the nutation part. The approximations of the precession parameters are given below in the general form $a_0 + a_1 T + \sum (C_i \cos 2\pi T/P_i + S_i \sin 2\pi T/P_i)$. T is the time from J2000.0 in years. P terms are the periods in years. The unit of the cosine/sine amplitudes of the periodic parts is arcseconds. All the coefficients are provided in Appendix B.

3.2 Precession Parameters

The long-term approximations for the precession of the ecliptic $P_{\rm A}$ and $Q_{\rm A}$ are presented as

$$\begin{aligned} P_{\rm A} &= 5540'' - 1.98'' \times 10^{-4} T + \textstyle\sum_{i=1}^{26} C_i \cos(2\pi T/P_i) + S_i \sin(2\pi T/P_i)\,, \\ Q_{\rm A} &= -1608'' - 2.06'' \times 10^{-4} T + \textstyle\sum_{i=1}^{26} C_i \cos(2\pi T/P_i) + S_i \sin(2\pi T/P_i)\,, \end{aligned} \tag{4}$$

where T is in TDB years, and the main periods P_i with the amplitudes C_i and S_i are given in Table 1. In the first column, the names of some special frequencies s_i are from Laskar (1985). s_i is the secular frequency related to the node of the planet i in the solar system, and can be obtained from the Lagrange solution of the autonomous system of order 1 by an analytical treatment. Comparisons of our numerical solutions for P and Q with their approximations (given in Eq. (4)) are depicted in Figures 5 and 6 from -1 Myr to 1 Myr. The difference in the two solutions is less than $200''$ over the whole period.

Table 1 The Main Periodic Terms in $P_{\rm A}$ and $Q_{\rm A}$

	$P_{\rm A}$		$Q_{\rm A}$		
Term	C_i [″]	S_i[″]	C_i [″]	S_i [″]	P[yr]
$-s_3$	–3720	1259	–1290	–3698	68975
$-s_1$	657	–2585	2508	736	235535
$-s_4$	–2068	–302	288	–2056	72488
$-s_2$	–855	–570	548	–838	192342
$-s_6$	438	338	–334	435	49178

Table 2 The Main Periodic Terms in $p_{\rm A}$ and $\epsilon_{\rm A}$

	$p_{\rm A}$		$\epsilon_{\rm A}$		
Term	C_i[″]	S_i [″]	C_i [″]	S_i [″]	P[yr]
$p+s_3$	–6651	–2197	738	–2216	40938
$p+s_4$	–3349	540	–175	–1126	39803
$p+s_6$	1526	–1218	376	469	53789
$p+s_1$	227	874	–313	84	28832
$p+s_2$	–370	255	–91	–129	29639

The long-term approximations for the general precession $p_{\rm A}$ and obliquity $\epsilon_{\rm A}$ are

$$\begin{aligned} p_{\rm A} &= 6221'' + 50.44766'' T + \textstyle\sum_{i=1}^{30} C_i \cos(2\pi T/P_i) + S_i \sin(2\pi T/P_i)\,, \\ \epsilon_{\rm A} &= 83953'' - 8.9'' \times 10^{-5} T + \textstyle\sum_{i=1}^{20} C_i \cos(2\pi T/P_i) + S_i \sin(2\pi T/P_i)\,, \end{aligned} \tag{5}$$

where T is in TCG years, and the cosine/sine amplitudes of the main periodic parts are given in Table 2. $p_{\rm A}$ in Table 2 designates the main precession frequency as give by Laskar et al. (2004).

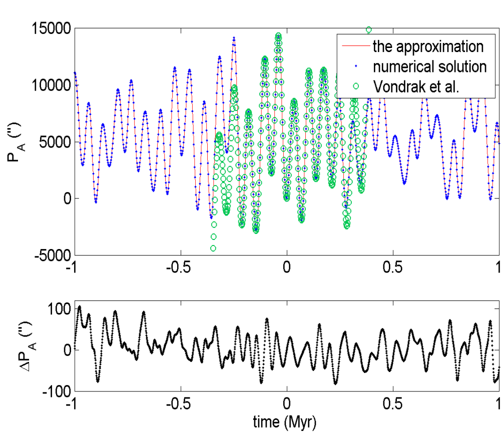

Fig. 5 *Top panel*: comparison of our numerical solution for P_{A} (*dotted line*) with its approximation given by Eq. (4) (*solid line*). The model of Vondrák et al. (2011) (*circles*) is also plotted. *Bottom panel*: the difference between the numerical solution and its approximation (*dotted line*).

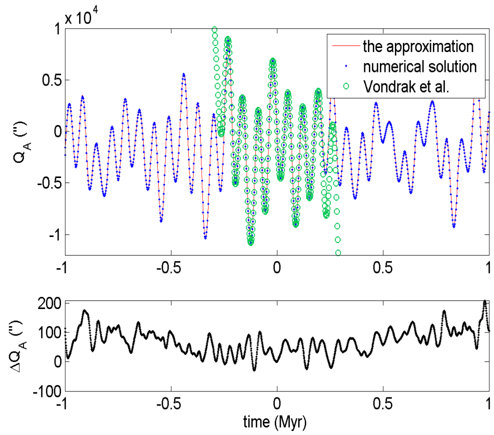

Fig. 6 *Top panel*: comparison of our numerical solution for Q_{A} (*dotted line*) with its approximation given by Eq. (4) (*solid line*). The model of Vondrák et al. (2011) (*circles*) is also plotted. *Bottom panel*: the difference between the numerical solution and its approximation (*dotted line*).

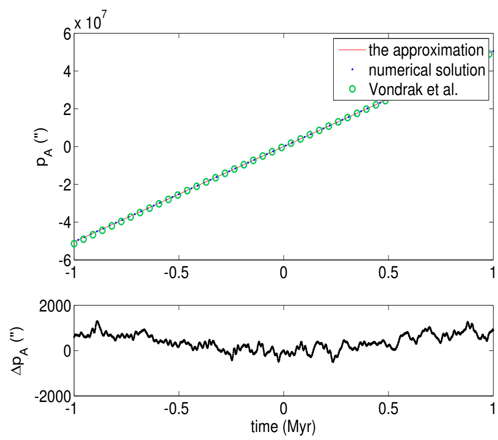

Fig. 7 *Top panel*: comparison of our numerical solution of p_{A} (*dotted line*) with its approximation given by Eq. (5) (*solid line*). The model of Vondrák et al. (2011) (*circles*) is also plotted. *Bottom panel*: the difference between the numerical solution and its approximation (*dotted line*).

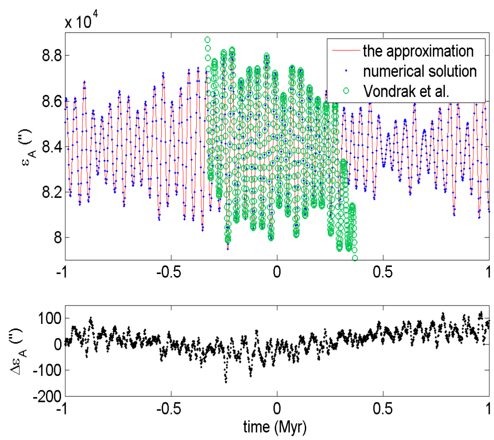

Fig. 8 *Top panel*: comparison of our numerical solution of ϵ_{A} (*dotted line*) with its approximation given by Eq. (5) (*solid line*). The model of Vondrák et al. (2011) (*circles*) is also plotted. *Bottom panel*: the difference between the numerical solution and its approximation (*dotted line*).

The comparisons of these approximations with the complete solutions are given in Figures 7 and 8. Twenty periodic terms are used for the obliquity ϵ_{A} to keep the difference less than $200''$ over the whole time span, but ten more periodic terms are needed to make the difference between the solution for p_{A} and its approximation smaller than $1300''$. The parameters p_{A} and ϵ_{A} mix the motion of the equator in the GCRS and the movement of the ecliptic of date. In addition, it is known that the calculation of a moving ecliptic in relativity framework presents a serious problem, when it is used in the GCRS.

In this paper, we simply transfer the orientation of the ecliptic in the BCRS to the GCRS in a 'Newtonian manner' providing some sort of definition for a GCRS ecliptic. p_{A} and ϵ_{A} are not regarded as the primary precession quantities. We will give the expressions of other parameters, such as X_{A} and Y_{A}, which are the precession part of the coordinates of the CIP unit vector in the GCRS to represent the precession of the equator in a future publication.

Vondrák et al. (2011) mention that their model is only valid within the interval ± 200 millennia from J2000.0, whereas outside this interval of time, the errors grow rapidly as can be seen in the figures above. Similarly, our model has a range of validity of ± 1 Myr around J2000.0. The difference between these two models is about $150''$ for P_{A}, Q_{A}, ϵ_{A} and several degrees for p_{A} at the end of the ± 200 millennia time span. Compared with the P03 theory, the accuracy of our model is several arcseconds for ± 2000 years around J2000.0.

4 RELATIVISTIC EFFECTS IN THE PRECESSION

Our program calculates the precession for both the Newtonian and the post-Newtonian cases. A series of numerical calculations were made to compare these two cases in order to identify the effects due to relativity on the precession of the Earth.

Figures 9 and 10 show the relativistic effects on the precession of the ecliptic parameters P_{A} and Q_{A}. Due to the first post-Newtonian effect of the Sun, the differences of P_{A} and Q_{A} are increasing with time and reach about $1\,300''$ in ± 1 Myr. These differences also have periodic parts, and the main period is about $195\,000$ yr. Because the precession of the ecliptic is due to planetary perturbations, the complicated motion of the solar system makes it hard to easily understand this period. The task of finding a reasonable explanation will be undertaken by analytical methods in the near future. Furthermore, the influence of the relativistic effects is smaller than $10''$ within ± 0.2 Myr. This shows that the model of Vondrák et al. (2011) that does not consider effects from General Relativity is suitable in this interval of time. Some other relativistic effects are too small to be considered in this work, such as the first post-Newtonian effect of the planets.

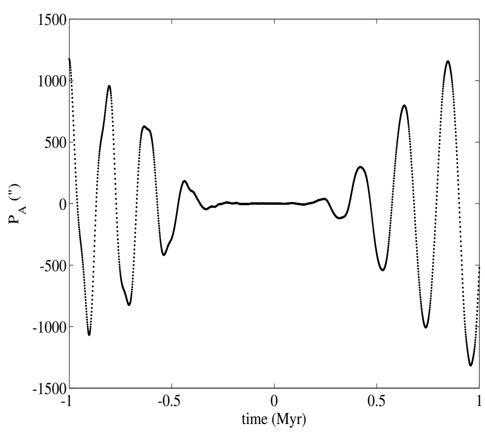

Fig. 9 The relativistic effect on the precession parameter P_{A} from -1 Myr to 1 Myr.

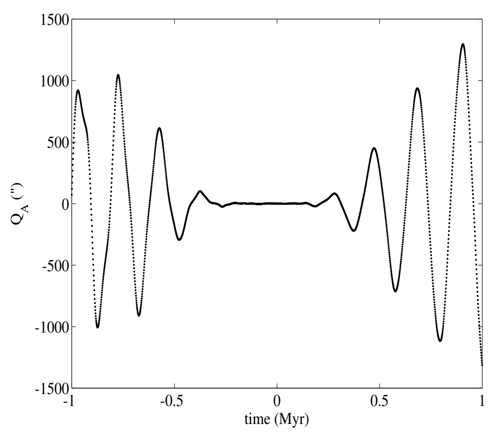

Fig. 10 The relativistic effect on the precession parameter Q_{A} from -1 Myr to 1 Myr.

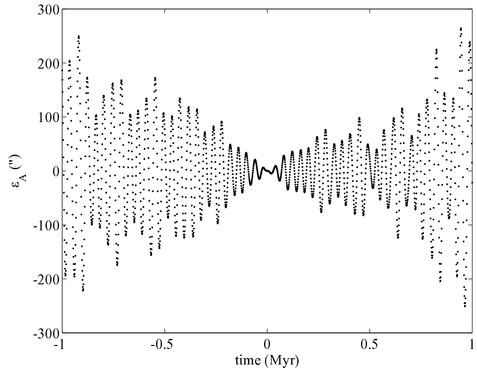

Fig. 11 The effect of the geodetic precession on the precession parameter ϵ_{A} from -1 Myr to 1 Myr.

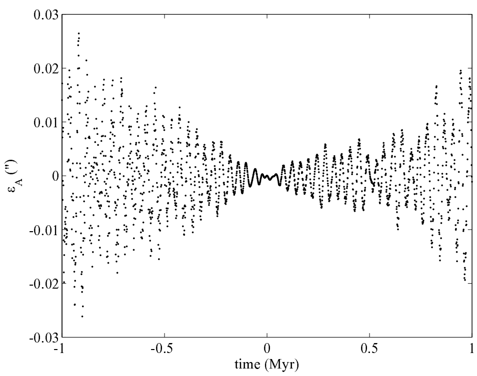

Fig. 12 The effect of the post-Newtonian inertial torque on the precession parameter ϵ_{A} from -1 Myr to 1 Myr.

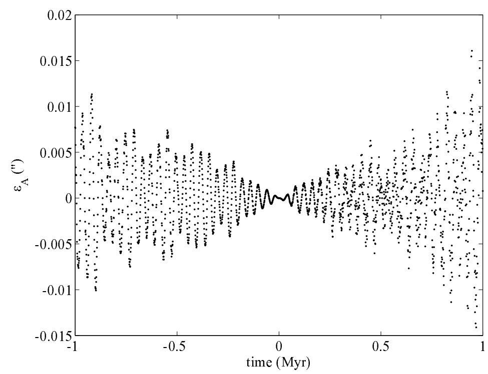

Fig. 13 The effect of the relativistic scaling and time scales on the precession parameter ϵ_{A} from -1 Myr to 1 Myr.

For the rotation of the Earth, the geodetic precession is well-known and considered by all previous works. Traditionally, geodetic precession is artificially added to a purely Newtonian solution which was already shown to not be correct (Klioner et al. 2010). Our result was integrated in a more rigorous relativistic framework and several relativistic features were included: (1) the post Newtonian inertial torque, (2) rigorous treatment of geodetic precession/nutation, (3) four time scales, TDB, TCB, TT and TCG, which are all evaluated at the geocenter, and (4) correct relativistic scaling of constants and parameters. To know the relative importance of the different contributions of relativity, we first repeated the Newtonian dynamical solution. Then another code was written to integrate the post-Newtonian equations of Earth's rotation. All these relativistic effects can be switched on or off independently of each other.

Figure 11 shows the effect of the geodetic precession on the obliquity of the Earth. The influence is about $300''$ in ± 1 Myr. It is the most important relativistic effect in Earth's rotation. The effect of the post-Newtonian inertial torque and the effect of the relativistic scaling and time scales are respectively depicted in Figures 12 and 13. Although these relativistic effects (except for the geodetic precession) accumulate with time, they are still too small to be considered in most cases over this time span. The amplitude of these effects for the obliquity of the Earth is less than one tenth of an arcsecond within ± 1 Myr.

5 CONCLUSIONS

The aim of the work described in this paper is to compute the Earth's precession in a way that it consistent with General Relativity, for a long time interval. The motion of the solar system was integrated in a numerically similar way as Quinn et al. (1991) and Klioner et al. (2010). Our framework is basically relativistic since it employs the post-Newtonian theory of Earth's rotation in the limit of rigidly rotating multipoles as designed by Klioner et al. (2010). The motion of the solar system and Earth's rotation axis in the interval ± 1 Myr from J2000.0 was calculated by our integrator, and approximations for the precession parameters $P_{\rm A}$, $Q_{\rm A}$, $p_{\rm A}$ and $\epsilon_{\rm A}$ are provided. Our solutions have very small discrepancies with respect to P03 near J2000.0, about several arcseconds within ± 2000 yr, and display good consistency with other long-term precession theories. This work, for the first time, involves a fully relativistic framework of Earth's rotation. The problem of a moving ecliptic in the framework of relativity will be investigated further.

The relativistic features considered by our work are: (1) the first post-Newtonian effects related to the Sun, (2) the geodetic precession/nutation, (3) the post-Newtonian inertial torque, and (4) several relativistic reference systems with corresponding time scales and relativistic scaling of parameters. We integrated the equations of translational and rotational motion with rigorous treatment of relativistic effects. This approach differs from the standard way to add the relativistic corrections to the purely Newtonian solution. The relativistic effects are treated as an additional force or torque in the equations of motion. All the motions are integrated in the corresponding proper reference system: the motion of the solar system in the BCRS and the Earth's rotation in the GCRS. The treatment of these systems follows the relevant IAU resolutions. Finally, the influences of these relativistic effects on the precession are obtained and discussed. It provides a reference to pursue and improve the long-term precession theory and Earth's rotation theory in the framework of relativity.

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Appendix A: MAIN CONSTANTS

Table A.1 Main Constants Used in Our Solutions

Symbol	Value	Name	Ref.
c	2.99792458×10^{8} m s^{-1}	Speed of light	IAU2009
$L_{\rm G}$	$6.969290134 \times 10^{-10}$	1-d(TT)/d(TCG)	IAU2009
$L_{\rm B}$	$1.550519768 \times 10^{-8}$	1-d(TDB)/d(TCB)	IAU2009
$L_{\rm C}$	$1.48082686741 \times 10^{-8}$	Average value of 1-d(TCG)/d(TCB)	IAU2009
$\rm TDB_0$	-6.55×10^{-5} s	TDB-TCB at T_0= 2443144.5003725	IAU2009
au	$1.49597870700 \times 10^{11}$ m	Astronomical unit	IAU2009
$\rm GM_S$	$1.32712440041 \times 10^{20}$ m^{3}s^{-2} [TDB-compatible]	Heliocentric gravitational constant	IAU2009
$\rm GM_E$	$3.986004415 \times 10^{14}$ m^{3}s^{-2} [TT-compatible] $3.986004356 \times 10^{14}$ m^{3}s^{-2} [TDB-compatible]	Geocentric gravitational constant	IAU2009

Table A.1 — *Continued.*

Symbol	Value	Name	Ref.
a_{S}	$696000000\,\mathrm{m}$	Equatorial radius of the Sun	
a_{E}	$6.3781366 \times 10^6\ \mathrm{m}$	Equatorial radius of the Earth	IAU2009
$J_{2\mathrm{S}}$	2.0×10^{-7}	Dynamical form factor of the Sun	IERS2010
$J_{2\mathrm{E}}$	1.0826359×10^{-3}	Dynamical form factor of the Earth	IAU2009
H	3273795×10^{-9}	Dynamical flattening of the Earth	IAU2006
$M_{\mathrm{M}}/M_{\mathrm{E}}$	$1.23000371 \times 10^{-2}$	Moon-Earth mass ratio	IAU2009
$M_{\mathrm{S}}/M_{\mathrm{Me}}$	6.0236×10^6	Sun-Mercury mass ratio	IAU2009
$M_{\mathrm{S}}/M_{\mathrm{Ve}}$	4.08523719×10^5	Sun-Venus mass ratio	IAU2009
$M_{\mathrm{S}}/M_{\mathrm{Ma}}$	3.09870359×10^6	Sun-Mars mass ratio	IAU2009
$M_{\mathrm{S}}/M_{\mathrm{J}}$	1.047348644×10^3	Sun-Jupiter mass ratio	IAU2009
$M_{\mathrm{S}}/M_{\mathrm{Sa}}$	3.4979018×10^3	Sun-Saturn mass ratio	IAU2009
$M_{\mathrm{S}}/M_{\mathrm{U}}$	2.290298×10^4	Sun-Uranus mass ratio	IAU2009
$M_{\mathrm{S}}/M_{\mathrm{N}}$	1.941226×10^4	Sun-Neptune mass ratio	IAU2009
$M_{\mathrm{S}}/M_{\mathrm{P}}$	1.36566×10^8	Sun-Pluto mass ratio	IAU2009
k_2	0.305	k_2 of the Earth	Lambeck 1988
Δt	$638\,\mathrm{s}$	Time lag of the Earth	

Appendix B: ALL PERIODIC TERMS IN THE EARTH PRECESSION

Table B.1 The Periodic Terms in P_{A} and Q_{A}

		P_{A}		Q_{A}			
i	Term	C_i ($''$)	S_i ($''$)	C_i ($''$)	S_i ($''$)	P (yr)	f_i (arcsec yr^{-1})
1	$-s_3$	–3720	1260	–1290	–3698	68975	18.789505
2	$-s_1$	657	–2585	2508	736	235535	5.502369
3	$-s_4$	–2068	–302	288	–2056	72488	17.878769
4	$-s_2$	–855	–570	548	–838	192342	6.737991
5	$-s_5$	438	338	–334	435	49178	26.35311
6		309	255	–225	289	67341	19.245403
7		217	322	–191	5	424863	3.050395
8		168	–313	288	183	65723	19.719017
9		–278	130	–112	–294	173673	7.462318
10		–278	–79	89	–285	75817	17.093767
11		–77	258	–157	–194	255871	5.065059
12		–24	124	–106	–33	64138	20.206412
13		29	3	–91	187	496536	2.610082
14		–135	–153	176	–151	70820	18.300011
15		–85	124	–257	187	1080090	1.1999
16		153	–276	395	–117	1309223	0.9899
17		14	–12	77	–94	663722	1.952624
18		55	–11	46	20	214239	6.049326
20		81	39	–41	92	77777	16.663106
21		–55	–16	19	–61	80440	16.111345

Table B.2 The Periodic Terms in p_{A} and ϵ_{A}

		p_{A}		ϵ_{A}			
i	Term	C_i ($''$)	S_i ($''$)	C_i ($''$)	S_i ($''$)	P (yr)	f_i (arcsec yr^{-1})
1	$p+s_3$	–6653	–2199	739	–2217	40938	31.657719
2	$p+s_4$	–3349	541	–175	–1126	39803	32.560229
3	$p+s_6$	1526	–1218	376	469	53789	24.094137
4	$p+s_1$	227	874	–313	84	28832	44.949527
5	$p+s_2$	–370	256	–91	–129	29639	43.726687
6		518	–353	110	174	41557	31.186031
7		324	542	–174	107	42171	30.73167
8		–482	200	–72	–158	38875	33.337421
9		–46	–201	63	–17	42847	30.24716
10		–140	–45	16	–50	30127	43.017971
11		–224	404	–143	–69	40316	32.145966
12		181	–98	38	55	38379	33.768806
13		–121	59	–24	–35	37783	34.301326
14		–9	–73	27	–6	28550	45.394546
15		35	–42	15	13	27300	47.472248
16		63	–35	15	16	37225	34.815702
17		56	–64	15	12	20459	63.347185
18		18	–77	18	3	20151	64.314835
19		–8	41	–9	–13	170984	7.57967
20		51	9	–2	16	29197	44.387641
21		3425	–2525			1309223	0.9899
22		–2951	1938			991814	1.306696
23		2117	–704			716770	1.808112
24		877	–993			416787	3.109503
25		–805	226			554293	2.338115
26		–710	–52			371201	3.491368
27		448	–33			324763	3.990599
28		–217	111			122237	10.602338
29		224	–55			94370	13.733109
30		–228	37			287695	4.504774

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