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Reduced transition probabilities for ^{4}He radiative capture reactions at astrophysical energies

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Abstract The reduced transition probabilities from an electric quadrupole B(E_2) and reduced transition probabilities from a magnetic dipole B(M_1) between the ground state and the first excited state have been calculated for the ^{3}He(α,γ)^{7}Be, ^{8}Be(α,γ)^{12}C and ^{12}C(α,γ)^{16}O radiative capture reactions with the M_3Y potential. These reactions are important in stellar evolution. The calculated B(M_1) and B(E_2) for ^{7}Be nuclei are found to be 1.082×10^{-3} e^2 fm^2 and 1.921 e^2 fm^4 from transitions 3/2$^-$ to 1/2$^-$, respectively. The obtained values for reduced transition probabilities B(E_2) for the ^{12}C and ^{16}O nuclei from transitions 0$^+$ to 2$^+$ are 12.54 e^2 fm^4 and 14.18 e^2 fm^4, respectively. The results are in satisfactory agreement with available experimental data.

Key words: nuclear reactions — abundances — methods: numerical

1 INTRODUCTION

Nuclear reactions are reactions that occur between nuclei, and between nuclei and other fundamental particles that generate all elements heavier than hydrogen. In astrophysically relevant nuclear reactions, two opposite reaction mechanisms are important, formation of a compound nucleus and direct reactions. Only a few levels exist for low excitations of a compound nucleus, because the direct mechanism cannot be neglected and can even be dominant in primordial and stellar nucleosynthesis. The main problem here is really to know the reaction rates at energies required for modeling stellar nucleosynthesis.

There are a large number of reactions which are not yet known with the required accuracy in astrophysics. For example, the reaction ^{7}Be(p,γ)^{8}Be plays a major role in the production of high energy neutrinos from the β-decay of ^{8}B in our Sun. These neutrinos come directly from the center of the Sun and are ideal probes of the Sun's structure. This is the most important reaction in nuclear astrophysics (Bahcall 1989). In another example, the reaction ^{12}C(α,γ)^{16}O is extremely relevant for the fate of massive stars and determines if the remnant of a supernova explosion becomes a black hole or a neutron star.

In theoretical calculations, Dubovichenko et al. have considered radiative capture processes in ^{4}He–^{3}He and ^{4}He–^{12}C channels ^{7}Be of ^{16}O nucleus on the basis of a two-cluster potential model with cluster interactions which contain forbidden states and reproduce phase shifts of elastic scattering and some characteristics of bound states (Dubovichenko & Dzhazairov-Kakhramanov 1998; Dubovichenko 1998). As the hydrogen supply is exhausted, other reaction pathways must be found to keep the star burning. The next most likely reactions involve the burning of helium nuclei, again because of the Coulomb barrier. During helium burning in a red giant star, many nuclear reactions are competing for the helium nuclei: the one, two and triple-^{4}He process. The one-^{4}He process fuses ^{4}He nuclei ^{7}Be and the two-^{4}He process fuses two ^{4}He -cluster nuclei ^{8}Be. The triple-^{4}He process fuses three ^{4}He -cluster nuclei carbon, and the other one combines the carbon with another particle to produce ^{16}O or the ^{4}He -^{12}C radiative capture reaction. It was shown that the model used has enabled researchers to describe the total cross sections of these photodisintegration processes. More recently, Bertulani presented a computer program capable of calculating bound and continuum state observables for a nuclear system, such as reduced transition probabilities, phase shifts, photo-disintegration cross sections, radiative capture cross sections, and astrophysical S-factors (Bertulani 2003). The code is based on a model with a type of potential that can be used to calculate nuclear reaction rates in numerous astrophysical reactions. In order to calculate the direct capture cross sections, one needs to solve the many-body problem for bound and continuum states that are relevant for the capture process. A model based on potential can be applied to obtain single-particle energies and wavefunctions. In numerous situations, this solution is good enough to obtain results for cross sections that can be compared with experimental data.

This paper is organized as follows. A brief review of multipole matrix elements and reduced transition probabilities is presented in Section 2. The relevant formalism and parameters, electric and magnetic multipole matrix elements and reduced transition probabilities are defined in this section. The results of applying the model with asymptotic wavefunctions are corroborated in more realistic calculations using wavefunctions generated from the M$_3$Y potentials in comparison with the Woods-Saxon potentials and experimental data, as discussed in Section 3. Summary and conclusions follow in Section 4.

2 BRIEF REVIEW OF MULTIPOLE MATRIX ELEMENTS AND REDUCED TRANSITION PROBABILITIES

The computer code RADCAP calculates various quantities related to radiative capture reactions. The bound state wavefunctions of the final nuclei are given by $\Psi_{JM}(\boldsymbol{r})$ and the ground state wavefunction is normalized so that $\int d^3r \; |\Psi_{JM}(\boldsymbol{r})|^2 = 1$.

The wavefunctions are calculated using the central potential ($V_0(r)$), spin-orbit potential ($V_{\rm S}(r)$) and Coulomb potential ($V_{\rm C}(r)$). The potentials $V_0(r)$ and $V_{\rm S}(r)$ are given by

$$\begin{aligned} V_0(r) &= V_0 \, f_0(r) \,, \\ V_{\rm S}(r) &= -\, V_{\rm S0} \left(\frac{\hbar}{m_\pi c}\right)^2 \frac{1}{r} \frac{d}{dr} f_{\rm S}(r) \,, \\ f_i(r) &= \left[1 + \exp\left(\frac{r - R_i}{a_i}\right)\right]^{-1} \,, \end{aligned} \tag{1}$$

where V_0, $V_{\rm S0}$, R_0, a_0, $R_{\rm S0}$ and $a_{\rm S0}$ are adjusted so that the ground state energy E_B or the energy of an excited state is reproduced.

The M3Y potential is given by a direct term as well as an exchange term, with different ranges, represented by a delta interaction (Bertsch et al. 1977; Kobos et al. 1984)

$$t(s) = A\frac{e^{-\beta_1 s}}{\beta_1 s} + B\frac{e^{-\beta_2 s}}{\beta_2 s} + C\delta(s) \,, \tag{2}$$

where $A = 7999$ MeV, $B = -2134$ MeV, $C = -276$ MeV fm^3, $\beta_1 = 4$ fm^{-1} and $\beta_2 = 2.5$ fm^{-1}.

The central part, which is obtained by folding the integration in Equation (2) with the ground state densities, and the spin-orbit part of the M_3Y potential is given by

$$\begin{aligned} V_0^{M3Y}(r) &= \lambda_0 V^{M3Y}(r) = \lambda_0 \int d^3r_1\, d^3r_2\, \rho_i(r_1)\rho_j(r_2)\, t(s)\,, \\ V_{\rm S}^{M_3Y}(r) &= -\lambda_{\rm S0} \left(\frac{\hbar}{m_\pi c}\right)^2 \frac{1}{r}\frac{d}{dr} V^{M_3Y}(r)\,, \end{aligned} \tag{3}$$

where ρ_i and ρ_j are the ground state densities of the two nuclei and $s = |\boldsymbol{r} + \boldsymbol{r}_2 - \boldsymbol{r}_1|$.

The radial Schrödinger equation for calculating the bound state is given by solving

$$-\frac{\hbar^2}{2m_{ab}} \left[\frac{d^2}{dr^2} - \frac{l\,(l+1)}{r^2}\right] u_{lj}^J(r) + \left[V_0(r) + V_C(r) + \langle \boldsymbol{s}\cdot\boldsymbol{l}\rangle\ V_{\rm S0}(r)\right] u_{lj}^J(r) = E_i u_{lj}^J(r) \tag{4}$$

with $\langle \boldsymbol{s}\cdot\boldsymbol{l}\rangle = \left[j(j+1) - l(l+1) - s(s+1)\right]/2$.

The electric and magnetic dipole transitions are derived by introducing the following operators (Bohr & Mottelson 1969)

$$\begin{aligned} \mathcal{O}_{E\lambda\mu} &= e_\lambda\, r^\lambda Y_{\lambda\mu}(\widehat{\boldsymbol{r}})\,, \\ \mathcal{O}_{M1\mu} &= \sqrt{\frac{3}{4\pi}}\mu_N \left[e_M l_\mu + \sum_{i=a,b} g_i\,(s_i)_\mu\right], \end{aligned} \tag{5}$$

where $e_\lambda = Z_b e\left(-\frac{m_a}{m_c}\right)^\lambda + Z_a e\left(\frac{m_b}{m_c}\right)^\lambda$ and $e_M = \left(\frac{m_a^2 Z_a}{m_c^2} + \frac{m_b^2 Z_b}{m_c^2}\right)$ are the effective electric and magnetic charges, respectively. l_μ and s_μ are the spherical components of order μ ($\mu = -1, 0, 1$) of the orbital and spin angular momentum ($\boldsymbol{l} = -i\boldsymbol{r}\times\boldsymbol{\nabla}$ and $\boldsymbol{s} = \boldsymbol{\sigma}/2$) and g_i are the gyromagnetic factors of particles a and b. μ_N is also the nuclear magneton.

The matrix element for the transition $J_0M_0 \longrightarrow JM$ is given by (Bohr & Mottelson 1969; Edmonds 1960)

$$\begin{aligned} &\langle JM\, |\mathcal{O}_{E\lambda\mu}|\, J_0M_0\rangle = \langle J_0M_0\lambda\mu|JM\rangle\ \frac{\langle J\,\|\mathcal{O}_{E\lambda}\|\, J_0\rangle}{\sqrt{2J+1}}, \\ &\langle J\,\|\mathcal{O}_{E\lambda}\|\, J_0\rangle = (-1)^{j+I_a+J_0+\lambda}\ \left[(2J+1)(2J_0+1)\right]^{1/2} \left\{\begin{array}{ccc} j & J & I_a \\ J_0 & j_0 & \lambda \end{array}\right\} \langle lj\,\|\mathcal{O}_{E\lambda}\|\, l_0j_0\rangle_J\,, \end{aligned} \tag{6}$$

where the subscript J is a reminder that the matrix element is spin dependent. For $l_0 + l + \lambda = \mathrm{odd}$, the reduced matrix element is null and for $l_0 + l + \lambda = $ even, it is given by

$$\langle lj\,\|\mathcal{O}_{E\lambda}\|\, l_0j_0\rangle_J = \frac{e_\lambda}{\sqrt{4\pi}}\ (-1)^{l_0+l+j_0-j}\ \frac{\hat{\lambda}\hat{j}_0}{\hat{j}} \left\langle j_0\frac{1}{2}\lambda 0|j\frac{1}{2}\right\rangle \int_0^\infty dr\, r^\lambda\, u_{lj}^J(r)\ u_{l_0j_0}^{J_0}(r)\,. \tag{7}$$

At very low energies, the magnetic transitions will be much smaller than the electric transitions. The M_1 contribution has to be considered in the cross sections for neutron photodisintegration or radiative capture. The M_1 transitions, in the case of sharp resonances, for the $J = 1^+$ state in ^{8}B at $E_R = 630$ keV above the proton separation threshold play a role (Kim et al. 1987).

Regarding the reduced matrix elements of the M_1 transition, for $l \neq l_0$ the magnetic dipole matrix element is zero and for $l = l_0$, it is given by (Lawson 1980)

$$\begin{aligned}\langle lj \,\|\mathcal{O}_{M1}\|\, l_0 j_0\rangle_J &= (-1)^{j+I_a+J_0+1}\sqrt{\tfrac{3}{4\pi}}\,\widehat{J}\widehat{J}_0\left\{\begin{array}{ccc} j & J & I_a \\ J_0 & j_0 & 1\end{array}\right\}\mu_N \\ \times\Big\{\tfrac{1}{\widehat{l}_0}e_M &\left[\tfrac{2\widetilde{j}_0}{\widehat{l}_0}\left(l_0\delta_{j_0,\,l_0+1/2}+(l_0+1)\,\delta_{j_0,\,l_0-1/2}\right)+(-1)^{l_0+1/2-j}\,\tfrac{\widehat{j}_0}{\sqrt{2}}\delta_{j_0,\,l_0\pm1/2}\delta_{j,\,l_0\mp1/2}\right] \\ &+g_N\tfrac{1}{\widehat{l}_0^2}\left[(-1)^{l_0+1/2-j_0}\,\widetilde{j}_0\delta_{j,\,j_0}-(-1)^{l_0+1/2-j}\,\tfrac{\widehat{j}_0}{\sqrt{2}}\delta_{j_0,\,l_0\pm1/2}\delta_{j,\,l_0\mp1/2}\right] \\ &+g_a\,(-1)^{I_a+j_0+J+1}\,\widehat{J}_0\widehat{J}\widehat{I}_a\widetilde{I}_a\left\{\begin{array}{ccc} I_a & J & j_0 \\ J_0 & I_a & 1\end{array}\right\}\Big\}\int_0^\infty dr\; u_{lj}^J(r)\; u_{l_0j_0}^{J_0}(r)\,, \qquad (8)\end{aligned}$$

where $g_N = 5.586(-3.826)$ for the proton (neutron) and $\mu_a = g_a\mu_N$ is the magnetic moment of the core nucleus.

The reduced transition probability $dB((E,B)\lambda)/dE$ of the nucleus, where i changes into $j+k$, contains information on the structure of the initial ground state and the interaction with the final continuum state. The reduced transition probability for a specific electromagnetic transition $(E,B)\lambda$ to a final state with momentum $\hbar k$ in the continuum is given by (Bertulani 2003)

$$\begin{aligned}&\frac{dB}{dE}((E,B)\lambda, J_i s \to k J_f s) = \\ &\frac{2J_f+1}{2J_i+1}\sum_{j_f l_f}\left|\sum_{j_i l_i j_c}\langle k J_f j_f l_f s j_c||\mathcal{M}((E,B)\lambda)||J_i j_i l_i s j_c\rangle\right|^2\frac{\mu k}{(2\pi)^3\hbar^2}. \qquad (9)\end{aligned}$$

The electric excitations (E) with a multipole operator are given by

$$\mathcal{M}(E\lambda\mu) = Z_{\text{eff}}^{(\lambda)} e r^\lambda Y_{\lambda\mu}(\hat{r}), \qquad (10)$$

where $Z_{\text{eff}}^{(\lambda)} = Z_b\left(\frac{m_c}{m_b+m_c}\right)^\lambda + Z_c\left(-\frac{m_b}{m_b+m_c}\right)^\lambda$ is the effective charge number.

For proton radiative capture, the effective charge numbers for E_1 and E_2 have to consider contributions to the cross sections from Coulomb breakup, photodisintegration and radiative capture. In the case of neutron radiative capture, the E_1 transition dominates the low-lying electromagnetic strength and the E_2 contribution can be neglected.

The initial and final states are given by the following wavefunctions (Bertulani 2003)

$$\begin{aligned}\Phi_i(\boldsymbol{r}) &= \langle \boldsymbol{r}|J_i j_i l_i s j_c\rangle = \frac{1}{r}\sum_{m_i m_c}(j_i\, m_i\, j_c\, m_c|J_i\, M_i) f^{j_c}_{J_i j_i l_i}(r)\mathcal{Y}^{l_i s}_{j_i m_i}(\hat{r})\phi_{j_c m_c}\,, \\ \Phi_f(\boldsymbol{r}) &= \langle \boldsymbol{r}|\boldsymbol{k} J_f j_f l_f s j_c\rangle \\ &= \frac{4\pi}{kr}\sum_{m_f m_c}(j_f\, m_f\, j_c\, m_c|J_f\, M_f) g^{j_c}_{J_f j_f l_f}(r) i^{l_f} Y^*_{l_f m_f}(\hat{k})\mathcal{Y}^{l_f s}_{j_f m_f}(\hat{r})\phi_{j_c m_c}\,, \qquad (11)\end{aligned}$$

where $f^{j_c}_{J_i j_i l_i}(r)$ and $g^{j_c}_{J_f j_f l_f}(r)$ are the radial wavefunctions and $\phi_{j_c m_c}$ is also the wavefunction of the core. Spinor spherical harmonics are denoted by $\mathcal{Y}^{ls}_{jm} = \sum_{m_l m_s}(l\, m_l\, s\, m_s|j\, m)Y_{lm}(\hat{r})\chi_{s m_s}$.

The reduced matrix element in Equation (9) can be expressed as (Bertulani 2003)

$$\langle k J_f j_f l_f s j_c||\mathcal{M}(E\lambda)||J_i j_i l_i s j_c\rangle = \frac{4\pi Z^{(\lambda)}_{\text{eff}} e}{k} D^{J_f j_f l_f}_{J_i j_i l_i}(\lambda s j_c)\,(-i)^{l_f} I^{J_f j_f l_f}_{J_i j_i l_i}(\lambda j_c)\,, \qquad (12)$$

where the angular momentum coupling coefficient $D^{J_f j_f l_f}_{J_i j_i l_i}(\lambda s j_c)$ and the radial integral $I^{J_f j_f l_f}_{J_i j_i l_i}(\lambda j_c)$ are given by

$$\begin{aligned} D^{J_f j_f l_f}_{J_i j_i l_i}(\lambda s j_c) &= (-1)^{s+j_i+l_f+\lambda}(-1)^{j_c+J_i+j_f+\lambda}(l_i\,0\,\lambda\,0|l_f\,0) \\ &\quad \sqrt{2j_i+1}\sqrt{2l_i+1}\sqrt{2J_i+1}\sqrt{2j_f+1} \\ &\quad \sqrt{\frac{2\lambda+1}{4\pi}}\left\{\begin{array}{ccc} l_i & s & j_i \\ j_f & \lambda & l_f \end{array}\right\}\left\{\begin{array}{ccc} j_i & j_c & J_i \\ J_f & \lambda & j_f \end{array}\right\}, \\ I^{J_f j_f l_f}_{J_i j_i l_i}(\lambda j_c) &= \int_0^\infty dr\, g^{j_c *}_{J_f j_f l_f}(r) r^\lambda f^{j_c}_{J_i j_i l_i}(r) \end{aligned} \tag{13}$$

with the asymptotic radial wavefunctions for the bound state

$$f^{j_c}_{J_i j_i l_i}(r) \to C^{j_c}_{J_i j_i l_i} W_{-\eta_i, l_i+1/2}(2qr). \tag{14}$$

Here, the asymptotic form of the continuum state represents the scattering state

$$\begin{aligned} g^{j_c}_{J_f j_f l_f}(r) &\to \exp\left[i(\sigma_{l_f}+\delta^{j_c}_{J_f j_f l_f})\right] \\ &\quad \times\left[\cos(\delta^{j_c}_{J_f j_f l_f})\, F_{l_f}(\eta_f;kr)+\sin(\delta^{j_c}_{J_f j_f l_f})\, G_{l_f}(\eta_f;kr)\right], \end{aligned} \tag{15}$$

where $C^{j_c}_{J_i j_i l_i}$, $W_{-\eta_i, l_i+1/2}$, F_{l_f}, G_{l_f} and $\eta_f = \eta_i/x$ are the asymptotic normalization coefficient, Whittaker function, regular Coulomb wavefunctions, irregular Coulomb wavefunctions and the Sommerfeld parameter, respectively (Abramowitz & Stegun 1965).

3 RESULTS AND DISCUSSION

We calculated the transition probabilities B(E_2) for the electric quadrupole and reduced transition probabilities B(M_1) for the magnetic dipole between the ground state and the first excited state energy for radiative capture reactions by the M_3Y potential. Our calculations for an electric quadrupole transitions B(E_2) and dipole magnetic transitions B(M_1) have been checked by comparison with experiment and other works using the Wood-Saxon potential model in Table 1. The results of theoretical models are in agreement with the basic assumptions of our model, namely that we can describe ^{7}Be, ^{8}Be, ^{12}C and ^{16}O as being formed by clusters in which all the electromagnetic strength arises from excitation of the relative motion. By including the cluster structure, the calculated static quadruple and dipole moments improve the agreement with experimental results. The calculated B(M_1) and B(E_2) between ground state and first excited state energy for radiative capture of ^{7}Be nuclei are found to be 1.082×10^{-3} e^2 fm^2 and 1.921 e^2 fm^4 from transitions 3/2$^-$ to 1/2$^-$, respectively. For ^{8}Be nuclei, these values from transitions 2$^+$ to 1$^+$ are also found to be 4.608×10^{-3} e^2 fm^2 and 3.885 e^2 fm^4, respectively. The obtained values for reduced transition probabilities B(E_2) for the ^{12}C and ^{16}O nuclei from transitions 0$^+$ to 2$^+$ are 12.54 e^2 fm^4 and 14.18 e^2 fm^4, respectively. The results are in satisfactory agreement with available experiment data.

Our calculations of ^{3}He(α,γ)^{7}Be radiative capture, low lying $3/2^-$ and $1/2^-$ resonances in ^{7}Be, ^{8}Be(α,γ)^{12}C radiative capture, low lying 0$^+$ and 2$^+$ resonances in ^{12}C and ^{12}C(α,γ)^{16}O radiative capture and low lying 0$^+$ and 2$^+$ resonances in ^{16}O have not yet been clearly experimentally identified.

The contribution from B(M_1) is comparable with the contribution from B(E_2) in the radiative capture and in the photodisintegration cross section, but they are much smaller than the contribution from B(E_1). However, important information about their structure comes from the E1 transitions which are characterized by the quantity B(E_1). The operator of the E_1 transition is mainly an isovector operator and this is very important, as this means that by analyzing elements of the E_1 transition matrix, we can obtain information about the proton. The nuclear potentials are important for evaluating the continuum wavefunctions because they affect the response in a nonnegligible way.

Table 1 Results for Reduced Transition Probabilities with Various Methods

Reaction	$B(E_2)$ (e^2 fm^4)	$B(M_1)$ (e^2 fm^2)	Type of calculation
$^3He(\alpha,\gamma)^7Be(3/2^- \rightarrow 1/2^-)$	1.05	1.082×10^{-3}	Wood-Saxon
	1.92	1.084×10^{-3}	This work
	–	5.181×10^{-3}	Exp.
$^8Be(\alpha,\gamma)^{12}C$ $(0^+ \rightarrow 2^+)$	12.54	–	Wood-Saxon
	8.45	–	This work
	7.58	–	Exp.
$^{12}C(\alpha,\gamma)^{16}O$ $(0^+ \rightarrow 2^+)$	14.18	–	Wood-Saxon
	9.34	–	This work
	7.42	–	Exp.

4 SUMMERY AND CONCLUSIONS

We have developed calculations for the reduced matrix elements of $B(E_2)$ and $B(M_1)$ interactions, between the ground state and the first excited state energy for the α–^{3}He, α–^{8}Be and α–^{12}C radiative capture reactions by the M_3Y potential, in astrophysical energies. The calculated $B(M_1)$ and $B(E_2)$ for ^{7}Be nuclei are found to be 1.082×10^{-3} e^2 fm^2 and 1.921 e^2 fm^4 from transitions 3/2$^-$ to 1/2$^-$, respectively. The obtained values for reduced transition probabilities $B(E_2)$ for the ^{12}C and ^{16}O nuclei from transitions 0^+ to 2^+ are 12.54 e^2 fm^4 and 14.18 e^2 fm^4, respectively. The results are in satisfactory agreement with available experimental data.

For future calculations, after computing the reduced matrix elements for (E_2) and $B(M_1)$ transitions, we are going to calculate the cross section for photodisintegration and radiative capture processes, because of their astrophysical applications. In this work, we have compared our results with experimental measurements of the S-factor. The agreement that we have found is quite good. Both the inclusion of the three-nucleon interaction and the addition of more excited states in the target will be addressed in the future. Our results contain the bulk of physics behind the investigated astrophysical radiative capture processes.

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