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Astronomical relativistic reference systems with multipolar expansion: the global one *

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Abstract With the rapid development of techniques for astronomical observations, the precision of measurements has been significantly increasing. Theories describing astronomical relativistic reference systems, which are the foundation for processing and interpreting these data now and in the future, may require extensions to satisfy the needs of these trends. Besides building a framework compatible with alternative theories of gravity and the pursuit of higher order post-Newtonian approximation, it will also be necessary to make the first order post-Newtonian multipole moments of celestial bodies be explicitly expressed in the astronomical relativistic reference systems. This will bring some convenience into modeling the observations and experiments and make it easier to distinguish different contributions in measurements. As a first step, the global solar system reference system is expressed as a multipolar expansion and the post-Newtonian mass and spin moments are shown explicitly in the metric which describes the coordinates of the system. The full expression of the global metric is given.

Key words: reference systems — gravitation

1 INTRODUCTION

Recent years have witnessed the rapid development of techniques for astronomical observations, causing the precision of measurements to significantly increase. One example is the space astrometry mission *Gaia*, which was launched by the European Space Agency (ESA) in 2013 (see Lindegren et al. 2008; Lindegren 2010, for recent reviews). It will obtain accurate astrometric data for $\sim 10^9$ objects from 6th to 20th magnitude. The accuracies for single stars down to 15th magnitude typically range from 8 to 25 microarcseconds (μas). With such a high performance, *Gaia* will be able to detect the relative positional change of a star due to the first order post-Newtonian (1PN) effects from the spherically symmetric parts of gravitational fields of the Sun and some giant planets (Klioner 2003). In some cases where the observed source is very close to the surfaces of Jupiter and Saturn, the higher order multipole moments might cause 1PN light bending up to the level from several tens to hundreds of μas (Klioner 1991; Kopeikin 1997; Klioner 2003), which are also observable by *Gaia*.

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Future space missions may even go further by measuring distances of laser links and angles among these links with unprecedented precision, such as the Télémetrie InterPlanétaire Optique (TIPO) (Samain 2002), the Laser Astrometric Test Of Relativity (LATOR) (Turyshev et al. 2004), the Astrodynamical Space Test of Relativity using Optical Devices (ASTROD) (Ni 2008), the Search for Anomalous Gravitation using Atomic Sensors (SAGAS) (Wolf et al. 2009), the Phobos Laser Ranging (PLR) (Turyshev et al. 2010) and the Beyond Einstein Advanced Coherent Optical Network (BEACON) (Turyshev et al. 2009). Some of them might be able to measure not only 1PN effects caused by the quadrupole moment of the Sun but also effects of the second order post-Newtonian (2PN) light deflection resulting from intrinsic nonlinearity of gravity with high precision.

On the surface of the Earth, time keeping and dissemination equipment are also undergoing great improvements such as optical clocks (e.g. Chou et al. 2010) and optical fiber networks (e.g. Predehl et al. 2012). These technologies will be able to measure the Earth's gravitational potential to new levels of precision by gravitational time dilation at the scale of daily life (Chou et al. 2010) and might bring some subtle effects due to the multipole moments of the Earth into their thresholds in the not-so-distant future.

Although, for processing and interpreting these data now and in the future, the International Astronomical Union (IAU) 2000 and subsequent Resolutions[1] on reference systems in the solar system for astrometry, celestial mechanics and metrology in the framework of general relativity (GR) (Soffel et al. 2003) provide a solid foundation, extensions might be required in some specific observations and measurements. To model the light propagation in those observations and experiments accessing 2PN GR effects, some efforts are dedicated to making the IAU Resolutions include all these contributions (e.g. Minazzoli & Chauvineau 2009). Meanwhile, some works are devoted to establishing self-consistent astronomical relativistic reference systems compatible with alternative relativistic theories of gravity, such as the scalar-tensor theory (Kopeikin & Vlasov 2004), setting up a framework for testing GR. Under these systems, the 2PN theory of light propagation is studied in astronomical observations and experiments using large bodies in the solar system (e.g. Minazzoli & Chauvineau 2011; Deng & Xie 2012). Astronomical relativistic reference systems for gravitational subsystems are also introduced for the advanced theory of lunar motion and for a new generation of lunar laser ranging (Kopeikin & Xie 2010; Xie & Kopeikin 2010).

When higher order post-Newtonian approximation for light propagation is considered, it will also be necessary to ensure the 1PN multipole moments of celestial bodies are explicitly expressed in the astronomical relativistic reference systems. Because, in some cases like the LATOR mission, the light bending caused by the quadrupole at 1PN order can be comparable with those due to the monopole at 2PN order (Klioner 2003). It also helps to distinguish effects from the 1PN multipole moments as well as the intrinsic nonlinearity of gravity at 2PN order. However, the IAU Resolutions on astronomical relativistic reference systems are written in the forms of integrals without showing explicit dependence on the mass and spin multipole moments of each local gravitating body, which may cause some inconvenience in modeling the observations, experiments and data analysis. To achieve this purpose, it is necessary to apply the techniques of multipolar expansion of the gravitational field, which have been intensively studied by many researchers (e.g. Sachs 1961; Pirani 1965; Bonnor & Rotenberg 1966; Epstein & Wagoner 1975; Wagoner 1979; Thorne 1980; Blanchet & Damour 1986; Blanchet 1987; Tao & Huang 1998).

Thus, in this work, I will focus on astronomical relativistic reference systems with multipolar expansion. More specifically, this approach ensures the 1PN multipole moments are expressed explicitly in the mathematical description of the reference systems within the framework of the scalar-tensor theory. As a first step, only the solar system barycentric reference system — the global one — will be considered here. Local reference systems with multipolar expansion will be presented in subsequent works.

[1] Resolutions adopted at the IAU General Assemblies: *http://www.iau.org/administration/resolutions/general_assemblies/*

The rest of the paper is organized as follows. Section 2 is devoted to debriefing primary concepts in astronomical relativistic reference systems. In Section 3, I present the outline of the techniques of multipolar expansion for astronomical relativistic reference systems (a demonstration is given in Appendix B, see online version). The full mathematical description of the solar system barycentric reference system with multipolar expansion and its two special cases are shown in Appendix C (see online version). Finally, in Section 4, I summarize the results.

2 BASICS OF ASTRONOMICAL RELATIVISTIC REFERENCE SYSTEMS

Theories of astronomical relativistic reference systems have been intensively studied (e.g. Kopejkin 1988; Brumberg & Kopejkin 1989; Brumberg 1991; Damour et al. 1991, 1992, 1993; Klioner 1993; Klioner & Voinov 1993; Klioner & Soffel 1998; Kopeikin & Vlasov 2004; Kopeikin & Xie 2010; Xie & Kopeikin 2010). The following part of this section will only give an overview of the primary concepts and necessary mathematical description (see Kopeikin et al. 2011; Soffel & Langhans 2013, for recent reviews and more details).

A reference system is a mathematical construction which gives "names" to spacetime events and a reference frame is a realization of the reference system. A well-defined reference system is the solid and robust foundation for a reference frame which can be materialized by astronomical catalogs and/or dynamical ephemerides of celestial bodies. One leading purpose of classical astrometry in the Newtonian framework is to establish an inertial celestial reference frame. However, this Newtonian concept of absolute space and time is abandoned in GR. In the 4-dimensional curved spacetime, time and space are two parts of a single event. The curvature of spacetime determines motion of matter and the matter, in turn, affects geometry of the spacetime.

An astronomical relativistic reference system is a mathematical description which assigns coordinates (four real numbers) x^μ ($\mu = 0, 1, 2, 3$) for an event within it. Among four coordinates, x^0 is the time coordinate: $t = c^{-1}x^0$ is the coordinate time where c is the speed of light; and the remaining three x^i ($i = 1, 2, 3$) are space coordinates. The coordinates $x^\alpha = (ct, x^i)$ as a whole are described by the metric tensor $g_{\mu\nu}(x^\alpha)$ which is a solution of the field equations of Einstein's GR or other alternative relativistic theories of gravity. The metric tensors of reference systems and the coordinate transformations between them hold all of the properties of the reference systems. Although all reference systems are mathematically equivalent, using some specific systems can largely simplify calculations in modeling astronomical and astrophysical processes.

In the solar system, an adequate relativistic description of a gravitational body's motion is not conceivable without a self-consistent theory of astronomical relativistic reference systems, because the solar system has a hierarchical structure with a diversity in various masses of the bodies and the presence of planetary satellite systems which form a set of gravitationally bounded subsystems. The Sun is the most massive body in the system, but giant planets, like Jupiter and Saturn, can still make it revolve at some distance around the solar system barycenter (SSB). Thus, a global solar system barycentric reference system is required to describe the orbital motion of bodies in the solar system and model the light propagation from distant celestial objects. On the other hand, rotational motion of a body is more natural for describing the local reference systems associated with each of the bodies. A local reference system of a body is also adequate for describing its figure and satellites' motion. Sometimes, a planet may have natural satellites with non-negligible masses which form a gravitational subsystem. It is convenient to introduce a local reference system associated with the barycenter of the subsystem, which leads to a natural decomposition of orbital motion of the subsystem around SSB and relative motion inside the subsystem.

In 2000, IAU adopted new resolutions which laid down a self-consistent general relativistic foundation for applications in modern geodesy, fundamental astrometry, celestial mechanics and spacetime navigation in the solar system. These resolutions combine two independent approaches to the theory of relativistic reference systems including the global one and local ones in the solar system developed in a series of publications by Brumberg and Kopeikin (BK formalism) (Kopejkin 1988; Brumberg & Kopejkin 1989; Brumberg 1991) and Damour, Soffel and Xu (DSX formalism) (Damour et al. 1991, 1992, 1993, 1994).

To make the IAU Resolutions fully compatible with modern ephemerides of the solar system (e.g. Pitjeva 2005; Folkner 2010; Fienga et al. 2011) which employ the generalized Einstein-Infeld-Hoffman (EIH) equations (Einstein et al. 1938) with two parameterized post-Newtonian (PPN) parameters β and γ, some efforts (Klioner & Soffel 2000; Kopeikin & Vlasov 2004) have been contributed. They can go back to the IAU Resolutions when $\beta = 1$ and $\gamma = 1$. I will follow the approach of Kopeikin & Vlasov (2004) in this work.

The metric tensor $g_{\mu\nu}(x^\alpha)$ under 1PN approximation for any reference system can be formally written as

$$g_{00} = -1 + \epsilon^2 N + \epsilon^4 L + \mathcal{O}(\epsilon^5) , \quad (1)$$

$$g_{0i} = \epsilon^3 L_i + \mathcal{O}(\epsilon^5) , \quad (2)$$

$$g_{ij} = \delta_{ij} + \epsilon^2 H_{ij} + \mathcal{O}(\epsilon^4) , \quad (3)$$

where $\epsilon \equiv 1/c$ and N, L, L_i and H_{ij} are coefficients of the metric. These coefficients can be solved from the field equations of Einstein's GR or other alternative relativistic theories of gravity with certain boundary conditions.

In particular, to solve the metric tensor for the solar system barycentric reference system, it is assumed that the solar system is isolated and there are no masses outside it. The considered number of bodies in the system depends on the required accuracy. Therefore, the spacetime of the solar system is asymptotically flat at infinity with the metric tensor $g_{\mu\nu}$ approaching the Minkowskian metric $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. In addition, "no-incoming-radiation" conditions are also imposed on the metric tensor to prevent the appearance of non-physical solutions (see Kopeikin & Vlasov 2004, for details). Its coordinates x^α cover the entire spacetime of the solar system and their origin coincides with the SSB at any instant of time. The law of conservation of angular momentum in the solar system can make the spatial axes of the global coordinates non-rotating in space either kinematically or dynamically (Brumberg & Kopejkin 1989). A reference system is called kinematically non-rotating if its spatial orientation does not change with respect to the Minkowskian spacetime at infinity as time goes on. A dynamically non-rotating system is defined by the condition that equations of motion of test particles moving with respect to the system do not have any terms that can be interpreted as the Coriolis or centripetal forces. With these assumptions and conditions, the metric tensor $g_{\mu\nu}(x^\alpha)$ can be obtained in the 1PN approximation within the framework of the scalar-tensor theory (Kopeikin & Vlasov 2004) and the solutions of its coefficients are given in Appendix A (see online version) in the form of integrals. Theoretically, this metric can be taken to model observations and experiments; however, its dependence on integrals makes this expression inconvenient and non-intuitive in practice.

Thus, in the next section, these integrals will be multipolarly expanded and expressed in terms of local mass and spin multipole moments of each bodies. This would make the metric tensor easier to deal with and show the physical contribution of multipole moments more clearly.

3 MULTIPOLAR EXPANSION OF GLOBAL REFERENCE SYSTEM

To realize the purpose of this work, I need to apply the techniques of relativistic multipolar expansion of the gravitational field, which involves some parameters of the so-called mulitpole moments.

In the Newtonian framework, multipole moments are uniquely defined as coefficients in a Taylor expansion of the gravitational potential in powers of the radial distance from the origin of a reference system to a field point. They can be functions of time in the most general astronomical situations.

Multipolar expansion in GR is quite different (see Thorne 1980, for a review). Because of the non-linearity of the gravitational interaction, a proper definition of relativistic multipole moments is much more complicated. This issue has been intensively and widely studied (e.g. Sachs 1961; Pirani 1965; Bonnor & Rotenberg 1966; Epstein & Wagoner 1975; Wagoner 1979; Thorne 1980; Blanchet & Damour 1986; Blanchet 1987; Tao & Huang 1998). It was shown that, in GR, the multipolar expansion of the gravitational field of an isolated gravitating system is characterized by only two independent sets: mass-type and current-type multipole moments (Thorne 1980; Blanchet & Damour 1986; Blanchet 1987).

In the scalar-tensor theory of gravity, the multipolar expansion becomes even more complicated due to the scalar field. It introduces an additional set of multipole moments which are caused by the scalar field (see Kopeikin & Vlasov 2004, for details). In this work, I will follow and apply the techniques of multipolar expansion and definitions of multipole moments which have been studied in great detail and used in Kopeikin & Vlasov (2004).

These required techniques are rather straightforward. All of the integrals in $g_{\mu\nu}(x^\alpha)$ [see Eqs. (A.11)–(A.17) and (A.19)] for the global reference system can be written in the form (Kopeikin & Vlasov 2004)

$$\mathbf{I}_n^{(C)}\{f\}(t,\boldsymbol{x}) = \int_{V_{\mathrm{C}}} f(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|^n \mathrm{d}^3x' , \tag{4}$$

where n is an integer with values of either -1 or 1. It suggests that multipolar expansion of these integrals needs three steps:

– Step 1. Taylor expand the integral (4) using the fact that the characteristic size of the body C is less than the characteristic distance between the field point, $\boldsymbol{x}$, and the body C, $\boldsymbol{x}_{\mathrm{C}}$, i.e. $|\boldsymbol{x}'-\boldsymbol{x}_{\mathrm{C}}| < R_{\mathrm{C}}$, where $\boldsymbol{R}_{\mathrm{C}} = \boldsymbol{x}-\boldsymbol{x}_{\mathrm{C}}$ and $R_{\mathrm{C}} = |\boldsymbol{R}_{\mathrm{C}}|$. Here $\boldsymbol{x}_{\mathrm{C}}$ represents the position of the center of mass of the body C with respect to the global system and it changes with the global time due to its orbital motion. See Figure 1 for the geometry of the vectors $\boldsymbol{x}$, $\boldsymbol{x}'$, $\boldsymbol{x}_{\mathrm{C}}$ and $\boldsymbol{R}_{\mathrm{C}}$.
– Step 2. Convert the global coordinates $\boldsymbol{x}'$ of a matter element inside body C into the local coordinates $\boldsymbol{Z}'_{\mathrm{C}}$ with respect to the center of mass of body C: $\boldsymbol{Z}'_{\mathrm{C}} = \boldsymbol{x}'-\boldsymbol{x}_{\mathrm{C}} + \mathcal{O}(\epsilon^2)$ (see eq. (11.2.3) in Kopeikin & Vlasov 2004, for details). See Figure 1 for the geometry of the vector $\boldsymbol{Z}'_{\mathrm{C}}$.
– Step 3. Collect and rearrange the expansion according to the definitions of mass and spin multipole moments (see eqs. (6.3.1) and (6.3.8) in Kopeikin & Vlasov 2004, for these definitions).

To demonstrate this procedure, the multipolar expansion of $U_{\mathrm{C}}(t,\boldsymbol{x})$ [see Eq. (A.11)] is shown in Appendix B as an example.

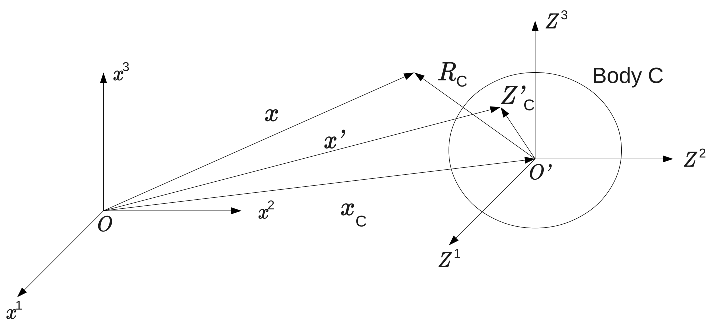

Fig. 1 The geometry of the vectors $\boldsymbol{x}$, $\boldsymbol{x}'$, $\boldsymbol{x}_{\mathrm{C}}$, $\boldsymbol{R}_{\mathrm{C}}$ and $\boldsymbol{Z}'_{\mathrm{C}}$.

After applying it straightforwardly on all of the integrals, the global metric tensor $g_{\mu\nu}$ of the solar system barycentric reference system can be written as

$$g_{\mu\nu} = \eta_{\mu\nu} + h^{(\mathcal{I})}_{\mu\nu} + h^{(\mathcal{I}^2)}_{\mu\nu} + h^{(\mathcal{S})}_{\mu\nu} + h^{(\mathcal{F})}_{\mu\nu} + h^{(\mathcal{B})}_{\mu\nu} + \mathcal{O}(\epsilon^5)\,, \tag{5}$$

where $h^{(\mathcal{I})}_{\mu\nu}$ is the contribution from one-body interactions, $h^{(\mathcal{I}^2)}_{\mu\nu}$ originates from two-body interactions, $h^{(\mathcal{S})}_{\mu\nu}$ is due to spins, $h^{(\mathcal{F})}_{\mu\nu}$ contains scaling function $A_{\rm C}$ and kinematic rotation $F^{\rm km}_{\rm C}$, and $h^{(\mathcal{B})}_{\mu\nu}$ is caused by bad moments. Their full expressions can be found in Appendix C. $h^{(\mathcal{F})}_{\mu\nu}$ can be eliminated by redefining mass multipole moments and by assuming local reference systems are kinematically non-rotating (see Appendix C for details). It is worth mentioning that $h^{(\mathcal{B})}_{\mu\nu}$ is gauge-dependent so that it can be eliminated by a coordinate transformation of the time component as

$$t' = t + \epsilon^3\lambda\,, \tag{6}$$

where

$$\lambda = 2(\gamma+1)\sum_{\rm C}\sum_{l=0}^{\infty}\frac{(-1)^l(2l+1)}{(2l+3)(l+1)!}G\mathcal{R}^{\langle L\rangle}_{\rm C}\left(\frac{1}{R_{\rm C}}\right)_{,\langle L\rangle}\,. \tag{7}$$

Here, $\mathcal{R}^{\langle L\rangle}_{\rm C}$ is a so-called "bad" moment defined in Equation (B.18) (Damour et al. 1992)[2]. The components of the new metric $g'_{\mu\nu}$ are

$$g'_{ij} \;=\; g_{ij} + \mathcal{O}(\epsilon^4)\,, \tag{8}$$

$$g'_{0i} \;=\; g_{0i} - \epsilon^3\lambda_{,i} + \mathcal{O}(\epsilon^5)\,, \tag{9}$$

$$g'_{00} \;=\; g_{00} - \epsilon^4 2\lambda_{,t} + \mathcal{O}(\epsilon^5)\,. \tag{10}$$

The issue of coordinate transformations and gauges in the relativistic astronomical reference systems is practically important and it has been investigated in detail in several works (e.g. Tao & Huang 1998; Tao et al. 2000; Tao 2006).

4 CONCLUSIONS

With advances in techniques for astronomical observations and experiments, the theories of astronomical relativistic reference systems might require extensions to satisfy the needs of new high-precision measurements. One direction is to ensure the 1PN multipole moments of celestial bodies are explicitly expressed in the reference systems. Since the effects of both these moments and nonlinearity of gravity are accessible for future space missions, it will bring some convenience for modeling the observations and experiments and make it easier to distinguish different contributions in measurements.

Therefore, as a first step, the global solar system reference system is expressed as a multipolar expansion and the 1PN mass and spin moments are shown explicitly in their metric which describes the coordinates of the system. The full expression of the global metric is given (see Appendix C for details of main results). These results might be used in modeling future high-precision time transfer (e.g. Petit & Wolf 1994; Wolf & Petit 1995; Blanchet et al. 2001; Petit & Wolf 2005; Nelson 2011; Deng 2012; Deng & Xie 2013b,a; Pan & Xie 2013, 2014).

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[2] Angle brackets surrounding a group of Roman indices denote the symmetric trace-free (STF) part of the corresponding three-dimensional object (see appendix A of Blanchet & Damour 1986, for details). Multi-index notations denote $L \equiv i_1 i_2 \cdots i_l$ and comma denotes a partial derivative. Therefore, $Y_{,L} = \partial^l Y/\partial x^{i_1}\partial x^{i_2}\cdots\partial x^{i_l}$.

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Appendix A: METRIC TENSOR FOR GLOBAL SOLAR SYSTEM BARYCENTRIC REFERENCE SYSTEM

The metric tensor $g_{\mu\nu}(x^\alpha)$ of the global solar system reference system can be solved in the 1PN approximation within the framework of the scalar-tensor theory as (Kopeikin & Vlasov 2004)

$$g_{00} = -1 + \epsilon^2 N + \epsilon^4 L + \mathcal{O}(\epsilon^5)\,, \tag{A.1}$$

$$g_{0i} = \epsilon^3 L_i + \mathcal{O}(\epsilon^5)\,, \tag{A.2}$$

$$g_{ij} = \delta_{ij} + \epsilon^2 H_{ij} + \mathcal{O}(\epsilon^4), \tag{A.3}$$

where $\epsilon \equiv c^{-1}$ and

$$\varphi = U(t,\boldsymbol{x})\,, \tag{A.4}$$

$$N = 2\,U(t,\boldsymbol{x})\,, \tag{A.5}$$

$$L = 2\Psi(t,\boldsymbol{x}) - 2(\beta-1)\varphi^2(t,\boldsymbol{x}) - 2U^2(t,\boldsymbol{x}) - \frac{\partial^2\chi(t,\boldsymbol{x})}{\partial t^2}\,, \tag{A.6}$$

$$L_i = -2(1+\gamma)\,U_i(t,\boldsymbol{x})\,, \tag{A.7}$$

$$H_{ij} = 2\gamma\delta_{ij}U(t,\boldsymbol{x})\,, \tag{A.8}$$

in which $\boldsymbol{x} \equiv x^i$ $(i = 1,2,3)$ and

$$\Psi(t,\boldsymbol{x}) \equiv \left(\gamma + \frac{1}{2}\right)\Psi_1(t,\boldsymbol{x}) - \frac{1}{6}\Psi_2(t,\boldsymbol{x}) + (1+\gamma-2\beta)\Psi_3(t,\boldsymbol{x}) + \Psi_4(t,\boldsymbol{x}) + \gamma\Psi_5(t,\boldsymbol{x})\,, \tag{A.9}$$

Gravitational potentials U, U^i, χ and Ψ_k $(k = 1,...,5)$ can be represented as linear combinations of the gravitational potentials of each body in the gravitational system

$$U = \sum_C U_{\mathrm{C}}, \quad U_i = \sum_C U^i_{\mathrm{C}}, \quad \Psi_k = \sum_C \Psi_{Ck}, \quad \chi = \sum_C \chi_{\mathrm{C}}, \tag{A.10}$$

where the summation index C numerates the bodies in the system, whose gravitational field contributes to the calculations. The gravitational potentials of the body C are defined as integrals taken only over the spatial volume V_{C} of this body

$$U_C(t,\boldsymbol{x}) = G\int_{V_C} \frac{\rho^*(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x', \tag{A.11}$$

$$U^i_C(t,\boldsymbol{x}) = G\int_{V_C} \frac{\rho^*(t,\boldsymbol{x}')v^i(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x', \tag{A.12}$$

$$\Psi_{C1}(t,\boldsymbol{x}) = G\int_{V_C} \frac{\rho^*(t,\boldsymbol{x}')v^2(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x', \tag{A.13}$$

$$\Psi_{C2}(t,\boldsymbol{x}) = G\int_{V_C} \frac{\rho^*(t,\boldsymbol{x}')h(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x', \tag{A.14}$$

$$\Psi_{C3}(t,\boldsymbol{x}) = G\int_{V_C} \frac{\rho^*(t,\boldsymbol{x}')\varphi(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x', \tag{A.15}$$

$$\Psi_{C4}(t,\boldsymbol{x}) = G\int_{V_C} \frac{\rho^*(t,\boldsymbol{x}')\Pi(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x', \tag{A.16}$$

$$\Psi_{C5}(t,\boldsymbol{x}) \;=\; G\int_{V_C}\frac{\pi^{kk}(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x', \tag{A.17}$$

where ρ^* is the invariant density (Fock 1959), Π is the specific internal energy of matter, $\pi^{\alpha\beta}$ is the anisotropic tensor of stress, φ is the perturbation of the scalar field and $h(t,\boldsymbol{x}) = H_{ii}(t,\boldsymbol{x})$. Potential χ is determined as a particular solution of the inhomogeneous equation

$$\nabla^2\chi = -2U \tag{A.18}$$

with the right side defined over the whole space and it is

$$\chi_{\mathrm{C}}(t,\boldsymbol{x}) = -G\int_{V_{\mathrm{C}}}\rho^*(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|\mathrm{d}^3x'. \tag{A.19}$$

Mathematically, all of the integrals in Equations (A.11)–(A.17) and (A.19) can be written in the form (Kopeikin & Vlasov 2004)

$$\mathbf{I}_n^{(C)}\{f\}(t,\boldsymbol{x}) = \int_{V_{\mathrm{C}}} f(t,\boldsymbol{x}')|\boldsymbol{x}-\boldsymbol{x}'|^n\mathrm{d}^3x', \tag{A.20}$$

where n is an integer with values of either -1 or 1.

Appendix B: DEMONSTRATION OF MULTIPOLAR EXPANSION: THE CASE OF U_{C}

This section of the appendixes is devoted to demonstrating the procedure of multipolar expansion for the integrals in Equations (A.11)–(A.17) and (A.19). Since it is valid for each of them, we only take U_{C} as an example and show the details of how to apply it. For other integrals, what is needed is just to repeat it. There are three steps.

- Step 1. Taylor expand the integral (4) using the fact that the characteristic size of the body C is less than the characteristic distance between the field point, $\boldsymbol{x}$, and the body C, $\boldsymbol{x}_{\mathrm{C}}$, i.e. $R'_{\mathrm{C}} < R_{\mathrm{C}}$, where $\boldsymbol{R}'_{\mathrm{C}} = \boldsymbol{x}' - \boldsymbol{x}_{\mathrm{C}}$, $\boldsymbol{R}_{\mathrm{C}} = \boldsymbol{x} - \boldsymbol{x}_{\mathrm{C}}$ and $R'_{\mathrm{C}} = |\boldsymbol{R}'_{\mathrm{C}}|$, $R_{\mathrm{C}} = |\boldsymbol{R}_{\mathrm{C}}|$. Here $\boldsymbol{x}_{\mathrm{C}}$ represents the position of the center of mass of the body C with respect to the global system. With the help of

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \frac{1}{|\boldsymbol{x}-\boldsymbol{x}_{\mathrm{C}}-(\boldsymbol{x}'-\boldsymbol{x}_{\mathrm{C}})|} = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\partial_L\left(\frac{1}{R_{\mathrm{C}}}\right)R_{\mathrm{C}}'^{\langle L\rangle}, \tag{B.1}$$

  in which angle brackets surrounding a group of Roman indices denote the STF part of the corresponding three-dimensional object (see appendix A of Blanchet & Damour 1986, for details) and multi-index notations denote $L \equiv i_1 i_2\cdots i_l$ and $\partial_L \equiv \partial_{i_1}\cdots\partial_{i_l}$, the integral in Equation (A.11) can be Taylor expanded as

$$\int_{V_{\mathrm{C}}}\frac{\rho^*(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\mathrm{d}^3x' = \sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\left(\frac{1}{R_{\mathrm{C}}}\right)_{,L}\int_{V_{\mathrm{C}}}\rho^{*\prime}R_{\mathrm{C}}'^{\langle L\rangle}\mathrm{d}^3x', \tag{B.2}$$

  where a comma denotes partial derivative.
- Step 2. Convert the global coordinates $\boldsymbol{x}'$ of a matter element inside the body C into the local coordinates $\boldsymbol{Z}'_{\mathrm{C}}$ with respect to the center of mass of the body C.
  In the local reference system of the body C, the local coordinates of a field point in the vacuum are $(c\Xi_{\mathrm{C}},\boldsymbol{Z}_{\mathrm{C}})$ and the coordinates of a matter element inside the body are $(c\Xi_{\mathrm{C}},\boldsymbol{Z}'_{\mathrm{C}})$, where Ξ_{C} is its local coordinate time, $\boldsymbol{Z}_{\mathrm{C}}$ is the position vector of the field point and $\boldsymbol{Z}'_{\mathrm{C}}$ is the position

vector pointing to the matter elements. These local coordinates have relationships with the global coordinates given as (Kopeikin & Vlasov 2004)

$$\Xi_{\rm C} \;=\; t+\epsilon^2\xi^0_{\rm C}\,, \tag{B.3}$$
$$Z^i_{\rm C} \;=\; R^i_{\rm C}+\epsilon^2\xi^i_{\rm C}\,, \tag{B.4}$$
$$Z'^i_{\rm C} \;=\; R'^i_{\rm C}+\epsilon^2[\xi'^i_{\rm C}+\mathcal{V}'^i_{\rm C}(R'^k_{\rm C}-R^k_{\rm C})v^k_{\rm C}]\,, \tag{B.5}$$

where $R^i_{\rm C}=x^i-x^i_{\rm C}(t)$, $R'^i_{\rm C}=x'^i-x^i_{\rm C}(t)$, $\mathcal{V}'^i_{\rm C}=v'^i-v^i_{\rm C}$,

$$\xi^0_{\rm C} \;=\; -(\mathcal{A}_{\rm C}+v^k_{\rm C}R^k_{\rm C})+\mathcal{O}(\epsilon^2), \tag{B.6}$$
$$\xi^i_{\rm C} \;=\; \left(\frac{1}{2}v^i_{\rm C}v^k_{\rm C}+D^{ik}_{\rm C}+F^{ik}_{\rm C}\right)R^k_{\rm C}+D^{ijk}_{\rm C}R^j_{\rm C}R^k_{\rm C}+\mathcal{O}(\epsilon^2), \tag{B.7}$$
$$\xi'^i_{\rm C} \;=\; \left(\frac{1}{2}v^i_{\rm C}v^k_{\rm C}+D^{ik}_{\rm C}+F^{ik}_{\rm C}\right)R'^k_{\rm C}+D^{ijk}_{\rm C}R'^j_{\rm C}R'^k_{\rm C}+\mathcal{O}(\epsilon^2), \tag{B.8}$$
$$F^{ij}_{\rm C} \;=\; -\varepsilon^{ijk}\mathcal{F}^k_{\rm C}, \tag{B.9}$$
$$D^{ij}_{\rm C} \;=\; +\delta^{ik}\gamma\bar{U}_C(\boldsymbol{x}_C)-\delta^{ik}A_C, \tag{B.10}$$
$$D^{ijk}_{\rm C} \;=\; \frac{1}{2}(a^j_C\delta^{ik}+a^k_C\delta^{ij}-a^i_C\delta^{jk}). \tag{B.11}$$

Therefore, from Equations (B.5) and (B.8), it has

$$R'^i_C \;=\; Z'^i_C-\epsilon^2\left[\left(\frac{1}{2}v^i_Cv^k_C+D^{ik}_C+F^{ik}_C\right)Z'^k_C+D^{ijk}_CZ'^j_CZ'^k_C\right] +\epsilon^2\mathcal{V}'^i_Cv^k_C(R^k_C-Z'^k_C)+\mathcal{O}(\epsilon^4), \tag{B.12}$$

which will be a link connecting the global and local coordinates of a matter element.

– Step 3. Collect and rearrange the expansion according to the definitions of mass and spin multipole moments.
According to Kopeikin & Vlasov (2004), the mass multipole moments $\mathcal{I}^{\langle L\rangle}_C$ are defined as

$$\mathcal{I}^{\langle L\rangle}_C \;=\; \int_{V_C}\sigma_C(\Xi_C,\boldsymbol{Z}'_C)Z'^{\langle L\rangle}_C d^3Z'_C+\frac{\epsilon^2}{2(2l+3)}\left[\frac{d^2}{d\Xi^2_C}\int_{V_C}\sigma_C(\Xi_C,\boldsymbol{Z}'_C)Z'^{\langle L\rangle}_C Z'^2_C d^3Z'_C -4(1+\gamma)\,\frac{2l+1}{l+1}\frac{d}{d\Xi_C}\int_{V_C}\sigma^i_C(\Xi_C,\boldsymbol{Z}'_C)Z'^{<iL>}_C d^3Z'_C\right]-\epsilon^2\int_{V_C}d^3Z'_C\,\sigma_C(\Xi_C,\boldsymbol{Z}'_C) \times\left\{A_C+(2\beta-\gamma-1)P_C+\sum_{k=1}^{\infty}\frac{1}{k!}\left[Q^K_C+2(\beta-1)P^K_C\right]Z'^K_C\right\}Z'^{\langle L\rangle}_C, \tag{B.13}$$

in which

$$\sigma_C(\Xi_C,\boldsymbol{Z}'_C) \;=\; \rho^*_C(\Xi_C,\boldsymbol{Z}'_C)\left\{1+\epsilon^2\left[\left(\gamma+\frac{1}{2}\right)\mathcal{V}^2_C(\Xi,\boldsymbol{Z}'_C)+\Pi_C(\Xi_C,\boldsymbol{Z}'_C)-(2\beta-1)U_C(\Xi_C,\boldsymbol{Z}'_C)\right]\right\} +\epsilon^2\gamma\pi^{kk}_C(\Xi_C,\boldsymbol{Z}'_C), \tag{B.14}$$

and the definitions of the spin moments $S^{\langle L\rangle}_C$ are

$$S^{\langle L\rangle}_C=\int_{V_C}\varepsilon^{pq<i_l}Z'^{L-1>p}_C\sigma^q_C(\Xi_C,\boldsymbol{Z}'_C)d^3Z'_C, \tag{B.15}$$

in which $\sigma^i_C(\Xi_C,\boldsymbol{Z}'_C)=\rho^*_C(\Xi_C,\boldsymbol{Z}'_C)\mathcal{V}^i_C(\Xi_C,\boldsymbol{Z}'_C)$.

With Equation (B.12) and these two definitions, Equation (B.2) can be collected and rearranged further as

$$
\begin{aligned}
& G\int_C \frac{\rho^*(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}d^3x' \\
= \; & G\sum_{l=0}^{\infty}\frac{(-1)^l}{l!}\left(\frac{1}{R_C}\right)_{,L}\Bigg\{\mathcal{I}_C^{\langle L\rangle}+\epsilon^2\Big[A_C+(2\beta-\gamma-1)P_C\Big]\mathcal{I}_C^{\langle L\rangle} \\
& -\epsilon^2\int_{V_C}\rho_C^{*}{}'\Big[(\gamma+\frac{1}{2})\mathcal{V}_C'^2+\Pi_C'+\gamma\frac{\pi_C'^{kk}}{\rho_C^{*}{}'}\Big]Z_C'^{\langle L\rangle}d^3Z_C'+\epsilon^2(2\beta-1)\int_{V_C}\rho_C^{*}{}'U_C'Z_C'^{\langle L\rangle}d^3Z_C' \\
& -\frac{\epsilon^2}{2(2l+3)}\Big[\ddot{\mathcal{N}}_C^{\langle L\rangle}-4(1+\gamma)\,\frac{2l+1}{l+1}\dot{\mathcal{R}}_C^{\langle L\rangle}\Big]+\epsilon^2\sum_{k=1}^{\infty}\frac{1}{k!}Q_C^K\int_C\rho_C^{*}{}'Z_C'^{<K>}Z_C'^{\langle L\rangle}d^3Z_C' \\
& +\epsilon^2 2(\beta-1)\sum_{k=1}^{\infty}\frac{1}{k!}P_C^K\int_C\rho_C^{*}{}'Z_C'^{<K>}Z_C'^{\langle L\rangle}d^3Z_C'+\epsilon^2\Big(-\frac{l}{2}v_C^kv_C^{<i_l}\mathcal{I}_C^{L-1>k} \\
& +lF_C^{k<i_l}\mathcal{I}_C^{L-1>k}-lD_C^{k<i_l}\mathcal{I}_C^{L-1>k}-l\mathcal{I}_C^{jk<L-1}D_C^{i_l>jk}-v_C^k\dot{\mathcal{I}}_C^{k\langle L\rangle}+v_C^kR_C^k\dot{\mathcal{I}}_C^{\langle L\rangle}\Big)\Bigg\} \\
& +\epsilon^2G\sum_{l=1}^{\infty}\frac{(-1)^l}{(l+1)!}\left(\frac{1}{R_C}\right)_{,L}v_C^k\dot{\mathcal{I}}_C^{<kL>}-\epsilon^2G\sum_{l=1}^{\infty}\frac{(-1)^l l}{(l+1)!}\varepsilon_{kpq}v_C^k\left(\frac{1}{R_C}\right)_{,pL-1}\mathcal{S}_C^{<qL-1>} \\
& +\epsilon^2G\sum_{l=1}^{\infty}\frac{(-1)^l(2l-1)}{(2l+1)l!}v_C^k\left(\frac{1}{R_C}\right)_{,kL-1}\mathcal{R}_C^{<L-1>}+\mathcal{O}(\epsilon^4), \qquad \text{(B.16)}
\end{aligned}
$$

where $\mathcal{N}_C^{\langle L\rangle}$ and $\mathcal{R}_C^{\langle L\rangle}$ are called "bad" moments defined as (Damour et al. 1992)

$$\mathcal{N}_C^{\langle L\rangle}=\int_{V_C}\rho_C^{*}{}'Z_C'^2Z_C'^{\langle L\rangle}d^3Z_C', \qquad \text{(B.17)}$$

and

$$\mathcal{R}_C^{\langle L\rangle}=\int_{V_C}\rho_C^{*}{}'\mathcal{V}_C'^kZ_C'^{<kL>}d^3Z_C'. \qquad \text{(B.18)}$$

This expression is not simplified further because many terms will cancel out by terms from other integrals after repeating the same approach and collecting them together.

Appendix C: GLOBAL METRIC AS A MULTIPOLAR EXPANSION

Full expressions of the global metric $g_{\mu\nu}$ with a multipolar expansion can be written as

$$
\begin{aligned}
g_{00} &= -1+h_{00}^{(\mathcal{I})}+h_{00}^{(\mathcal{I}^2)}+h_{00}^{(\mathcal{S})}+h_{00}^{(\mathcal{F})}+h_{00}^{(\mathcal{B})}+\mathcal{O}(\epsilon^5), && \text{(C.1)}\\
g_{0i} &= h_{0i}^{(\mathcal{I})}+h_{0i}^{(\mathcal{S})}+h_{0i}^{(\mathcal{B})}+\mathcal{O}(\epsilon^5), && \text{(C.2)}\\
g_{ij} &= \delta_{ij}+h_{ij}^{(\mathcal{I})}+\mathcal{O}(\epsilon^4), && \text{(C.3)}
\end{aligned}
$$

where

$$
\begin{aligned}
h_{00}^{(\mathcal{I})} = \; & \epsilon^2 2\sum_C\sum_{l=0}^{\infty}\frac{(2l-1)!!}{l!}\frac{G\mathcal{I}_C^{\langle L\rangle}}{R_C^{2l+1}}R_C^{\langle L\rangle}+\epsilon^4\sum_C\sum_{l=0}^{\infty}\frac{[2(2\gamma+1)l+6\gamma+5](2l-1)!!}{(2l+3)l!}\frac{G\mathcal{I}_C^{\langle L\rangle}}{R_C^{2l+1}}v_C^2R_C^{\langle L\rangle} \\
& -\epsilon^4\sum_C\sum_{l=0}^{\infty}\frac{(2l+1)!!}{l!}\frac{G\mathcal{I}_C^{\langle L\rangle}}{R_C^{2l+3}}v_C^kv_C^mR_C^{<kmL>}-\epsilon^4\sum_C\sum_{l=0}^{\infty}\frac{(2l+1)(2l+1)!!}{(2l+5)l!}\frac{G\mathcal{I}_C^{<kL>}}{R_C^{2l+3}}v_C^kv_C^mR_C^{<mL>}
\end{aligned}
$$

$$+\epsilon^4 4(\gamma+1)\sum_C\sum_{l=0}^{\infty}\frac{(2l-1)!!}{(l+1)!}\frac{G\dot{\mathcal{I}}_C^{<kL>}}{R_C^{2l+1}}v_C^k R_C^{\langle L\rangle}-\epsilon^4\sum_C\sum_{l=0}^{\infty}\frac{(2l-3)!!}{l!}\frac{G\ddot{\mathcal{I}}_C^{\langle L\rangle}}{R_C^{2l-1}}R_C^{<L>}, \quad (C.4)$$

$$\begin{aligned} h_{00}^{(\mathcal{I}^2)} &= -\epsilon^4 2\beta\sum_C\sum_D\sum_{l,k=0}^{\infty}\frac{(2l-1)!!(2k-1)!!}{l!k!}\frac{G^2\mathcal{I}_C^{<L>}\mathcal{I}_D^{<K>}}{R_C^{2l+1}R_D^{2k+1}}R_C^{<L>}R_D^{<K>} \\ &-\epsilon^4 2\gamma\sum_C\sum_{D\neq C}\sum_{l,k=0}^{\infty}\frac{(l+1)(2l-1)!!(2k-1)!!}{l!k!}\frac{G^2\mathcal{I}_C^{<L>}\mathcal{I}_D^{<K>}}{R_C^{2l+1}R_{CD}^{2k+1}}R_C^{<L>}R_{CD}^{<K>} \\ &+\epsilon^4\sum_C\sum_{D\neq C}\sum_{l,k,p=0}^{\infty}\frac{(-1)^p(2l-1)!!(2k+2p+1)!!}{l!k!p!\mathcal{M}_C}\frac{G^2\mathcal{I}_C^{<L>}\mathcal{I}_C^{<P>}\mathcal{I}_D^{<K>}}{R_C^{2l+1}R_{CD}^{2k+2p+3}}R_C^{<mL>}R_{CD}^{<mKP>} \\ &+\epsilon^4 2\sum_C\sum_{D\neq C}\sum_{l,k,p=0}^{\infty}\frac{(-1)^p(l+2)(2l+1)(2l-1)!!(2k+2p+1)!!}{(2l+3)l!k!p!\mathcal{M}_C}\frac{G^2\mathcal{I}_C^{<mL>}\mathcal{I}_C^{<P>}\mathcal{I}_D^{<K>}}{R_C^{2l+1}R_{CD}^{2k+2p+3}}R_C^{<L>}R_{CD}^{<mKP>}, \end{aligned} \quad (C.5)$$

$$h_{00}^{(\mathcal{S})} = -\epsilon^4 4(\gamma+1)\sum_C\sum_{l=0}^{\infty}\frac{(2l+1)!!}{(l+2)l!}\frac{G\mathcal{S}_C^{<qL>}}{R_C^{2l+3}}\varepsilon_{kpq}v_C^k R_C^{<pL>}, \quad (C.6)$$

$$h_{00}^{(\mathcal{F})} = \epsilon^4 2\sum_C\sum_{l=0}^{\infty}\frac{(l+1)(2l-1)!!}{l!}\frac{G\mathcal{I}_C^{<L>}}{R_C^{2l+1}}A_C R_C^{<L>}+\epsilon^4 2\sum_C\sum_{l=0}^{\infty}\frac{(2l+1)!!}{l!}\frac{G\mathcal{I}_C^{<kL>}}{R_C^{2l+3}}F_C^{km}R_C^{<mL>}, \quad (C.7)$$

$$h_{00}^{(\mathcal{B})} = \epsilon^4 4(\gamma+1)\sum_C\sum_{l=0}^{\infty}\frac{(2l+1)(2l+1)!!}{(2l+3)(l+1)!}\frac{G\mathcal{R}_C^{<L>}}{R_C^{2l+3}}v_C^k R_C^{<kL>}+\epsilon^4 4(\gamma+1)\sum_C\sum_{l=0}^{\infty}\frac{(2l+1)!!}{(2l+3)(l+1)!}\frac{G\dot{\mathcal{R}}_C^{<L>}}{R_C^{2l+1}}R_C^{<L>}, \quad (C.8)$$

$$h_{0i}^{(\mathcal{I})} = -\epsilon^3 2(\gamma+1)\sum_C\sum_{l=0}^{\infty}\frac{(2l-1)!!}{l!}\frac{G\mathcal{I}_C^{<L>}}{R_C^{2l+1}}R_C^{<L>}v_C^i-\epsilon^3 2(\gamma+1)\sum_C\sum_{l=0}^{\infty}\frac{(2l-1)!!}{(l+1)!}\frac{G\dot{\mathcal{I}}_C^{<iL>}}{R_C^{2l+1}}R_C^{<L>}, \quad (C.9)$$

$$h_{0i}^{(\mathcal{S})} = \epsilon^3 2(\gamma+1)\sum_C\sum_{l=0}^{\infty}\frac{(2l+1)!!}{(l+2)l!}\frac{G\mathcal{S}_C^{<qL>}}{R_C^{2l+3}}\varepsilon_{ipq}R_C^{<pL>}, \quad (C.10)$$

$$h_{0i}^{(\mathcal{B})} = -\epsilon^3 2(\gamma+1)\sum_C\sum_{l=0}^{\infty}\frac{(2l+1)(2l+1)!!}{(2l+3)(l+1)!}\frac{G\mathcal{R}_C^{<L>}}{R_C^{2l+3}}R_C^{<iL>}, \quad (C.11)$$

$$h_{ij}^{(\mathcal{I})} = \epsilon^2 2\gamma\delta_{ij}\sum_C\sum_{l=0}^{\infty}\frac{(2l-1)!!}{l!}\frac{G\mathcal{I}_C^{<L>}}{R_C^{2l+1}}R_C^{<L>}, \quad (C.12)$$

where $l!!$ means the double factorial of l, δ_{ij} is the Kronecker symbol, ε_{ijk} is the Levi-Civita symbol and dot means derivative with respect to time. Since these dots only appear at 1PN order, their difference between the derivative with respect to the global time and that against local times is at the 2PN order. If new mass multipole moments are defined as

$$\mathcal{I}_C^{\langle L\rangle}\Big|_{\text{new}} = [1+\epsilon^2(l+1)A_C]\mathcal{I}_C^{\langle L\rangle}, \quad (C.13)$$

then the first term of $h_{00}^{(\mathcal{F})}$ can be absorbed by the first term of $h_{00}^{(\mathcal{I})}$ and the other parts of $g_{\mu\nu}$ remain formally unchanged. If the local reference system associated with the body C is kinematically non-rotating, i.e. $F_C^{km}=0$, the second term of $h_{00}^{(\mathcal{F})}$ vanishes.

C.1. Special Cases

Two special cases can be obtained:

1. A single body with arbitrary mass and spin multipole moments. Its metric tensor $g_{\mu\nu}^{(1)}$ reads as

$$
\begin{aligned}
g_{00}^{(1)} &= -1+\epsilon^2 2\sum_{l=0}^{\infty}\frac{(2l-1)!!}{l!}\frac{G\mathcal{I}^{\langle L\rangle}}{R^{2l+1}}R^{\langle L\rangle}-\epsilon^4\sum_{l=0}^{\infty}\frac{(2l-3)!!}{l!}\frac{G\ddot{\mathcal{I}}^{\langle L\rangle}}{R^{2l-1}}R^{\langle L\rangle}\\
&-\epsilon^4 2\beta\sum_{l,k=0}^{\infty}\frac{(2l-1)!!(2k-1)!!}{l!k!}\frac{G^2\mathcal{I}^{\langle L\rangle}\mathcal{I}^{<K>}}{R^{2l+1}R^{2k+1}}R^{\langle L\rangle}R^{<K>}\\
&+\epsilon^4 4(\gamma+1)\sum_{l=0}^{\infty}\frac{(2l+1)!!}{(2l+3)(l+1)!}\frac{G\dot{\mathcal{R}}^{\langle L\rangle}}{R^{2l+1}}R^{\langle L\rangle}+\mathcal{O}(\epsilon^5), \qquad \text{(C.14)}
\end{aligned}
$$

$$
\begin{aligned}
g_{0i}^{(1)} &= -\epsilon^3 2(\gamma+1)\sum_{l=0}^{\infty}\frac{(2l-1)!!}{(l+1)!}\frac{G\dot{\mathcal{I}}^{<iL>}}{R^{2l+1}}R^{\langle L\rangle}+\epsilon^3 2(\gamma+1)\sum_{l=0}^{\infty}\frac{(2l+1)!!}{(l+2)l!}\frac{G\mathcal{S}^{<qL>}}{R^{2l+3}}\varepsilon_{ipq}R^{<pL>}\\
&-\epsilon^3 2(\gamma+1)\sum_{l=0}^{\infty}\frac{(2l+1)(2l+1)!!}{(2l+3)(l+1)!}\frac{G\mathcal{R}^{\langle L\rangle}}{R^{2l+3}}R^{<iL>}+\mathcal{O}(\epsilon^5), \qquad \text{(C.15)}
\end{aligned}
$$

$$
g_{ij}^{(1)} = \delta_{ij}+\epsilon^2 2\gamma\delta_{ij}\sum_{l=0}^{\infty}\frac{(2l-1)!!}{l!}\frac{G\mathcal{I}^{\langle L\rangle}}{R^{2l+1}}R^{\langle L\rangle}+\mathcal{O}(\epsilon^4). \qquad \text{(C.16)}
$$

   When $\gamma=\beta=1$, it can return to the previous results (e.g. Blanchet & Damour 1986; Blanchet 1987). This case might be a good approximation for modeling some measurements from the LATOR mission after cutting the summations of l to certain values, since LATOR will be able to measure the 1PN effects caused by the quadrupole moment of the Sun (Turyshev et al. 2004).

2. A system consisting of N spinning point masses. Its metric tensor $g_{\mu\nu}^{(N)}$ has the form

$$
\begin{aligned}
g_{00}^{(N)} &= -1+\epsilon^2 2\sum_C\frac{G\mathcal{M}_C}{R_C}+\epsilon^4 2(\gamma+1)\sum_C\frac{G\mathcal{M}_C}{R_C}v_C^2-\epsilon^4\sum_C\frac{G\mathcal{M}_C}{R_C^3}(v_C^kR_C^k)^2\\
&-\epsilon^4 2\beta\sum_C\sum_D\frac{G^2\mathcal{M}_C\mathcal{M}_D}{R_CR_D}-\epsilon^4 2\gamma\sum_C\sum_{D\neq C}\frac{G^2\mathcal{M}_C\mathcal{M}_D}{R_CR_{CD}}\\
&+\epsilon^4\sum_C\sum_{D\neq C}\frac{G^2\mathcal{M}_C\mathcal{M}_D}{R_CR_{CD}}R_C^mR_{CD}^m-\epsilon^4 2(\gamma+1)\sum_C\frac{G\mathcal{S}_C^q}{R_C^3}\varepsilon_{kpq}v_C^kR_C^p\\
&+\epsilon^4 2\sum_C\frac{G\mathcal{M}_C}{R_C}A_C+\mathcal{O}(\epsilon^5), \qquad \text{(C.17)}
\end{aligned}
$$

$$
g_{0i}^{(N)} = -\epsilon^3 2(\gamma+1)\sum_C\frac{G\mathcal{M}_C}{R_C}v_C^i+\epsilon^3(\gamma+1)\sum_C\frac{G\mathcal{S}_C^q}{R_C^3}\varepsilon_{ipq}R_C^p+\mathcal{O}(\epsilon^5), \qquad \text{(C.18)}
$$

$$
g_{ij}^{(N)} = \delta_{ij}+\epsilon^2 2\gamma\delta_{ij}\sum_C\frac{G\mathcal{M}_C}{R_C}+\mathcal{O}(\epsilon^4). \qquad \text{(C.19)}
$$

   If a sub-case is considered where $\gamma=\beta=1$, $g_{\mu\nu}^{(N)}$ identically matches the global metric shown in previous works (Will 1993; Soffel et al. 2003).