Research in Astron. Astrophys. **2011** Vol. **11** No. **8**, 981–996
http://www.raa-journal.org *http://www.iop.org/journals/raa*

Research in Astronomy and Astrophysics

Solar sail time-optimal interplanetary transfer trajectory design *

Sheng-Ping Gong, Yun-Feng Gao and Jun-Feng Li

School of Aerospace, Tsinghua University, Beijing 100084, China; *gongsp@tsinghua.edu.cn*

Received 2010 September 8; accepted 2010 November 22

Abstract The fuel consumption associated with some interplanetary transfer trajectories using chemical propulsion is not affordable. A solar sail is a method of propulsion that does not consume fuel. Transfer time is one of the most pressing problems of solar sail transfer trajectory design. This paper investigates the time-optimal interplanetary transfer trajectories to a circular orbit of given inclination and radius. The optimal control law is derived from the principle of maximization. An indirect method is used to solve the optimal control problem by selecting values for the initial adjoint variables, which are normalized within a unit sphere. The conditions for the existence of the time-optimal transfer are dependent on the lightness number of the sail and the inclination and radius of the target orbit. A numerical method is used to obtain the boundary values for the time-optimal transfer trajectories. For the cases where no time-optimal transfer trajectories exist, first-order necessary conditions of the optimal control are proposed to obtain feasible solutions. The results show that the transfer time decreases as the minimum distance from the Sun decreases during the transfer duration. For a solar sail with a small lightness number, the transfer time may be evaluated analytically for a three-phase transfer trajectory. The analytical results are compared with previous results and the associated numerical results. The transfer time of the numerical result here is smaller than the transfer time from previous results and is larger than the analytical result.

Key words: techniques: miscellaneous — solar sail — solar polar orbit

1 INTRODUCTION

In deep space, the force of solar radiation pressure is one of the predominant extra forces exerted on a solar sail. The solar radiation pressure force can be used for performing orbit maneuvers and interplanetary transfers without consuming any fuel. Compared with chemical propulsion, solar sails show great advantages in interplanetary transfer missions. A number of studies of solar sail-based interplanetary missions were carried out in the late 1970s, with an emphasis on a rendezvous with Comet Halley in 1986 (Wright & Warmke 1976; Friedman et al. 1978). Sauer (1976, 1977) presented detailed trajectory designs, based upon generalizations of the variational approach assuming the orbits of Earth and Mars to be circular and coplanar. Lebedev & Zhukov (1964) first investigated the time-optimal transfer from Earth to Mars using a solar sail. Jayaraman (1980) revisited the

* Supported by the National Natural Science Foundation of China.

minimum-time transfers between Earth's orbit and Mars's orbit using different characteristic accelerations. Jayaraman's solutions were different from those obtained by Lebedev & Zhukov (1964). His transfer times were about 10% larger and his sail orientation histories were significantly different. Wood et al. (1982) commented on Jayaraman (1980) and pointed out that the solutions in (Lebedev & Zhukov 1964) were correct and the transversality condition of variational calculus had been applied incorrectly in Jayaraman (1980). In fact, when an indirect method is employed in the optimization process, the long simulation time makes the solution a difficult task, especially because the boundary values are extremely sensitive to the value of the initial estimate of the adjoint variables, particularly for a high variation of orbital inclination or small sail characteristic acceleration (Dachwald & Seboldt 2003). An indirect method including only one phase is almost impossible for a high inclination mission using a small acceleration sail. In addition, a direct method has other disadvantages, since the whole trajectory requires a high number of parameters for the discretization process. Usually, a high inclination transfer trajectory is obtained by dividing the trajectory into a certain number of phases and by using, in each phase, a locally optimal control law (Macdonald et al. 2006). The advantage of such control laws lies in their capability to approximate the optimal trajectory through simple numerical simulations. However, an approach based on locally optimal control laws is not well suited for the problem discussed in this paper. One disadvantage is the difficulty of satisfying the boundary conditions of the final radius and the orbital inclination. These control laws are particularly useful for planetary escape missions (Macdonald & McInnes 2004; Mengali & Quarta 2005). Macdonald et al. used the blended locally optimal control laws to optimize interplanetary transfer trajectories (Macdonald et al. 2007).

A promising application of solar sails is to achieve heliocentric circular orbits with small radius and high inclination. One of these missions is represented by the Solar Polar Imager (SPI) mission, one of several Sun-Earth-related solar sail roadmap missions envisioned by NASA (Burch 1997). The SPI mission requires the spacecraft to evolve in a heliocentric circular orbit having radius of 0.48 AU (astronomical unit) and inclination of 75°. Spacecraft in this orbit can be used to measure the solar magnetic field, coronal mass ejections and solar irradiance in the Sun's polar regions. In 1996, JPL studied a single trajectory phase that had several close approaches to the Sun, which were constrained at the final orbital distance of 0.48 AU. The flight time for the baseline trajectory in the study was nearly three years for a characteristic sail acceleration of 1 mm s^{-2} (Goldstein et al. 1998). In order to find trajectories with shorter flight times, Sauer (1999) discussed a two phase trajectory scenario that is likely to be very close to optimal, where the initial phase of the trajectory delivers the spacecraft to a circular orbit at the desired orbital distance and at an inclination of 15°. Dachwald et al. (2006) and Mengali & Quarta (2009) divided all transfer trajectories into several phases. Their obtained results were similar. The radius is reduced during the first phase and the inclination is increased to an objective value during the second phase and during the last phase the radius is adjusted to the target value and the orbit is circularized. Dachwald et al. (2006) used the Evolutionary Neurocontrol optimization method to obtain shorter time results using the temperature as a constraint, where the "Hot" mission scenario generated a much faster transfer than the "Cold" scenario. Mengali & Quarta (2009) combined the locally optimal method with the globally optimal approach to produce an approximate solution in the form of interpolating functions. Each phase is studied in a globally optimal framework using an indirect approach and the final trajectory is simply obtained as an orderly sequence of the different phases.

This paper discusses the transfer time from the Earth's orbit to a circular orbit of given radius and inclination. The inclination cranking using a solar sail is a typical application of solar sail technology. The cranking maneuver is onerous in terms of flight time, especially when inclination variations of some tens of degrees are required. Therefore, it is convenient to perform the cranking maneuver when the solar distance is as small as possible, in order to maximize the sail thrust that varies as the inverse square of the solar distance. The cranking efficiency is dependent on the lightness number of the sail and the distance from the Sun. The inclination change over one loop is proportional to the lightness number of the solar sail. The period of a large radius is long and the cranking rate is small, which means that varying the inclination in an orbit with a small radius can save a great deal of time. Therefore, one of the efficient methods is to first transfer the spacecraft to a smaller radius for inclination cranking, then the radius can be increased to the desired value. This strategy can save time if the extra time required to adjust the radius is shorter than the time saved for inclination cranking. Without considering the temperature constraint on the sail, does a minimum radius exist for a minimum transfer time? What are the conditions for this minimum radius? Which parameters determine this minimum radius? This paper will try to answer these questions. There are cases where the total transfer time decreases as the minimum radius from the Sun decreases, which means that the minimum time is obtained when the sail tends to approach the Sun. A zero radius leads to the singularity of the dynamical equation and is also impossible in engineering practice. Therefore, there is no minimum time transfer trajectory in this case without any restriction on the radius. In this paper, we try to figure out the conditions for these cases. If the minimum time transfer trajectory exists, the dependence of the minimum distance on different parameters is discussed. For some cases, the minimum distance may be very small, which leads to a high temperature in the sail. In engineering practice, the extreme temperature of the sail exerts a constraint on the minimum distance and solar angle. Therefore, the minimum distance from the Sun during the transfer is very important for a solar sail transfer mission. One reason is that the minimum distance greatly influences the total flight time. Another reason is that it influences the equilibrium temperature of the sail. Finally, this paper gives the results of the transfer trajectory with minimum radius constraints. The transfer time of a three-phase transfer trajectory is evaluated analytically and the results are compared with the results in literature and numerical results in this paper.

2 MINIMUM TIME TRANSFER TRAJECTORY OF AN IDEAL SOLAR SAIL

Consider the problem of minimum time heliocentric transfer of a solar sail from an initial circular orbit of radius r_0 and inclination i_0 to a target circular orbit with radius r_f and inclination i_f. An ideal plane solar sail is assumed in this paper. The lightness number of the sail is used to describe the solar radiation pressure force that can be written as

$$\boldsymbol{F} = \beta \frac{\mu}{R^2} \cos^2 \alpha \boldsymbol{n}, \tag{1}$$

where μ is a solar gravitational constant, computed by multiplying the gravitational constant by the mass of the Sun, R is the distance from the Sun, $\boldsymbol{n}$ is the unit vector along the sail's normal direction and α is the angle between the sail normal and the direction of sunlight. Here β is the lightness number of the sail, which is used to describe the acceleration capability of the sail.

A two-body model is used. Perturbation forces are not considered and only the solar gravity and solar radiation pressure force are exerted on the solar sail. A system of nondimensional units is introduced for convenience. The distance unit is taken as the distance from the Sun to Earth, while the time unit is chosen such that the solar gravitational parameter is unitary. With such a choice, the dynamical equation of motion in the ecliptic's inertial frame can be given by

$$\begin{cases} \dot{\boldsymbol{R}} = V, \\ \dot{\boldsymbol{V}} = -\frac{1}{R^3}\boldsymbol{R} + \beta \frac{1}{R^4}(\boldsymbol{R} \cdot \boldsymbol{n})^2 \boldsymbol{n}. \end{cases} \tag{2}$$

The sail has to escape Earth before its journey to an interplanetary orbit. Either an impulse maneuver is used before sail deployment or the solar radiation pressure force is used to escape Earth. The interplanetary transfer does not include the geocentric orbit and the sail is assumed to depart from the heliocentric orbit of Earth. The departure position and velocity of the sail are the same as those of Earth, namely, $C_\infty = 0$. In this case, the departure time greatly influences the transfer time for some missions. Therefore, the departure time is free and optimized in order to find the minimum transfer time. Then, the constraints of the initial time can be described as

$$\boldsymbol{\Psi}\left[t_0, \boldsymbol{R}(t_0), \boldsymbol{V}(t_0)\right] = \begin{bmatrix} \boldsymbol{R}(t_0) - \boldsymbol{m}(t_0) \\ \boldsymbol{V}(t_0) - \boldsymbol{n}(t_0) \end{bmatrix} = \mathbf{0}, \tag{3}$$

where $\boldsymbol{m}(t_0)$ and $\boldsymbol{n}(t_0)$ are respectively the position and velocity of Earth at the initial time.

The object's position and velocity are dependent on the mission types. For examples, a flyby mission requires only the position to be the same as the object planet, but a rendezvous mission requires both the position and velocity to agree with those of the planet. For the Solar Polar Imager mission, the objective is to transfer the sail to a circular orbit with fixed radius and inclination. The final time constraints of all these different missions are written in a general form as

$$\boldsymbol{\Pi}\left[t_f, \boldsymbol{R}(t_f), \boldsymbol{V}(t_f)\right] = \mathbf{0}. \tag{4}$$

The objective function of the minimum time transfer optimization problem is given by

$$J = -\int_0^t \kappa dt, \tag{5}$$

where κ is a positive weight constant.

The Hamiltonian function of the system is defined as

$$\boldsymbol{H} = -\kappa + \boldsymbol{\lambda}_R(t) \cdot \boldsymbol{V} + \boldsymbol{\lambda}_V(t) \cdot \left[-\frac{\mu}{R^3}\boldsymbol{R} + \beta\frac{\mu}{R^4}(\boldsymbol{R}\cdot\boldsymbol{n})^2\boldsymbol{n} \right]. \tag{6}$$

The time derivatives of the adjoint variables, obtained from the Euler-Lagrange equations, are

$$\begin{cases} \dot{\lambda}_R = -\frac{\partial H}{\partial \boldsymbol{R}} = \frac{\mu}{R^3}\boldsymbol{\lambda}_V - \frac{3\mu}{R^5}(\boldsymbol{R}\cdot\lambda_V)\boldsymbol{R} - 2\beta\frac{\mu}{R^4}(\boldsymbol{R}\cdot\boldsymbol{n})(\boldsymbol{n}\cdot\lambda_V)\left[\boldsymbol{n} - \frac{2(\boldsymbol{R}\cdot\boldsymbol{n})\boldsymbol{R}}{R^2}\right], \\ \dot{\lambda}_V = -\frac{\partial H}{\partial \boldsymbol{V}} = -\boldsymbol{\lambda}_R. \end{cases} \tag{7}$$

The optimal values of the control variables are obtained by invoking Pontryagin's maximum principle (Pontryagin et al. 1962), that is, by maximizing H at any time. By imposing the necessary condition, one has

$$\boldsymbol{n}(t) = \arg\max H(t, \boldsymbol{n}, \boldsymbol{\lambda}). \tag{8}$$

In the low-thrust trajectory optimization, $\boldsymbol{\lambda_V}$ is called the "Prime Vector." To maximize the Hamiltonian function, one has to align the direction of low-thrust along the "Prime Vector." This optimal control law is not suitable for solar sails because of two reasons. The first one is that the solar radiation pressure force cannot always be aligned along the "Prime Vector" because the solar radiation pressure force cannot continuously be sun-ward. The second reason is that the direction of the solar radiation pressure force determines its magnitude, which means that aligning the direction of solar radiation pressure force along the "Prime Vector" does not generate a maximum Hamiltonian function, even if it is possible for the alignment. Maximizing the Hamiltonian function means adjusting the sail attitude to maximize the projection of the solar radiation pressure force along the "Prime Vector." In fact, it is a kind of local optimization problem (McInnes 1999), based on which one knows that the normal vector lies in the plane spanned by the sunlight and the "Prime Vector." In the plane, the angle between the sunlight and the sail normal is given by

$$\tan\alpha = \frac{-3 \pm \sqrt{9 + 8\tan^2\tilde{\alpha}}}{4\tan\tilde{\alpha}}, \tag{9}$$

where $\tilde{\alpha}$ is the angle between the "Prime Vector" and the direction of sunlight.

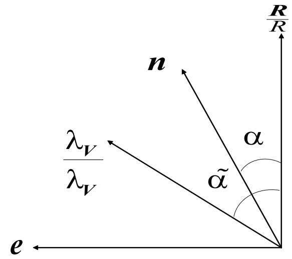

Fig. 1 Optimal control law description.

As shown in Figure 1, the optimal control law can be written in the vector form as

$$\begin{cases} \boldsymbol{n} = \cos\alpha \frac{\boldsymbol{R}}{R} + \sin\alpha \boldsymbol{e}, \\ \frac{\boldsymbol{\lambda}_V}{\lambda_V} = \cos\widehat{\alpha} \frac{\boldsymbol{R}}{R} + \sin\widehat{\alpha} \boldsymbol{e}, \end{cases} \tag{10}$$

where $\boldsymbol{e}$ is the unit vector perpendicular to the sunlight direction in the plane spanned by sunlight and the "Prime Vector."

The boundary conditions are obtained as

$$\boldsymbol{\lambda}_R(t_0) = -\boldsymbol{\gamma}_0 \cdot \frac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{R}(t_0)}, \tag{11}$$

$$\boldsymbol{\lambda}_V(t_0) = -\boldsymbol{\gamma}_0 \cdot \frac{\partial \boldsymbol{\Psi}}{\partial \boldsymbol{V}(t_0)}, \tag{12}$$

$$\boldsymbol{\lambda}_R(t_f) = \boldsymbol{\gamma}_f \cdot \frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{R}(t_f)}, \tag{13}$$

$$\boldsymbol{\lambda}_V(t_f) = \boldsymbol{\gamma}_f \cdot \frac{\partial \boldsymbol{\Pi}}{\partial \boldsymbol{V}(t_f)}, \tag{14}$$

where $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}_f$ are Lagrange multipliers related to the initial and final time constraints.

Substitution of the expressions of the initial and final time constraints into the equations yields the results for boundary conditions. The minimum transfer time is obtained by enforcing the transversality condition as

$$\begin{aligned} H(t_0) &= \boldsymbol{\gamma}_0 \cdot \tfrac{\partial \boldsymbol{\Psi}}{\partial t_0} = -\left[\boldsymbol{\gamma}_{R0} \cdot \tfrac{\partial \boldsymbol{m}(t_0)}{\partial t_0} + \boldsymbol{\gamma}_{V0} \cdot \tfrac{\partial \boldsymbol{n}(t_0)}{\partial t_0}\right] \\ &= -\left[\boldsymbol{\gamma}_{R0} \cdot \boldsymbol{V}(t_0) + \boldsymbol{\gamma}_{V0} \cdot \boldsymbol{a}(t_0)\right] = -\left[\boldsymbol{\gamma}_{R0} \cdot \boldsymbol{V}(t_0) - \boldsymbol{\gamma}_{V0} \cdot \tfrac{1}{R^3(t_0)} \boldsymbol{R}(t_0)\right] \\ &= \boldsymbol{\lambda}_R(t_0) \cdot \boldsymbol{V}(t_0) - \boldsymbol{\lambda}_V(t_0) \cdot \tfrac{1}{R^3(t_0)} \boldsymbol{R}(t_0), \end{aligned} \tag{15}$$

$$H(t_f) = -\boldsymbol{\gamma}_f \cdot \frac{\partial \boldsymbol{\Pi}}{\partial t_f}. \tag{16}$$

The initial adjoint variables are free, while the transversality condition of the initial time is determined by the initial adjoint variables. The boundary conditions and transversality condition of the final time are dependent on the final time constraint.

Missions such as the Solar Polar Imager require the sail to transfer the payload to a new heliocentric orbit. Usually, the inclination and radius of the object's orbit are taken into account. Then, the target orbit is described by the orbit parameters r_f, e_f, and i_f. Therefore, the corresponding final constraints are written as

$$\boldsymbol{\Pi}\left[t_f, \boldsymbol{R}(t_f), \boldsymbol{V}(t_f)\right] = \begin{bmatrix} r(t_f) - r_f \\ e(t_f) - e_f \\ i(t_f) - i_f \end{bmatrix}. \tag{17}$$

In this case, the final position adjoint variables and velocity adjoint variables are constrained. The final boundary conditions are given by

$$\boldsymbol{\lambda}_R\left(t_f\right) = \begin{bmatrix} \frac{\partial r(t_f)}{\partial \boldsymbol{R}(t_f)} & \frac{\partial e(t_f)}{\partial \boldsymbol{R}(t_f)} & \frac{\partial i(t_f)}{\partial \boldsymbol{R}(t_f)} \end{bmatrix} \begin{bmatrix} \gamma_{af} & \gamma_{ef} & \gamma_{if} \end{bmatrix}^{\mathrm{T}}, \tag{18}$$

$$\boldsymbol{\lambda}_V\left(t_f\right) = \begin{bmatrix} \frac{\partial r(t_f)}{\partial \boldsymbol{V}(t_f)} & \frac{\partial e(t_f)}{\partial \boldsymbol{V}(t_f)} & \frac{\partial i(t_f)}{\partial \boldsymbol{V}(t_f)} \end{bmatrix} \begin{bmatrix} \gamma_{af} & \gamma_{ef} & \gamma_{if} \end{bmatrix}^{\mathrm{T}}. \tag{19}$$

The final transversality condition is given by

$$H\left(t_f\right) = -\boldsymbol{\gamma}_f \cdot \frac{\partial \boldsymbol{\Pi}}{\partial t_f} = 0. \tag{20}$$

In order to provide frequent polar observations, a circular polar orbit with a 3:1 resonance with the Earth at a solar distance of 0.48 AU is desired. The 0.48 AU circular polar orbits were considered in a previous study from JPL. This paper focuses on transferring the trajectories to a radius of 0.48 AU and a different inclination. The departure time does not influence the optimal transfer time for this kind of mission since the target orbit has no requirements on the direction of the orbital plane. Therefore, this paper assumes that the sail departs from Earth at an MJD (Modified Julian Date) of 5478. Earth is assumed to evolve in a Keplerian orbit and its orbital elements are given in Table 1.

Table 1 Orbit Elements of Earth

MJD	a (AU)	e	i (°)	Ω (°)	ω (°)	M (°)
2455	0.9999880495	1.67168116×10^{-2}	0.8854353×10^{-3}	175.406477	287.6157755	257.606837

3 THE METHOD OF SOLVING THE OPTIMAL CONTROL PROBLEM

The minimum time required to transfer the sail to a different inclination is obtained using an indirect method when the optimal solution has been found. The initial values of the adjoint variables are estimated in order to obtain the optimal control law. One reason that makes selecting the initial adjoint variables difficult is that the ranges of the adjoint variables cannot be determined. Scaling the adjoint variables makes the optimization much easier. Since the equations with the adjoint variables are homogeneous, a solution to the equation multiplied by a factor will also be a solution of the equation. Therefore, the adjoint variables can be enforced within a unit sphere. The Hamiltonian function has to be scaled to match the transversality conditions, which can be achieved by adjusting the constant κ in the Hamiltonian function. After normalization, the adjoint variables and κ can be transformed into six angle variables.

$$\begin{cases} \boldsymbol{\lambda}_R\left(t_0\right) = \cos\alpha_1\cos\alpha_2[\cos\alpha_3\cos\alpha_4, \cos\alpha_3\sin\alpha_4, \sin\alpha_3]^{\mathrm{T}}, \\ \boldsymbol{\lambda}_V\left(t_0\right) = \cos\alpha_1\cos\alpha_2[\cos\alpha_5\cos\alpha_6, \cos\alpha_5\sin\alpha_6, \sin\alpha_5]^{\mathrm{T}}, \\ \alpha_{1,2} \in \left(0, \frac{\pi}{2}\right), \alpha_{3,5} \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \alpha_{4,6} \in (0, 2\pi), \\ \kappa = \sin\alpha_1. \end{cases} \tag{21}$$

Now, the optimal control problem is transformed into a problem of solving algebraic equations. The free parameters include the departure time t_0, the arrival time t_f, and six angle variables related to initial adjoint variables. They are collected in a vector as

$$\boldsymbol{Y} = \left[\, t_0 \;\; t_f \;\; \alpha_1 \;\; \alpha_2 \;\; \alpha_3 \;\; \alpha_4 \;\; \alpha_5 \;\; \alpha_6 . \,\right]^{\mathrm{T}} \tag{22}$$

The equality constraints include the final state variable constraints, final adjoint variable constraints, and transversality conditions. The total number of final state variable constraints and final adjoint variable constraints is six. Therefore, there are eight free parameters for eight constraints. Good initial values of these free parameters are necessary for the iterative calculation to obtain an accurate solution. The particle swarm optimization method is used to obtain initial values of the free parameters. The optimization results of the particle swarm optimization are taken as an initial estimation for the local optimization method.

As discussed above, the optimal solution with respect to the transfer missions discussed in this paper may not exist, which means there are no free parameters that exactly satisfy the constraints. One way to handle this problem is to enforce either a minimum distance or maximum temperature constraint for the maximization. The process of enforcing the constraint makes it difficult to obtain the optimal solution. Instead, the parameter optimization is used to obtain different feasible solutions by changing the weight coefficient. In this case, the final adjoint variable constraints and transversality conditions are not enforced. The free parameters are optimized to satisfy the final state variable constraints and minimize the transfer time. The transfer trajectory problem is transformed into a parameter optimization problem for computing the feasible trajectories as follows. Find the parameter vector $\boldsymbol{Y}$ that minimizes the transfer time subject to dynamics Equations (2) and (7) and constraint Equation (17). The augmented Lagrange method is used to convert the constrained optimization problem into an unconstrained one. The objective function for the unconstrained optimization problem can be written as

$$J = \kappa \left(t_f - t_0 \right) + \left| \boldsymbol{\Pi} \right| . \tag{23}$$

Without enforcing the first-order necessary conditions, the feasible solution is not unique. The coefficient κ is adjusted to obtain different transfer trajectories. A large κ generates a short transfer time. The optimization process stops when the final constraints are within a specified accuracy. In this paper, the optimization continues until the final constraint uncertainties are less than 10^{-5} AU for radius, 10^{-5} for eccentricity, and 10^{-5} radians for inclination.

4 NUMERICAL RESULTS AND DISCUSSION

The maneuverability of the sail increases with the sail's lightness number. The minimum time for transfer exists for a large lightness number solar sail. A case is given to illustrate that the minimum time transfer trajectory exists for a large lightness number solar sail. As shown in Figure 2, the transfer trajectory satisfying the first-order conditions is shown for the case of $\beta = 0.73$, $r_f = 0.48$ AU and $i_f = 75°$. The great maneuverability of the sail transports the spacecraft to the target orbit in less than one loop. The maneuverability of the solar sail increases as the sail approaches the Sun and decreases as the sail moves away from it. It can be seen from the figure that the sail flies outwards at the beginning of the transfer duration and the transfer trajectory becomes highly elliptical. This means that the sail decreases its maneuverability at the beginning in order to obtain the minimum transfer time. From this example, it can be concluded that minimum transfer trajectories exist for large lightness number solar sails. As the lightness number decreases, the acceleration ability of the sail decreases. The sail needs to approach the Sun to increase its maneuverability and to decrease the transfer time for some small lightness number and high inclination missions. To guarantee the existence of the optimal solution, a given inclination corresponds to a minimum lightness number or a given lightness number corresponds to a maximum inclination. For a sail with a given lightness

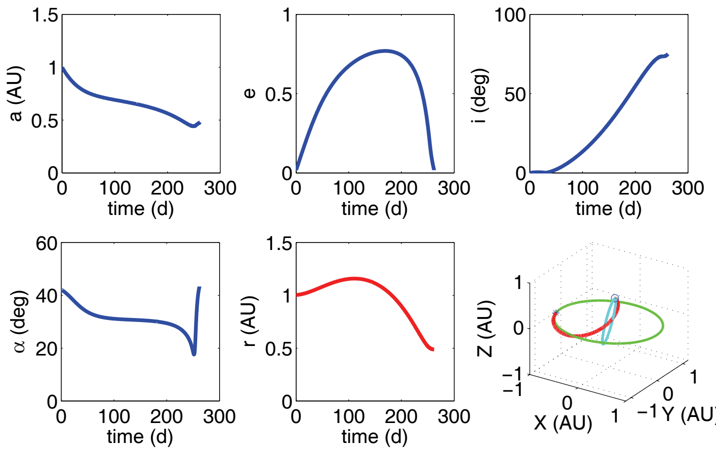

Fig. 2 Transfer parameters of an optimal solution for $\beta = 0.73$.

number, if the target inclination is above the maximum value, the minimum-time transfer trajectory satisfying the first-order optimality conditions may not be obtained using the above indirect method. It cannot be strictly proven that the minimum-time transfer trajectories do not exist. In this paper, a numerical criterion of judging the non-existence of an optimal solution is defined as follows: firstly, the optimal solution cannot be obtained using the indirect method; secondly, the transfer time decreases with minimum radius $r_{\min}$ from the Sun during the transfer. If the optimal solution of the control problem cannot be found by the indirect method, the parameter optimization without enforcing the adjoint variable constraints and transversality conditions is used to obtain a feasible solution. The weight coefficient is varied to obtain different solutions. When the sail approaches the Sun, the integration becomes very slow and the optimization becomes difficult. In addition, the minimum admissible radius is typically constrained by structural requirements and, in particular, by the temperature limit of the sail film. The sail film's equilibrium temperature increases as $r_{\min}$ is reduced and it may eventually exceed the maximum allowable value of either the payload or the solar sail's bearing structure. Therefore, once the maximum admissible temperature is fixed, there exists a maximum allowable minimum distance from the Sun. The existence of such an upper value provides an additional constraint that must be properly taken into account during the performance evaluation process. Here the temperature constraint is not considered when the relationship between $r_{\min}$ and the transfer time is discussed. The coefficient κ is increased to decrease the transfer time and minimum distance until the minimum distance becomes about 0.1 AU. If the time still decreases when $r_{\min}$ arrives at 0.1 AU, it is assumed that no optimal solution exists.

Using the above method to judge the existence of the optimal solution, the boundary conditions of the optimal solution may be derived in the space of lightness number and target inclination. The existence of optimal solutions is restricted to the lightness number for large inclinations. For a given lightness number, the target inclination is increased gradually to find the optimal solution until the solution does not exist. Then, this inclination is recorded and the parameter optimization method is used to find feasible transfer trajectories. When the optimal solution cannot be found by an indirect method, the numerical results show that the transfer time always decreases, as the minimum distance continually decreases. The boundary inclinations of different lightness numbers for AU are calculated, as shown in Figure 3. The boundary inclination increases with the lightness number almost linearly. The parameter space is divided by a boundary line and the optimal solution does not exist in the region above the line. Furthermore, the optimal solution exists for all inclinations when

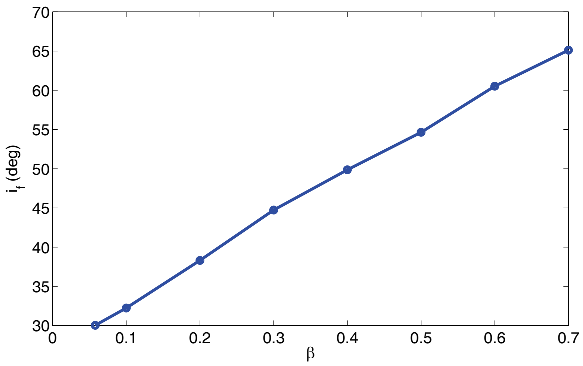

Fig. 3 Boundary of existence of the optimal solution.

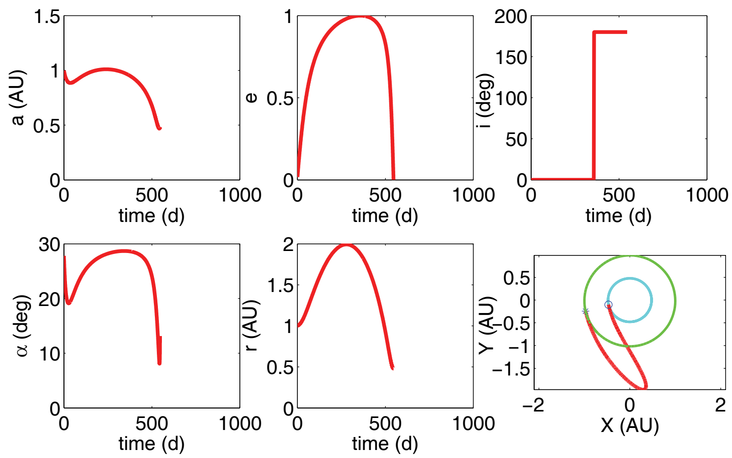

Fig. 4 Optimal reversal transfer trajectory.

the lightness number is above a critical value, including the reversal trajectory, where the inclination is changed from 0° to 180°. The critical lightness number is determined numerically to be about 0.73.

Figure 4 gives the parameters of the minimum time transfer trajectory to a reverse circular orbit using a solar sail of $\beta = 0.75$. The transfer trajectory lies in the ecliptic plane and no out-of-plane component of the solar radiation pressure force is used to change the orbit inclination. The solar radiation pressure force is used to reverse the angular momentum of the sail. That is why the inclination changes from 0° to 180° instantaneously. The reversal trajectory using a solar sail to move towards outer space has been discussed by Vulpetti (1997), where a constant sail attitude is used to reverse the angular momentum around the Sun. This kind of trajectory is not discussed here.

The transfer time for the optimal solution in the region below the boundary line is calculated to investigate the relationship between the transfer time and the target inclination. As shown in Figure 5, the transfer time and minimum distance during the transfer for different target inclinations are given for the case of $\beta = 0.058$ and $r_f = 0.48$ AU. The transfer time increases with target inclination, which is easy to understand since a larger target inclination requires a longer cranking time. The transfer time increases with inclination slowly when the inclination is small and the sail does not need to approach the Sun for small target inclinations. As the inclination increases, the

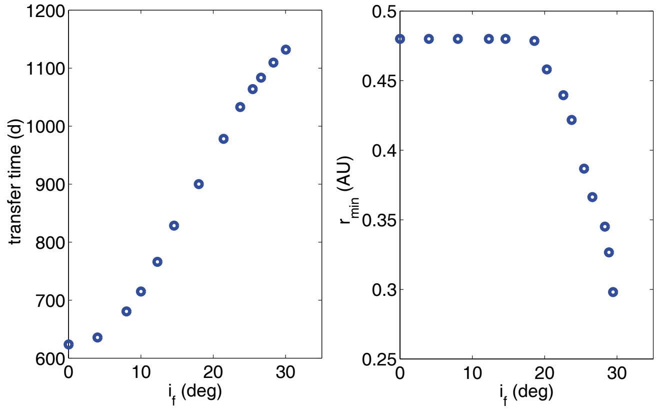

Fig. 5 Relationship between the minimum transfer time, minimum distance and inclination.

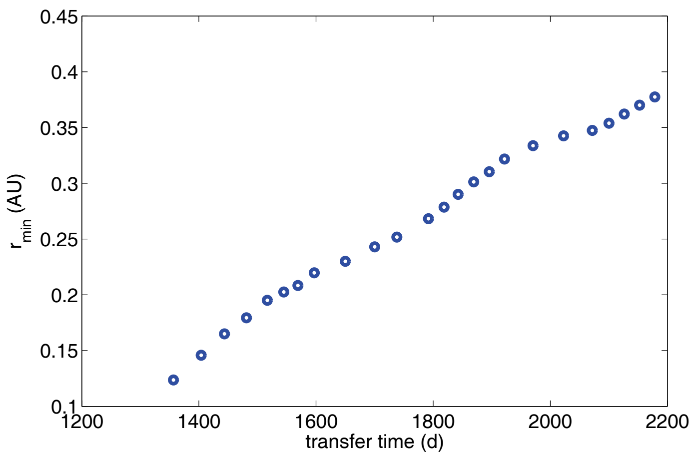

Fig. 6 Relationship between the transfer time and minimum distance.

increment rate of transfer time also increases, which may be seen from the slope of the dotted line. Meanwhile, the sail has to approach the Sun to obtain the minimum time transfer trajectory, and the minimum distance from the Sun decreases as the target inclination increases until the inclination arrives at the critical value. For the inclination above the critical value, the solution satisfying the first-order conditions cannot be found by the indirect method and the transfer time will decrease as the minimum distance from the Sun decreases.

The characteristic acceleration used in previous literatures for the Solar Polar Imager Mission is $0.35\ \mathrm{mm\,s^{-2}}$, which corresponds to a lightness number of about 0.058. The inclination of the SPI orbit is from $75°$ to $90°$, which is far above the critical inclination of $\beta = 0.058$. Therefore, a solar sail of $\beta = 0.058$ may not generate the minimum time transfer trajectory satisfying the first-order conditions without any constraint on the minimum distance or temperature. To compare with the results of Dachwald et al. (2006), a transfer trajectory with similar parameters is used for simulations.

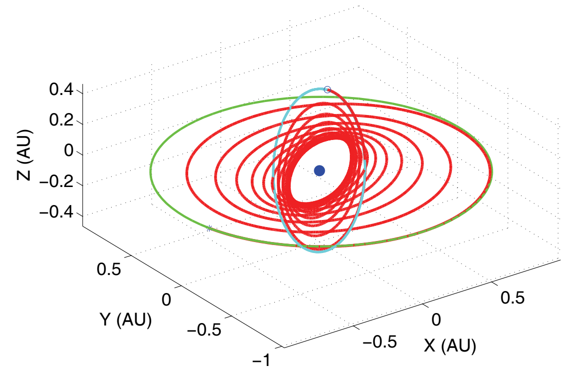

Fig. 7 Transfer trajectory of $\beta = 0.058$ for 0.48 AU and a 75° mission.

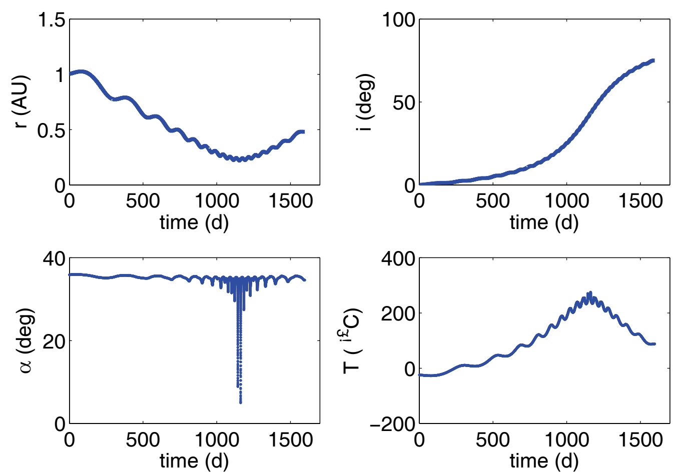

Fig. 8 Transfer parameters of the minimum distance of 0.22 AU.

Figure 6 gives the transfer time and minimum distance from the Sun generated by a different weight coefficient. The small minimum distance corresponds to small transfer time. The transfer duration may be from 3.4 yr to about 6 yr when the minimum distance changes from 0.1 AU to 0.4 AU. Further decrement will further decrease the transfer time. In the literature, a minimum distance of 0.22 AU for the 'Hot' mission scenario is used to obtain the transfer trajectory and the transfer time is about 4.7 yr. The weight coefficient is chosen to make the $r_{\min}$ be 0.22 AU. The corresponding transfer time is about 4.37 yr. The corresponding transfer trajectory is shown in Figure 7 and the transfer parameters are shown in Figure 8.

This paper adopts an ideal sail model while Dachwald et al. use a non-perfect reflection model, where a set of optical coefficients are used to parameterize the optical characteristics of the sail film. This difference in sail models is an important factor that contributes to the difference in transfer times. The transfer time is shorter when the highest equilibrium temperature of the sail here is higher because the pitch angle is much smaller when the sail is near the Sun. However, the time duration of high temperature is much shorter here. For example, the time duration of temperature above 200°C is less than 300 d while the duration in the literature is about 600 d. There are some other differences between both transfer trajectories. The transfer trajectories in the literature include three phases, and one orbit element is changed during each phase. The merit is that the computational effort is less. Different from the three-phase transfer trajectories, the transfer trajectory in this paper includes only one phase. There is no apparent phase division for the inclination and radius changes. It can be found from the transfer trajectory that the radius and inclination are changed simultaneously. The inclination is increased continuously during the entire transfer duration. At the beginning of the transfer trajectory, most of the solar radiation pressure force is used to reduce the radius and the increment rate of inclination is very small. When the sail approaches the Sun, the solar radiation pressure is mainly used to increase the inclination, where the variation of the inclination is very fast. For the component of solar radiation pressure force changing the radius, there is a critical point where the distance from the Sun reaches its minimum.

5 ANALYTICAL RESULTS OF TRANSFER TRAJECTORIES

It is found from the results of the minimum time or near minimum time transfer trajectories that the solar angle between the sunlight and sail normal stays almost constant during the whole transfer journey. Usually, a preliminary estimate of these parameters is obtained by dividing the mission into a certain number of phases and by using a locally optimal control law in each phase. In this section, the transfer time of this kind of three-phase transfer trajectories is evaluated analytically. The spiral trajectory of the sail may be described by the logarithmic spiral curve that is written as (McInnes (1999))

$$r\left(\theta\right) = r_0 \mathrm{e}^{\theta \tan\gamma}, \tag{24}$$

where γ is the angle between the solar sail velocity and the transverse direction.

The planar dynamical equation of motion of the sail in polar coordinates can be written as

$$\begin{cases} \frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -\frac{\mu}{r} + \beta\frac{\mu}{r^2}\cos^3\alpha, \\ r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt} = \beta\frac{\mu}{r^2}\cos^2\alpha\sin\alpha. \end{cases} \tag{25}$$

Substitution of the logarithmic spiral solution into the dynamical equation yields

$$r^3\left(\frac{d\theta}{dt}\right)^2 = \mu\left[1 - \beta\cos^2\alpha\left(\cos\alpha - \tan\gamma\sin\alpha\right)\right]\cos^2\gamma. \tag{26}$$

From Equation (26), the radial velocity and transverse velocity of the sail can be obtained as

$$v_\theta = r\dot{\theta} = \sqrt{\frac{\mu}{r}}\left[1 - \beta\cos^2\alpha\left(\cos\alpha - \tan\gamma\sin\alpha\right)\right]^{1/2}\cos\gamma. \tag{27}$$

The implicit relationship between the spiral angle γ, lightness number β and the pitch angle α can be obtained by combining the dynamical equations and the solar sail velocity components.

$$\frac{\sin\gamma\cos\gamma}{2 - \sin^2\gamma} = \frac{\beta\cos^2\alpha\sin\alpha}{1 - \beta\cos^3\alpha}. \tag{28}$$

For small spiral angles, it is found that Equation (28) may be approximated to obtain

$$\tan\gamma = \frac{2\beta\cos^2\alpha\sin\alpha}{1 - \beta\cos^3\alpha}. \tag{29}$$

Integration of the radial component of the sail velocity yields the transfer time between the initial orbit radius r_0 and a final radial distance r_f.

$$\Delta t = \frac{1}{3}\left|r_f^{3/2} - r_0^{3/2}\right|\left(\frac{2}{\beta\mu\sin\alpha\cos^2\alpha\tan\gamma}\right)^{1/2}. \tag{30}$$

When the radius is reduced, the pitch angle and spiral angle satisfy $\sin\alpha < 0, \tan\gamma < 0$. The formula is still valid for the inward spiral trajectory. The analytical solution of the locally optimal cranking maneuver may be obtained by integration of the variational equation of the inclination.

$$\frac{di}{df} = \beta\cos^2\alpha\sin\alpha\cos\delta\frac{\cos(f+\omega)}{1+e\cos f}. \tag{31}$$

For the locally optimal cranking maneuver, the clock angle is always 0° or 180° to maximize the component of solar radiation pressure force perpendicular to the orbit plane. Assuming the perigee of the argument is constant, the change of the inclination over one period is obtained as

$$\Delta i = 2\beta\cos^2\alpha\sin\alpha\left[\frac{\cos\omega}{e}(\pi - \frac{E_{\pi/2-\omega} - E_{-\pi/2-\omega}}{\sqrt{1-\mathrm{e}^2}}) + \frac{\sin\omega}{e}\ln\frac{1+e\sin\omega}{1-e\sin\omega}\right]. \tag{32}$$

If the orbit is circular, the integration can be simplified as

$$\Delta i = 2\beta\cos^2\alpha\sin\alpha\int_{-\pi/2}^{\pi/2}\cos f\,df = 4\beta\cos^2\alpha\sin\alpha. \tag{33}$$

For a small lightness number, the eccentricity of the spiral orbit is very small. Equation (33) gives a good approximation of the inclination variation over one orbit. Now the transfer time of transfer trajectories including three phases can be evaluated. The first phase is a transfer between an initial circular orbit of radius r_0 to a middle orbit of radius $r_{\min}$. The second phase is the cranking maneuver at the radius $r_{\min}$. The last phase is the transfer from the middle orbit to the target orbit of radius r_f. The time required for each phase may be calculated using Equations (30) and (33). The transfer time from the initial orbit to the middle orbit is given by

$$\Delta t_1 = \frac{1}{3}\left|r_0^{3/2} - r_{\mathrm{m}}^{3/2}\right|\left(\frac{2}{\beta\mu\cos^2\alpha\sin\alpha\tan\gamma}\right)^{1/2}. \tag{34}$$

The time required to change the inclination is obtained by using Equation (33).

$$\Delta t_2 = \frac{2\pi\left(i_f - i_0\right)}{4\beta\cos^2\alpha\sin\alpha}\sqrt{\frac{r_{\mathrm{m}}^3}{\mu}}. \tag{35}$$

Transfer time from the middle radius to the target radius is calculated similarly.

$$\Delta t_3 = \frac{1}{3}\left|r_f^{3/2} - r_{\mathrm{m}}^{3/2}\right|\left(\frac{2}{\beta\mu\cos^2\alpha\sin\alpha\tan\gamma}\right)^{1/2}. \tag{36}$$

Therefore, the total transfer time may be evaluated as

$$\begin{aligned}\Delta t &= \Delta t_1 + \Delta t_2 + \Delta t_3 \\ &= \tfrac{1}{3}\left(\left|r_0^{3/2} - r_{\mathrm{m}}^{3/2}\right| + \left|r_f^{3/2} - r_{\mathrm{m}}^{3/2}\right|\right)\left(\tfrac{2}{\beta\mu\cos^2\alpha\sin\alpha\tan\gamma}\right)^{1/2} + \tfrac{2\pi(i_f - i_0)}{4\beta\cos^2\alpha\sin\alpha}\sqrt{\tfrac{r_{\mathrm{m}}^3}{\mu}}.\end{aligned} \tag{37}$$

If the radius $r_{\min}$ of the middle orbit is chosen to satisfy $r_f \le r_{\mathrm{m}} \le r_0$, the total transfer time is written as

$$\Delta t = \frac{1}{3}\left(r_f^{3/2} + r_0^{3/2}\right)\left(\frac{2}{\beta\mu\cos^2\alpha\sin\alpha\tan\gamma}\right)^{1/2} + \frac{2\pi\left(i_f - i_0\right)}{4\beta\cos^2\alpha\sin\alpha}\sqrt{\frac{r_{\mathrm{m}}^3}{\mu}}. \tag{38}$$

If the radius of the middle orbit is smaller than the target orbit's radius, namely, $r_{\rm m} \leq r_f$, the total transfer time is given by

$$\Delta t = \frac{1}{3}\left(r_0^{3/2} + r_f^{3/2} - 2r_{\rm m}^{3/2}\right)\left(\frac{2}{\beta\mu\cos^2\alpha\sin\alpha\tan\gamma}\right)^{1/2} + \frac{2\pi\left(i_f - i_0\right)}{4\beta\cos^2\alpha\sin\alpha}\sqrt{\frac{r_{\rm m}^3}{\mu}}. \tag{39}$$

For both cases, the total transfer time increases with the inclination difference between the initial and target orbit, and decreases with the lightness number. These two conclusions are obvious since large lightness number means large maneuverability and small inclination variation requires small transfer time. In the first case, the transfer time increases with radius $r_{\rm m}$. Therefore, the minimum transfer time is achieved by taking $r_{\rm m} = r_f$, which means the cranking maneuver is executed in the orbit with the target radius. For the second case, the relationship between the total transfer time and the middle orbit radius may be achieved by deriving the total transfer time with respect to the middle orbit radius.

$$\frac{\partial\Delta t}{\partial r_{\rm m}} = -\left(\frac{2}{\beta\mu\cos^2\alpha\sin\alpha\tan\gamma}\right)^{1/2} r_{\rm m}^{1/2} + \frac{3\pi\left(i_f - i_0\right)}{4\beta\cos^2\alpha\sin\alpha}\sqrt{\frac{r_{\rm m}}{\mu}}. \tag{40}$$

Taking $\frac{\partial\Delta t}{\partial r_{\rm m}} = 0$ yields

$$i_f - i_0 = \frac{4\sqrt{1-\beta\cos^2\alpha}}{3\pi}. \tag{41}$$

Using the assumption that the light number is small, Equation (41) is simplified as $i_f - i_0 = 24.3°$. It means that the total transfer time decreases with the middle orbit radius when the inclination difference is smaller than 24.3°, and increases with the middle orbit radius when it is larger than 24.3°. Whether a smaller radius is required for a cranking maneuver is dependent on the inclination difference between the initial and target orbits. The minimum transfer time may be obtained by taking the target orbit as the middle orbit when the inclination difference is smaller than 24.3°. A smaller orbit radius than the target orbit radius is required to change the inclination when the inclination difference is larger than 24.3°. In addition, the total transfer time decreases as the middle orbit radius decreases and the transfer time arrives at its minimum when the middle radius tends to zero. However, this is impossible both numerically and practically. It can be found from Figure 3 that the critical inclination is about 28° when the lightness number tends to zero. Therefore, the analytical results provide a good evaluation for the critical inclination when the lightness number is small. To compare the analytical results with the numerical results and previous results, similar parameters are used to obtain the analytical evaluation. Using a solar sail of $\beta = 0.058$, the transfer time evaluation of the mission to a circular orbit of 0.48 AU and 75° for different middle radius is shown in Figure 9.

As the middle radius decreases, the time required to adjust the radius increases while the time for the cranking maneuver decreases. The overall result is that the total transfer time decreases as the middle radius decreases. Dachwald used the data fitting to obtain an analytical expression for the transfer duration, which is given by

$$\Delta t = 1374 - 1597a_{\rm cr,opt} + \frac{75}{0.0113a_{\rm cr,opt}^{-1.53}}, \tag{42}$$

where $a_{\rm cr,opt}$ is the optimal semi-major axis calculated by the constraint temperature, which is taken as the radius from the Sun in the literature since the eccentricity is very small. The analytical results, numerical results, and the results of Dachwald are given for comparison, as shown in Figure 10.

The transfer time of the analytical results is shortest and that of Dachwald is longest. One important reason for the analytical method generating a smaller transfer time is that the eccentricity of the

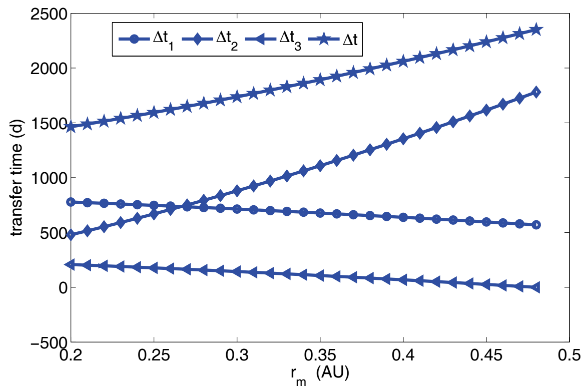

Fig. 9 Transfer time of different middle radius for $\beta = 0.058$.

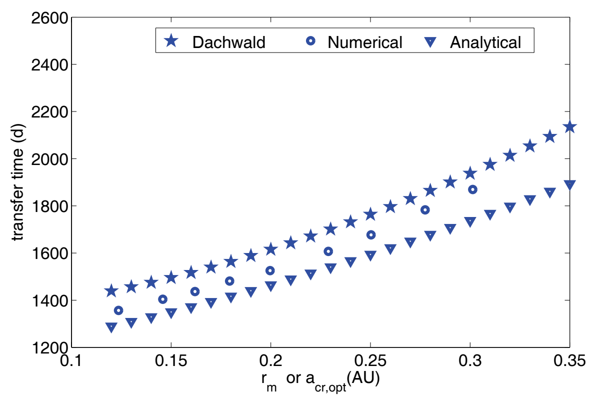

Fig. 10 Comparison of different results.

orbit is not taken as a constraint at the final time. The target orbit is a circular orbit but the final orbit of the logarithmic spiral transfer is not circular. One reason for the Dalchwald's method generating a larger transfer time is that an optical model is used for the sail. Though different methods are used to calculate the transfer time required, the time differences are within 0.15. Therefore, the analytical method may be used to evaluate the transfer for the primary design.

6 CONCLUSIONS

The time optimal control law for interplanetary transfer to a circular orbit of a given radius and inclination is derived for an ideal solar sail. The indirect method is used to obtain the solution of the optimal control problem. The results show that an optimal solution does not exist for a high inclination mission when a small lightness number solar sail is used to achieve the transfer. For a given target radius, the condition for existence of the optimal solution is dependent on the lightness number and target inclination. The boundary of the existing optimal solution is obtained through numerical methods. As the target inclination increases, the transfer time of the optimal solution increases and the minimum distance from the Sun decreases. When the optimal solution does not exist, the parameter optimization is used to obtain a feasible solution. The results show that the transfer time decreases as the minimum distance from the Sun decreases. The transfer time obtained by parameter optimization is smaller than that of Dachwald but the highest temperature is higher. Finally, the analytical expression of transfer time is derived for a three-phase transfer trajectory. The result approximates the numerical result well.

Acknowledgements This work was funded by the National Natural Science Foundation of China (Grant Nos. 10902056 and 10832004).

References

Burch, J. L. 1997, in Sun-Earth Connection Roadmap: Strategic Planning for the Years 2000-2020. TR, National Aeronautics and Space Administration

Dachwald, B., & Seboldt, W. in 54th International Astronautical Congress of the International Astronautical Federation (IAF), IAC-03-Q.5.06, Bremen, Germany

Dachwald, B., Ohndorf, A., & Wie, B. 2006, in AIAA/AAS Astrodynamics Specialist Conference and Exhibit, Keystone, Colorado

Friedman, L., Carroll, W., & Goldstein, R. 1978, AIAA paper, 78

Goldstein, B. E., Buffington, A. & Cummings, A. C., et al. 1998, in Proceedings of SPIE, 3442, 65

Jayaraman, T. S. 1980, Journal of Guidance and Control, 3, 536

Macdonald, M., Hughes, G. W., McInnes, C. R., et al. 2006, Journal of Spacecraft and Rockets, 43, 960

Macdonald, M., & McInnes, C. R. 2005, Journal of Guidance Control and Dynamics, 28, 315

Macdonald, M., McInnes, C., & Dachwald, B. 2007, Journal of Spacecraft and Rockets, 44, 273

McInnes, C. R. 1999, Solar Sailing: Technology, Dynamics and Mission Applications (London: Springer)

Mengali, G., & Quarta, A. A. 2005, Journal of Spacecraft and Rockets, 42, 954

Mengali, G., & Quarta, A. A., 2009, Journal of Guidance, Control, and Dynamics, 32, 456

Pontryagin, L. S., Boltyanskii, V. G., Gamkrelidze, R. V., & Mishchenko, E. F. 1962, The Mathematical Theory of Optimal Processes, Fizmatgiz, Moscow 1961, Authorized Translation from the Russian, Translator: Trirogoff, K. N., ed. Neustadt, L. W.

Sauer, Jr, C. 1976, in American Institute of Aeronautics and Astronautics and American Astronautical Society, Astrodynamics Conference, San Diego, Calif., AIAA, 9 Vol. 1

Sauer, C. G. Jr. 1977, in AAS/AIAA Astrodynamics Conference, 77

Sauer, C. G. Jr. 1999, Astrodynamics, 103, 547

Vulpetti, G. 1997, Acta Astronautica, 40, 733

Wood, L. J., Bauer, T. P., & Zondervan, K. P. 1982, Journal of Guidance, Control, and Dynamics, 5, 221

Wright, J. L., & Warmke, J. M. 1976, in American Institute of Aeronautics and Astronautics and American Astronautical Society, Astrodynamics Conference, San Diego, Calif, AIAA paper 76

Lebedev, V. N., & Zhukov, A. N. 1964, Cosmic Research, 2, 41