Research in Astron. Astrophys. **2013** Vol. **13** No. **4**, 471–478
http://www.raa-journal.org *http://www.iop.org/journals/raa*

***R**esearch in*
***A**stronomy and*
***A**strophysics*

A fast ellipsoid model for asteroids inverted from lightcurves *

Xiao-Ping Lu[1], Hai-Bin Zhao[2] and Zhong You[1]

[1] Faculty of Information Technology, Macau University of Science and Technology, Avenida Wai Long, Taipa, Macau, China; *xplu@must.edu.mo*
[2] Purple Mountain Observatory, Chinese Academy of Sciences, Nanjing 210008, China

Received 2012 August 20; accepted 2012 November 15

Abstract Research about asteroids has recently attracted more and more attention, especially focusing on their physical structures, such as their spin axis, rotation period and shape. The long distance between observers on Earth and asteroids makes it impossible to directly calculate the shape and other parameters of asteroids, with the exception of Near Earth Asteroids and others that have passed by some spacecrafts. Photometric measurements are still generally the main way to obtain research data on asteroids, i.e. the lightcurves recording the brightness and positions of asteroids. Supposing that the shape of the asteroid is a triaxial ellipsoid with a stable spin, a new method is presented in this article to reconstruct the shape models of asteroids from the lightcurves, together with other physical parameters. By applying a special curvature function, the method calculates the brightness integration on a unit sphere and Lebedev quadrature is employed for the discretization. Finally, the method searches for the optimal solution by the Levenberg-Marquardt algorithm to minimize the residual of the brightness. By adopting this method, not only can related physical parameters of asteroids be obtained at a reasonable accuracy, but also a simple shape model of an ellipsoid can be generated for reconstructing a more sophisticated shape model.

Key words: ellipsoid model — lightcurves — photometric — shape: asteroids

1 INTRODUCTION

With the development of technology, humans have much more ability to discover the universe now than in the past. The beginning of our solar system is still an interesting problem, which motivates research about asteroids because asteroids hold much information about the original formation of the solar system. Most asteroids found now lie between Mars and Jupiter, and are commonly called main belt asteroids (MBAs). Generally MBAs are orbiting the Sun in a stable configuration, with the exception of some larger bodies that pass nearby them and perturb their orbits. Due to the long distance of 2.1 $\sim$ 3.3 AU between the Earth and most of the asteroids, until now the main research data about asteroids have been photometric brightness with the position coordinates of asteroids, the Earth and the Sun, generally lightcurves, which are recorded by many ground-based observatories.

Shape models with other physical parameters such as the rotation period and spin axis, and especially the albedo of asteroids, are the most important information about asteroids. The total

* Supported by the National Natural Science Foundation of China.

distribution of the spin axis and rotation period of MBAs can help in researching the origin and evolution of the solar system. It is common for the rotation periods of MBAs to be stable with durations of some hours, less than one day. But recently it was found that thermal radiation from the Sun can change the asteroid's rate of rotation, called the YORP effect. From observations, Lowry et al. (2007) and Taylor et al. (2007) predicted that the rotation period of asteroid 54 509 will double in about 600 000 years which demonstrates that the periods of asteroids can change with a very small amplitude over a long term.

There are many methods to reconstruct the shape models of asteroids with other physical parameters. Russell (1906) first started the research about the shape of asteroids and concluded a pessimistic result that the shape could not be obtained by merely depending on the lightcurves observed at opposition. With the development of both mathematical theories and technologies applied to telescopes, more and more lightcurves viewed from various solar phase angles have been recorded by many observatories located all over the world and shape inversion algorithms can calculate the shape model with reasonable accuracy by using high-speed computers. Lumme & Bowell (1981a,b) presented an estimation of the scattering law in the surface of atmosphereless bodies and introduced a method based on spherical harmonics for determination of the asteroid's pole (Lumme et al. 1990). Hapke (1984) contributed further to the scattering law with consideration of bidirectional reflectance and the roughness of the surface. Based on the scattering law of Lumme and Bowell, Karttunen (1989); Karttunen & Bowell (1989) presented a method to generate lightcurves of triaxial ellipsoid models and discussed the variations in the convex and nonconvex models. However, they just noted that it was possible to determine the axial ratios of a triaxial ellipsoid model, but did not give a method to find the shape models from lightcurves. Cellino et al. (1989) adopted a model formed by merging eight octants of ellipsoids having different semiaxes and also generated the corresponding lightcurves without an inverse method. Adopting the sparse photometric data collected by the Hipparcos satellite, Cellino et al. (2009) presented a photometric inversion to find the physical properties of asteroids with the assumption that the shape model of the asteroid is a triaxial ellipsoid. Furthermore Kaasalainen et al. (1992a,b) built a very efficient method to reconstruct the arbitrary surface of asteroids and the inverse shape models were later confirmed by flyby observations in space (Kaasalainen et al. 2002).

Nevertheless, until now the ellipsoid model has played an important role. Firstly the photometric data observed by ground-based observatories are not sufficiently accurate because of atmospheric mist, CCD heat, incorrect operations and so on. Generally, the error associated with photometric data is about 2% and we cannot expect to obtain a very accurate shape model by only using lightcurves. Secondly, the ellipsoid model is simple but it can make the inversion easy and efficient with acceptable physical parameters while statistical research about the spin axis and period of asteroids can obtain the required data in a fast way. Thirdly the rough model inferred from the ellipsoid model may be the initial value for reconstructing a more accurate shape model by using Kaasalainen's method.

Under a similar definition as that employed in Kaasalainen's method and tiling the triangular facets in the Lebedev way, we present a fast method of computing the ellipsoid model in this article, which is organized as follows. In Section 2 we describe the technical details of the fast ellipsoid model, including the scattering law in Section 2.1 and the photometric integration by applying the curvature function in Section 2.2. Furthermore, the inverse problem with a discretized integration is presented in Section 2.3. Finally the equations for derivatives are given for reference in optimization in Section 2.4. To conclude, the summary and future plan are discussed in Section 3.

2 ELLIPSOID MODEL FOR ASTEROIDS

The disk-integrated photometric data, representing the brightness in lightcurves, contain much information about the asteroid which can be applied to calculate the related parameters. The periodic variation of brightness is mainly due to the variation of the shape as the asteroid spins around its axis. We assume that the shape of the asteroid is a triaxial ellipsoid, with three semiaxes $a \geq b \geq c > 0$, which spins with period P expressed in hours around its shortest axis whose spherical coordinate is denoted as (λ, β) in the J2000 ecliptic frame system. As is the convention, the brightness data recorded in lightcurves are reduced to unit distances between the asteroid, the Sun and the Earth, and are corrected according to the light-time (Durech et al. 2010). So we assume lightcurves have been processed in this way before applying our method.

2.1 The Scattering Law

The scattering behavior is an inevitable problem in the asteroid model. Lumme & Bowell (1981a,b) considered several physical parameters, such as the single-scattering albedo Ω_0, the asymmetry factor g, the volume density of the surface material D, and the roughness of the surface ϱ. Eventually, a sophisticated scattering model was built to simulate the behavior of the light reflecting from the Sun. Hapke (1984) took into account the opposition effect and the shadowing cast by the particles onto the surface. These scattering models can express the physical characteristics of the light reflection in a rational manner, but they are not efficient in reality due to uncertainty in the physical parameters. Kaasalainen et al. (2005) carried out photometry research with an artificial asteroid in laboratory experiments and confirmed that shape variation is the main cause of variation in brightness, not the scattering law. Furthermore, they found that it is hard to distinguish the difference between the scattering law and random error. In order to acquire the shape model in an efficient way, a simple scattering law is needed. Kaasalainen et al. (2001) presented a convenient method to simulate the scattering behavior by defining a linear combination of the single scattering factor S_{LS} (Lommel-Seeliger) and the multiple scattering factor S_{L} (Lambert). The scattering law can be expressed as

$$\begin{aligned} S(\mu, \mu_0, \alpha) &= f(\alpha)\Big[S_{LS}(\mu, \mu_0) + cS_L(\mu, \mu_0)\Big] \\ &= f(\alpha)\Big(\frac{\mu\mu_0}{\mu + \mu_0} + \gamma\,\mu\mu_0\Big), \end{aligned} \tag{1}$$

where μ and μ_0 are defined as follows under the definition of $\eta(\vartheta, \varphi)$ as the outward unit normal vector of the surface, and ω and ω_0 as the directions to the Earth and the Sun observed from the asteroid respectively,

$$\mu = \omega \cdot \eta, \qquad \mu_0 = \omega_0 \cdot \eta. \tag{2}$$

The phase function $f(\alpha)$ is a fitted function in the three-parameter form

$$f(\alpha) = A_0 \exp\left(-\frac{\alpha}{D}\right) + k\alpha + 1, \tag{3}$$

where A_0 and D are the amplitude and scale length of the opposition effect and k is the overall slope of the phase curve. The above scattering law with four parameters adopted in this article can perform efficiently in the shape inverse problem and realistically simulate the opposition effect.

2.2 The Photometric Integration

In order to reconstruct the shape model of asteroids, the synthetic brightness must be described in the direct problem. As mentioned above, the scattering law with four parameters can be adopted herein to generate the photometric brightness as a surface integral

$$L(\omega_0, \omega) = \iint_{E_+} S(\mu, \mu_0, \alpha)\, d\sigma, \tag{4}$$

where E_+ is the part of the asteroid's shape that is both illuminated by the Sun and visible from the Earth, i.e. $\mu, \mu_0 > 0$. Assuming that the shape model of asteroids is a triaxial ellipsoid, the integral (4)

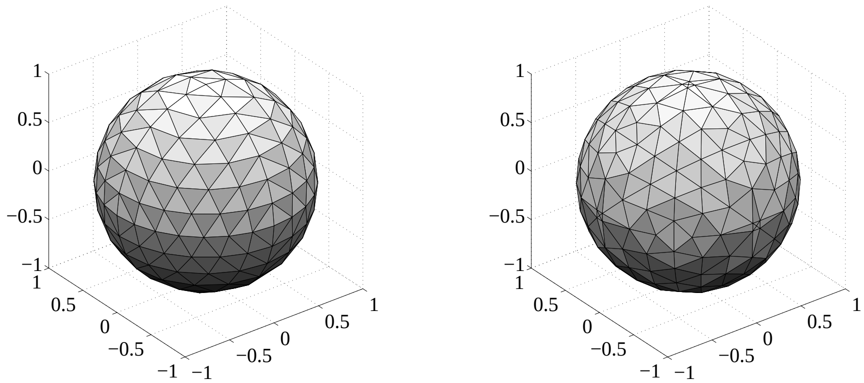

Fig. 1 The comparison of two different tessellation methods (*Left*: Triangulation; *Right*: Lebedev).

can be calculated numerically by traditional triangulation, tiling the approximately equal triangular facets on the surface, which is a linear algorithm of the number of tessellated facets. In Figure 2, it is shown that an error level of 10^{-4} needs more than 10^4 triangular facets tessellated on the surface of the ellipsoid.

Lebedev & Laikov (1999) presented a fast method to calculate the surface integral on the unit sphere S by unequally tiling the triangular facets. Kaasalainen et al. (2012) applied this technique to their method and confirmed that the Lebedev quadrature is efficient in terms of the surface integration. The different distributions of triangular facets of the traditional triangularization and Lebedev quadrature are shown in Figure 1.

Due to the fact that the Lebedev quadrature is based on the unit sphere, in our method a curvature function from the surface of ellipsoid E to the unit sphere S is built with the format

$$G(\theta,\phi) = \frac{\left|\frac{\partial \boldsymbol{r}}{\partial\theta} \times \frac{\partial \boldsymbol{r}}{\partial\phi}\right|}{\sin\theta}, \qquad \boldsymbol{r}(\theta,\phi) \in E. \tag{5}$$

With the curvature function $G(\theta,\phi)$ the brightness integral (4) can be transformed to the surface integral on the unit sphere S

$$L(\omega_0,\omega) = \iint_{S_+} S(\mu,\mu_0,\alpha)\, G(\theta,\phi)\, d\sigma, \tag{6}$$

where S_+ is the similar part of the unit sphere with $\mu, \mu_0 > 0$.

The curvature function under the standard parameterization of the ellipsoidal surface E

$$x_1 = a\sin\theta\cos\phi, \quad x_2 = b\sin\theta\sin\phi, \quad x_3 = c\cos\theta, \quad \theta\in[0,2\pi], \quad \phi\in[0,\pi], \tag{7}$$

has the form

$$G(\theta,\phi) = abc\sqrt{\left(\frac{\sin\theta\cos\phi}{a}\right)^2 + \left(\frac{\sin\theta\sin\phi}{b}\right)^2 + \left(\frac{\cos\theta}{c}\right)^2}. \tag{8}$$

Kaasalainen et al. (1992b) also presented the other curvature function with the form

$$G(\vartheta,\varphi) = \left(\frac{abc}{(a\sin\vartheta\cos\varphi)^2 + (b\sin\vartheta\sin\varphi)^2 + (c\cos\vartheta)^2}\right)^2 \tag{9}$$

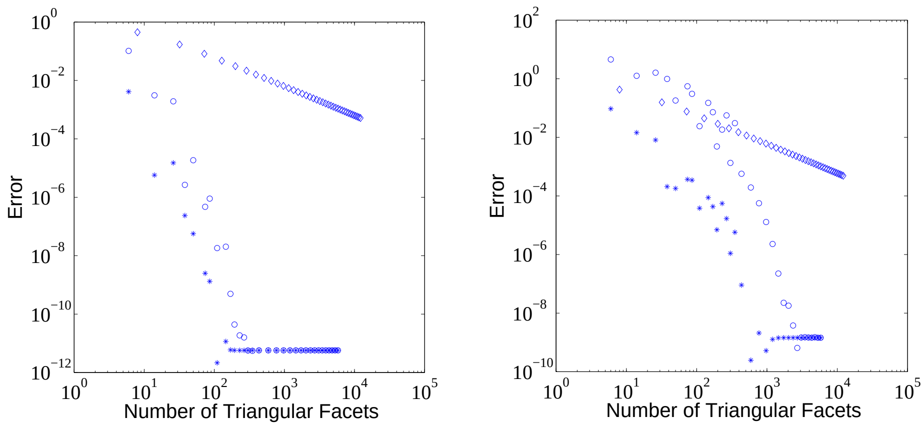

Fig. 2 The comparison of triangulation ('$\diamond$') and Lebedev quadratures with our curvature ('$*$') and Kaasalainen's curvature ('$\circ$') respectively for computing the surface area of ellipsoids with various semiaxes (*Left*: $a = 8$, $b = 7$, $c = 6$; *Right*: $a = 10$, $b = 2$, $c = 1.5$).

under the definition of (ϑ, φ) denoting the spherical coordinates of the normal vectors of the ellipsoidal surface.

Both of the curvature functions can perform well in computing the brightness integral (6). But the first one mentioned and adopted in our method is more efficient for extreme situations, such as the ellipsoid model with a large difference between its three semiaxes. The 'dog-bone' shaped asteroid (216) Kleopatra has been found to have a shape that is close to an elongated ellipsoid (Descamps et al. 2011). Figure 2 shows the comparison of our curvature function ('$*$') and Kaasalainen's curvature function ('$\circ$') to compute the surface area of ellipsoids with different semiaxes with the following formula, easily defining the scattering function $S(\mu, \mu_0, \alpha) = 1$ in the integral (6),

$$\text{Surface Area of Ellipsoids} = \iint_{S_+} G(\theta, \phi)\, d\sigma. \tag{10}$$

Apparently our curvature can obtain less error than Kaasalainen's curvature, especially in the case of elongated ellipsoids. Besides, the difference between the triangulation ('$\diamond$') and Lebedev quadrature ('$*$' or '$\circ$') is also compared in Figure 2. The dominant Lebedev quadrature can guarantee the efficiency of our fast method.

2.3 The Inverse Problem

With the definitions of both the scattering function and the brightness integration, the inverse problem of the shape model of asteroids can be illustrated as follows.

Supposing that all lightcurves in this article are formatted in the same way as DAMIT (Durech et al. 2010), the relative brightness (L) with respect to a Julian Day (t) and the positions of the Earth and Sun (ω, ω_0) in an ecliptic asteroid-centric coordinate system can be obtained directly from lightcurves. According to the aforementioned brightness integration, all the required parameters are three semiaxes (a, b, c), the spherical coordinate (λ, β) of the spin axis in the ecliptic system, the period (P), the four parameters (A_0, D, k, γ) of the scattering function and the initial phase angle (Φ_0) at the beginning epoch (t_0). So there are 11 parameters in all, as t_0 is generally set to be the first Julian date of all lightcurves.

The observed brightness will vary periodically with the rotation of the asteroid around its spin axis, in which it seems that the Sun and Earth spin around the stationary asteroid. Assuming the asteroid's coordinate system is the Cartesian frame with the spin axis being the z-axis and the long semiaxis being the x-axis, the origins of the ecliptic and asteroid coordinate systems are located at the same point at the center of the asteroid. Denoting $\tilde{\omega}$ and $\tilde{\omega_0}$ as the coordinate of the Earth and Sun in the asteroid coordinate system, the transformation between the ecliptic system and the asteroid system can be given as

$$\tilde{\omega} = R_3(\Phi)R_2(\beta)R_3(\lambda)\omega, \qquad \tilde{\omega_0} = R_3(\Phi)R_2(\beta)R_3(\lambda)\omega_0, \tag{11}$$

where R_2 and R_3 are the rotation matrices describing the y and z axes respectively with the forms

$$R_2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}, \qquad R_3(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{12}$$

As the asteroid rotates with an angular speed $2\pi/P$, phase angle Φ varies with respect to the epoch t (Julian Day) in the form

$$\Phi(t) = \Phi_0 + \frac{2\pi}{P}(t - t_0), \tag{13}$$

but if the YORP effect is taken into account, the corresponding form will be

$$\Phi(t) = \Phi_0 + \frac{2\pi}{P}(t - t_0) + \frac{1}{2}\nu(t - t_0)^2, \tag{14}$$

where $\nu = d\Omega/dt$ ($\Omega = 2\pi/P$) is the angular rate of change of the period P.

Let (θ_i, ϕ_i) be the discretized points of the Lebedev quadrature with a total number of facets N being tessellated on the surface of the unit sphere with a corresponding weight w_i, which can also be treated as the area of small facets. With the normal vector of any small facet of the ellipsoid shape

$$\boldsymbol{\eta}_i = \left(\frac{\sin\theta_i\cos\phi_i}{a}, \frac{\sin\theta_i\sin\phi_i}{b}, \frac{\cos\theta_i}{c}\right), \tag{15}$$

the brightness integral (6) can be discretized as

$$L(\omega, \omega_0) \approx \sum_{i=1}^{N}\left(S(\mu^{(i)}, \mu_0^{(i)}, \alpha)G(\theta_i, \phi_i)w_i\right), \tag{16}$$

where $\mu^{(i)}$ and $\mu_0^{(i)}$ are the inner products of $\tilde{\omega}$ and $\tilde{\omega_0}$ and the unit normal vector $\boldsymbol{\eta_i}/|\boldsymbol{\eta_i}|$ respectively. Furthermore, under the definition of normal vector (15), the curvature function (8) can be simplified in the form

$$G(\theta_i, \phi_i) = abc|\boldsymbol{\eta_i}|. \tag{17}$$

Finally, merging the scattering function $S(\mu, \mu_0, \alpha)$ in Equation (1), the discretized brightness integral (16) will be

$$L(\omega, \omega_0) \approx \sum_{i=1}^{N}\left(f(\alpha)(\tilde{\boldsymbol{\omega}} \cdot \boldsymbol{\eta_i})(\tilde{\boldsymbol{\omega_0}} \cdot \boldsymbol{\eta_i})\left[\frac{1}{(\tilde{\boldsymbol{\omega}} \cdot \boldsymbol{\eta_i}) + (\tilde{\boldsymbol{\omega_0}} \cdot \boldsymbol{\eta_i})} + \frac{\gamma}{|\boldsymbol{\eta_i}|}\right]abcw_i\right), \tag{18}$$

where $(\boldsymbol{x} \cdot \boldsymbol{y})$ is the inner product of two vectors $\boldsymbol{x}$ and $\boldsymbol{y}$.

The inverse problem of the ellipsoid model can be described to find the 11 parameters mentioned above to minimize

$$\chi^2 = \sum_i \left\|\frac{\boldsymbol{L}^{(i)}}{\langle \boldsymbol{L}^{(i)}\rangle} - \frac{\tilde{\boldsymbol{L}}^{(i)}}{\langle \tilde{\boldsymbol{L}}^{(i)}\rangle}\right\|^2, \tag{19}$$

where $\boldsymbol{L}^{(i)}$ and $\tilde{\boldsymbol{L}}^{(i)}$ denote the brightness vectors of observed data and the synthetic data containing all points in the ith lightcurve, while $\langle \boldsymbol{L}^{(i)}\rangle$ and $\langle \tilde{\boldsymbol{L}}^{(i)}\rangle$ denote the mean brightness of the two brightness vectors.

2.4 The Optimization Algorithm

There are many methods to find the best fit solution of the inverse problem (19), such as a genetic algorithm, methods of conjugate gradients and so on. Herein we employ the Levenberg-Marquardt (LM) method to search for the optimal solution, which works very well in practice and has almost become the standard in nonlinear least-squares routines (Marquardt 1963). The LM method can converge rapidly, but the obtained result is often a local optimal solution. There is a general method to complement its deficiency, i.e. searching for the best fit result with various initial values. The details of the LM method can be found in Press et al. (1992).

But it is necessary to compute the derivatives of χ^2 in the LM method, which means the derivatives of the discretized brightness integral (18) with respect to the 11 parameters have to also be computed. In order to facilitate easy programming, we give the formulas for reference.

Letting

$$\tilde{\mu}_i = \tilde{\boldsymbol{\omega}} \cdot \boldsymbol{\eta_i}, \tilde{\mu}_i = \tilde{\boldsymbol{\omega}} \cdot \boldsymbol{\eta_i}, S = \frac{\tilde{\mu}\tilde{\mu}_0}{\tilde{\mu} + \tilde{\mu}_0} + \frac{\gamma\tilde{\mu}\tilde{\mu}_0}{|\boldsymbol{\eta_i}|}. \tag{20}$$

The derivative of $L(\omega, \omega_0)$ with respect to the semimajor axis (a) has the form

$$\frac{\partial L}{\partial a} = \sum_{i=1}^{N}\left[f(\alpha)bcw_i S + abcw_i\left(\frac{\partial S}{\partial \mu}\frac{\partial \mu}{\partial a} + \frac{\partial S}{\partial \mu_0}\frac{\partial \mu_0}{\partial a} + \frac{\gamma\mu\mu_0 \sin^2\theta_i \cos^2\phi_i}{a^3|\boldsymbol{\eta_i}|^3}\right)\right]. \tag{21}$$

The other semiaxes (such as b and c) have similar formulas. The derivatives of $L(\omega, \omega_0)$ with respect to $(\lambda, \beta, P, \Phi_0)$ are

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{N}\left[f(\alpha)abcw_i\left(\frac{\partial S}{\partial \mu}R_3(\Phi)R_2(\beta)\frac{\partial R_3(\lambda)}{\partial \lambda}\boldsymbol{\omega} \cdot \boldsymbol{\eta_i} + \frac{\partial S}{\partial \mu_0}R_3(\Phi)R_2(\beta)\frac{\partial R_3(\lambda)}{\partial \lambda}\boldsymbol{\omega_0} \cdot \boldsymbol{\eta_i}\right)\right], \tag{22}$$

$$\frac{\partial L}{\partial \beta} = \sum_{i=1}^{N}\left[f(\alpha)abcw_i\left(\frac{\partial S}{\partial \mu}R_3(\Phi)\frac{\partial R_2(\beta)}{\partial \beta}R_3(\lambda)\boldsymbol{\omega} \cdot \boldsymbol{\eta_i} + \frac{\partial S}{\partial \mu_0}R_3(\Phi)\frac{\partial R_2(\beta)}{\partial \beta}R_3(\lambda)\boldsymbol{\omega_0} \cdot \boldsymbol{\eta_i}\right)\right], \tag{23}$$

$$\frac{\partial L}{\partial \Phi_0} = \sum_{i=1}^{N}\left[f(\alpha)abcw_i\left(\frac{\partial S}{\partial \mu}\frac{\partial R_3(\Phi)}{\partial \Phi}R_2(\beta)R_3(\lambda)\boldsymbol{\omega} \cdot \boldsymbol{\eta_i} + \frac{\partial S}{\partial \mu_0}\frac{\partial R_3(\Phi)}{\partial \Phi}R_2(\beta)R_3(\lambda)\boldsymbol{\omega_0} \cdot \boldsymbol{\eta_i}\right)\right]. \tag{24}$$

Due to the relation of P and Φ_0 in Equation (13),

$$\frac{\partial L}{\partial P} = \frac{\partial L}{\partial \Phi_0}\frac{\partial \Phi}{\partial P}. \tag{25}$$

The derivatives of $L(\omega, \omega_0)$ with respect to the three parameters A_0, D and k in the scattering function are easy to obtain by multiplying the corresponding derivatives of $f(\alpha)$ respectively,

$$\frac{\partial f}{\partial A_0} = \exp\left(-\frac{\alpha}{D}\right), \quad \frac{\partial f}{\partial D} = \frac{A_0\alpha}{D^3}\exp\left(-\frac{\alpha}{D}\right), \quad \frac{\partial f}{\partial k} = \alpha, \tag{26}$$

and the derivative with respect to γ is

$$\frac{\partial L}{\partial \gamma} = \sum_{i=1}^{N}\left[f(\alpha)abcw_i\frac{\mu\mu_0}{|\boldsymbol{\eta_i}|}\right]. \tag{27}$$

3 CONCLUSIONS

We have described a fast method to obtain the physical parameters and the shape models of asteroids based on the ellipsoidal shape. This method adopted the Lebedev quadrature to discretize the surface integral on the unit sphere, which can largely decrease the computational cost while maintaining a high accuracy. In addition, now that this method can compute the period and the orientation of the spin axis in an efficient way, we can rely on its result to refine the shape of asteroids, such as adopting Kaasalainen's method.

The related equations are presented in this article, such as the curvature function (8) and discretized brightness integral (18), and especially the complicated derivatives of χ^2. Here it should be noted that all the formulas are given in a general form with three semiaxis parameters a, b and c because this format is comprehensible and easy to compare with other formulas like Kaasalainen's curvature function. In computation, due to the relative brightness, the semiaxis parameters a, b and c are not exactly the same as the real asteroids in terms of size. So the ratios of the axes in the ellipsoid model a/b and a/c are more practical. By applying the fast method, it is easy to compute the ratios from the derived parameters or we can set $c = 1$ in the LM method to directly obtain the ratios. Besides, we can also replace the $\frac{2\pi}{P}$ in $\Phi(t)$ by its angular speed Ω to simplify the computation. The detailed numerical test and application to real asteroids will be presented in a future article.

Acknowledgements The authors thank the anonymous referee for comments. This work is funded by grant No. 019/2010/A2 from the Science and Technology Development Fund, MSAR. Haibin Zhao is grateful for the support of the National Natural Science Foundation of China (Grant Nos. 10503013, 11078006 and 10933004) and the Minor Planet Foundation of Purple Mountain Observatory. In addition, we sincerely appreciate Prof. Kaasalainen's constructive advice about this fast method.

References

Cellino, A., Hestroffer, D., Tanga, P., Mottola, S., & Dell'Oro, A. 2009, A&A, 506, 935
Cellino, A., Zappala, V., & Farinella, P. 1989, Icarus, 78, 298
Descamps, P., Marchis, F., Berthier, J., et al. 2011, Icarus, 211, 1022
Durech, J., Sidorin, V., & Kaasalainen, M. 2010, A&A, 513, A46
Hapke, B. 1984, Icarus, 59, 41
Kaasalainen, M., Lamberg, L., Lumme, K., & Bowell, E. 1992a, A&A, 259, 318
Kaasalainen, M., Lamberg, L., & Lumme, K. 1992b, A&A, 259, 333
Kaasalainen, M., Lu, X., & Vänttinen, A.-V. 2012, A&A, 539, A96
Kaasalainen, M., et al. 2002, Asteroids III, 139, Space Science Series (Univ. of Arizona Press)
Kaasalainen, M., Torppa, J., & Muinonen, K. 2001, Icarus, 153, 37
Kaasalainen, S., Kaasalainen, M., & Piironen, J. 2005, A&A, 440, 1177
Karttunen, H. 1989, A&A, 208, 314
Karttunen, H., & Bowell, E. 1989, A&A, 208, 320
Lebedev, V. I., & Laikov, D. N. 1999, Doklady Mathematics, 59, 477
Lowry, S. C., Fitzsimmons, A., Pravec, P., et al. 2007, Science, 316, 272
Lumme, K., & Bowell, E. 1981a, AJ, 86, 1694
Lumme, K., & Bowell, E. 1981b, AJ, 86, 1705
Lumme, K., Karttunen, H., & Bowell, E. 1990, A&A, 229, 228
Marquardt, D. W. 1963, Journal of the Society for Industrial & Applied Mathematics, 11, 431
Press, W. H., et al. 1992, Numerical Recipes: The Art of Scientific Computing; 2nd ed. (Cambridge: Cambridge Univ. Press)
Russell, H. N. 1906, ApJ, 24, 1
Taylor, P. A., Margot, J.-L., Vokrouhlický, D., et al. 2007, Science, 316, 274