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Radiating two-fluid universe coupled with rotation interacting with a scalar field

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Abstract The solution of three new interesting studies, a rotating anisotropic two-fluid universe coupled with radiation and a scalar field, are studied here, where the anisotropic pressure is generated by the presence of two non-interacting perfect fluids which are in relative motion with respect to each other. In this problem, special discussion is made of the physically interesting class of models in which one fluid is a perfect comoving radiative fluid which is taken to model the cosmic microwave background and the second a perfect non-comoving fluid which will model the observed material content of the universe. Studying the different aspects of these model universes, the role of the radiation and scalar fields in defining the physical and dynamical properties of these models is specifically discussed. Analysis of the rotational perturbations is also made, in the course of which the amount of anisotropy induced in the pressure distribution by a small deviation from the Friedmann metric is also investigated, and it is observed that such anisotropies could grow faster than the expansion of the universe. All the models obtained in this problem are found to be theoretically satisfactory and thereby substantiate the possibilities of the existence of such astrophysical objects in this universe.

Key words: cosmic microwave background — cosmology: miscellaneous — stars: formation

1 INTRODUCTION

Rotation plays an important role in the structure and equilibrium configurations of astrophysical objects. That is why during the last few years there has been considerable effort in introducing rotation into the general theory of relativity, so that it can be applied to realistic astrophysical situations. Recent theoretical work on more realistic equations of state of different stellar models reveals that some of these objects could have anisotropic pressure, at least in some parts. A small difference in the radial velocities of the neutron and the proton-electron components of the fluid could cause a significant amount of anisotropy. Existence of such differences in the four-velocities is probably normal during the early phases of the formation of neutron stars. However, whether or not this difference will last long enough to be significant is a subject of further research. Some studies on anisotropy and anisotropic fluid models were made by Herrera & Santosh (1997) and Herrera et al. (2001, 2002, 2004). Anisotropic pressure due to two imperfect fluids, which are weakly interacting, is also an interesting problem in cosmology. Anisotropic fluid models could have a wide range of applications in nature.

It was discovered by Smoot et al. (1977) that there is an observed motion of our galaxy relative to the microwave background radiation. Since the isotropy and homogeneity of both the cosmic microwave background and the observed matter are established with reasonable experimental accuracy, we have to study models which are spherically symmetric with respect to the rest frame associated with the motion of the two fluids, but which have anisotropic pressure. The conventional assumption is that the universe initially evolved from a radiation-like state to a matter-like state ("dust") at later times. The presently accepted view of the evolution of the universe is that, except for very early times, the universe is reasonably described by a "Friedmann-Robetson-Walker (FRW) model." However, in light of the defects in FRW models, we consider isotropic models. The considered models have sources of either perfect comoving radiation fluids or perfect comoving matter fluids; each model is applicable to different eras in the evolution of the universe. As we discuss below, anisotropic pressure, which can be generated through the use of a two-fluid model, where the energy-momentum tensor is composed of two non-interacting perfect fluids, is given as (Letelier 1980; Bayin 1982)

$$T_{\mu\nu} = (P_1 + \rho_1)U^{\mu}U^{\nu} - P_1 g^{\mu\nu} + (P_2 + \rho_2)W^{\mu}W^{\nu} - P_2 g^{\mu\nu}, \tag{1}$$

where $U^{\mu}U_{\mu} = W^{\mu}W_{\mu} = 1$.

Some solutions of two-fluid models were obtained by Alpher & Herman (1949), Chernin (1966), Cohen (1967), McIntosh (1968a,b), Nowotny (1976), Tolman (1934), Maniharsingh (1993) and Singh (2009). MacCallum (1979) mentioned that there are three main ways of generating anisotropic stresses in cosmology: the presence of electromagnetic fields, the presence of a viscous term, and the anisotropic stresses due to the anisotropic expansion of a cloud of collisionless particles. Payne (1970) used models of this type to investigate the effect of a cosmic microwave background with temperature greater than 3K. A fluid consisting of two perfect fluid components and having the energy-momentum tensor given by Equation (1) is expected to reach equilibrium through dissipative mechanisms. However, in some cases, the two perfect fluid components could be decoupled from each other, or could at least be weakly interacting.

In the centers of stars where the gravitational field is strong, a scalar field may have some effects on stellar configurations. Such an effect will become important only when the general relativistic effect itself becomes important. The zero-mass scalar field has acquired particular importance since Weinberg (1978) and Wilczek (1978) proposed the existence of a low-mass (≤ 1 MeV) scalar boson, the so-called axion. Such particles will explain the absence of charge conjugation and parity (CP) non-conservation in strong interactions in particle physics, as pointed out by Peccei & Quinn (1977). Moreover, any light particle has a potential for playing a major role in stellar energy loss; so there may exist a cosmic background of these particles. It is, therefore, expected that the stellar configuration will be appreciably affected by its own scalar field in the case of a neutron star, and more particularly in the case of pulsars. Thus the results of our investigation here will be applicable in the exploration of the behavior, characteristics and the properties of rotating astrophysical objects coupled with cosmic axion fields or scalar fields. Maniharsingh (1990, 1991, 1995), Singh & Bhamra (1987, 1990) and Singh (1987, 1988a,b, 1989a,b, 2010a,b) studied different rotating and non-rotating one-fluid models coupled with a scalar field and with either a radiation field or an electromagnetic field. Thus we study rotating two-fluid model universes coupled with a scalar field and a radiation field in order to be able to understand the hidden properties of such universes, which are not well-understood in the case of one-fluid model universes.

Application of the two-fluid model to cosmology is particularly motivated by observations (Fabbri et al. 1982; Lubin et al. 1983) which indicate that the radiation frame and the matter frame of the universe may not coincide. In the present epoch, considering that matter and radiation are decoupled, the energy-momentum tensor may very well be a reasonable representation of the matter content of the universe. Here in this case, the rotational perturbations of such models are examined in order to substantiate the possibility that the universe is endowed with some rotation. The nature and role of the metric rotation as well as that of matter rotation are studied and the effects of radiation and scalar fields on them are discussed. The periods of physical validity and the restriction on the radii of the models for real astrophysical situations are also studied.

2 DERIVATION OF FIELD EQUATIONS

The metric considered in this problem is

$$\begin{aligned} ds^2 \;=\; & dt^2 - \exp[h(r) + k(t)]dr^2 - \exp[k(t)](r^2 d\Theta^2 + r^2 \sin^2 \Theta d\Phi^2) \\ & + 2r^2 \Omega \exp[k(t)] \sin^2 \Theta d\Phi dt, \end{aligned} \tag{2}$$

where $h(r)$ is an arbitrary function of r, $k(t)$ is an arbitrary function of time t, and $\Omega(r,t)$ is the metric rotation function which is related to the local dragging of inertial frames.

The total energy-momentum tensor $T^{\mu\nu}$ is taken here to be the anisotropic fluid energy-momentum tensor plus the energy-momentum tensor for radially expanding radiation and that of a zero-mass scalar field, thus

$$\begin{aligned} T^{\mu\nu} \;=\; & (\rho_1 + P_1)U^\mu U^\nu - P_1 g^{\mu\nu} + (\rho_2 + P_2)W^\mu W^\nu - P_2 g^{\mu\nu} \\ & + \sigma z^\mu z^\nu + \frac{1}{\phi^2}\left(\phi^\mu \phi^\nu - \tfrac{1}{2} g^{\mu\nu} \phi^k \phi_k\right), \end{aligned} \tag{3}$$

where

$$U^\mu U_\mu = 1, \qquad W^\mu W_\mu = 1. \tag{4}$$

Here σ is the source density of the radiation field and z^μ are the components of radiation, ϕ is the scalar field and P_1, ρ_1, U^μ, P_2, ρ_2, and W^μ are respectively the pressure, density and 4-velocities of the two fluids.

The energy-momentum tensor given by Equation (3) can be put into the standard form by the transformation (Letelier 1980)

$$\left.\begin{aligned} \overline{U}^\mu &= \cos\overline{\alpha} U^\mu + \left[\tfrac{(\rho_2+P_2)}{(\rho_1+P_1)}\right]^{\frac{1}{2}} \sin\overline{\alpha} W^\mu, \\ \overline{W}^\mu &= \left[\tfrac{(\rho_1+P_1)}{(\rho_2+P_2)}\right]^{\frac{1}{2}} \sin\overline{\alpha} U^\mu + \cos\overline{\alpha} W^\mu. \end{aligned}\right\} \tag{5}$$

Now we note that this transformation leaves the quadratic form invariant

$$(\rho_1 + P_1)U^\mu U^\nu = (\rho_2 + P_2)W^\mu W^\nu. \tag{6}$$

Thus

$$T^{\mu\nu}(U, W) = T^{\mu\nu}(\overline{U}, \overline{W}). \tag{7}$$

Now we shall rotate U and W such that one becomes timelike, while the other is spacelike; in other words we choose $\overline{\alpha}$ such that

$$\overline{U}^\mu \, \overline{W}_\mu = 0. \tag{8}$$

From relations (5) and (8) we obtain

$$\tan(2\overline{\alpha}) = \frac{[(\rho_1 + P_1)(\rho_2 + P_2)]^{\frac{1}{2}}}{(\rho_1 + P_1) - (\rho_2 + P_2)} 2W^\mu U_\mu. \tag{9}$$

Thus $\overline{U}_\mu$ is a timelike vector and $\overline{W}_\mu$ is a spacelike vector. Also we define the following quantities

$$\left.\begin{aligned} V^\mu &= \frac{\overline{U}^\mu}{(\overline{U}^\nu \overline{U}_\nu)^{\frac{1}{2}}},\ X^\mu = \frac{\overline{W}^\mu}{(-\overline{W}^\nu \overline{W}_\nu)^{\frac{1}{2}}}, \\ \rho &= T^{\mu\nu} V_\mu V_\nu = (\rho_1 + P_1)\overline{U}^\alpha \overline{U}_\alpha - (P_1 + P_2), \\ P_r &= T^{\mu\nu} X_\mu X_\nu = (P_1 + P_2) - (\rho_2 + P_2)\overline{W}^\alpha \overline{W}_\alpha, \\ P_\perp &= P_1 + P_2. \end{aligned}\right\} \tag{10}$$

Then the energy-momentum tensor can be written in the standard form as

$$\begin{aligned} T^{\mu\nu} &= (\rho + P_\perp)\, V^\mu V^\nu - P_\perp g^{\mu\nu} + (P_r - P_\perp) X^\mu X^\nu \\ &\quad + \frac{1}{\phi^2}(\phi^\mu \phi^\nu - \frac{1}{2} g^{\mu\nu} \phi^k \phi_k) + \sigma z^\mu z_\nu, \end{aligned} \tag{11}$$

where

$$V^\mu V_\nu = 1\,, \quad X^\mu X_\mu = -1, \quad V^\mu X_\mu = 0, \quad z^\mu z_\mu = 0. \tag{12}$$

It may be noted that

$$\begin{aligned} P_r &= -\frac{1}{2}\,(\rho_1 - P_1 + \rho_2 - P_2) \\ &\quad + \frac{1}{2}\left\{[(\rho_1 + P_1) - (\rho_2 + P_2)]^2 + 4(U_\mu W^\mu)^2 (\rho_1 + P_1)(\rho_2 + P_2)\right\}^{\frac{1}{2}}, \end{aligned} \tag{13}$$

and

$$\begin{aligned} \rho &= \frac{1}{2}\,(\rho_1 - P_1 + \rho_2 - P_2) + \frac{1}{2}\left\{\left[\Big(\rho_1 + P_1\Big) + \Big(\rho_2 + P_2\Big)\right]^2\right. \\ &\quad \left. + 4(\rho_1 + P_1)(\rho_2 + P_2)\,\left[(U^\mu W_\mu)^2 - 1\right]\right\}^{\frac{1}{2}}. \end{aligned} \tag{14}$$

In the above equation, V^μ represents the 4-velocity vector of the “mean” fluid, while X^μ is a spacelike vector along the direction of anisotropy. Here, σ is the source density of the radiation field and z^μ are the components of the radiation satisfying the relations

$$z_i z^i = 0, \quad z_1 \neq 0, \quad z_2 = 0, \quad z_3 \neq 0, \quad z_4 \neq 0. \tag{15}$$

Also, ϕ is the scalar field which satisfies the relation

$$\frac{\partial}{\partial x^\nu}\left[\phi_\alpha (-g)^{\frac{1}{2}} g^{\alpha\nu}\right] - \frac{\phi_\alpha \phi_\nu}{\phi}(-g)^{\frac{1}{2}} g^{\alpha\nu} = 0. \tag{16}$$

We choose

$$U^4 \neq 0, \quad U^1 = U^2 = 0, \quad U^3 = \omega \exp\left(-\frac{\gamma}{2}\right), \tag{17}$$

where $\omega = \frac{d\Phi}{dt}$ is the angular velocity of matter.

$$X^4 = X^3 = X^2 = 0\,, \quad X^1 \neq 0\,. \tag{18}$$

Now we define the null vector z^i as $\frac{dx^i}{d\tau}$ (Vaidya 1966). This gives

$$z_4 = \exp\left(\frac{\gamma}{2}\right), \quad z_3 = r^2 \exp\left(k + \frac{\gamma}{2}\right)(\Omega - \omega)\sin^2\Theta. \tag{19}$$

Because of the complexity of the problem, in the following, we take up only the cases where $\gamma = 0$.

Now from Equations (15) and (18) we obtain

$$z^1 = \exp\left(-\frac{h}{2} - \frac{k}{2}\right), \quad z_1 = -\exp\left(\frac{h}{2} + \frac{k}{2}\right). \tag{20}$$

Thus Einstein's field equation

$$R_{\mu\nu} = 8\pi G\left(T_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}T\right)$$

gives, if we consider terms up to the first order in Ω (because of slow rotation),

$$\begin{aligned} 8\pi G P_r = {} & r^{-2}\exp(-h-k) - r^{-2}\exp(-k) - \ddot{k} - \frac{3}{4}\dot{k}^2 \\ & + 8\pi G\left[\sigma(z_1)^2\exp(-h-k) + \frac{\phi'^2}{\phi^2}\exp(-h-k)\right]. \end{aligned} \tag{21}$$

$$8\pi G P_\perp = -\frac{h'}{2r}\exp(-h-k) - \ddot{k} - \frac{3}{4}k^2, \tag{22}$$

$$8\pi G\rho = \left(\frac{h'}{r} - r^{-2}\right)\exp(-h-k) + r^{-2}\exp(-k) + \frac{3}{4}\dot{k}^2 + 8\pi G\left[\sigma(z_4)^2 + \frac{\dot{\phi}^2}{\phi^2}\right], \tag{23}$$

$$\sigma z_1 z_4 + \frac{\phi'\dot{\phi}}{\phi^2} = 0, \tag{24}$$

$$\frac{3}{2}\dot{k}\Omega' + \dot{\Omega}' = 0, \tag{25}$$

and

$$\Omega'' + \left(\frac{4}{r} - \frac{h'}{2}\right)\Omega' = 16\pi G(\rho + P_\perp + \sigma)(\Omega - \omega)\exp(h+k). \tag{26}$$

The overhead 'dot' and 'prime' denote differentiations with respect to t and r, respectively.

3 SOLUTIONS OF THE FIELD EQUATIONS

From Equation (16) we have

$$\left(\ddot{\phi} + \frac{3}{2}\dot{k}\dot{\phi} - \frac{\dot{\phi}^2}{\phi}\right)\exp(k) - \left(\phi'' - \frac{\phi'^2}{\phi} + \frac{2}{r}\phi' - \frac{h'}{2}\phi'\right)\exp(-h) = 0, \tag{27}$$

which gives

$$\frac{\dot{\phi}}{\phi} = a\exp\left(-\frac{3k}{2}\right), \tag{28}$$

and

$$\frac{\phi'}{\phi} = b\,r^{-2}\exp\left(\frac{h}{2}\right), \tag{29}$$

where a and b are arbitrary constants. Thus making use of the relations (19), (20), (28) and (29) in Equation (24), we obtain

$$\sigma = ab\,r^{-2}\exp(-2k). \tag{30}$$

Now inserting relations (22), (23) and (24) in Equation (26), we get

$$\begin{aligned} & \left[\Omega'' + \left(\frac{4}{r} - \frac{h'}{2}\right)\Omega'\right]\exp(-h) \\ & = 2(\Omega-\omega)\left[\left(\frac{h'}{2r}\right)\exp(-h) + r^{-2} - r^{-2}\exp(-h) - \exp(k)\ddot{k} - 8\pi G a^2\exp(-2k)\right]. \end{aligned} \tag{31}$$

Also from Equation (25), we obtain

$$\Omega(r,t) \;=\; M(r)\,\exp\left(-\frac{3k}{2}\right) \;+\; N(t)\,, \tag{32}$$

where $M(r)$ is an arbitrary function of r and $N(t)$ is an arbitrary function of time. Thus relations (31) and (32) together now give

$$\begin{aligned}&\left[\frac{M''}{M}+\left(\frac{4}{r}-\frac{h'}{2}\right)\frac{M'}{M}\right]\exp(-h)\\ &=\;2\left[\left(\frac{h'}{2r}\right)\exp(-h)+r^{-2}-r^{-2}\exp(-h)\;-\;\exp(k)\ddot{k}-8\pi Ga^2\exp(-2k)\right]\\ &\quad\times\left[1+\frac{N(t)\exp\left(3\frac{k}{2}\right)}{M(r)}-\frac{\omega\exp\left(3\frac{k}{2}\right)}{M(r)}\right].\end{aligned} \tag{33}$$

Since the left-hand side is only a function of r, the right-hand side must be a function of either only r or only t. Hence, we see that the form and value of ω will be restricted according to this condition. Now we take up the following cases.

3.1 Case I

In this case we assume that

$$\Omega\;-\;\omega\;=\;a_0\exp\left(-\frac{3}{2}\,k\right)\left[1\;-\;\dot{N}(t)\right], \tag{34}$$

where a_0 is an arbitrary constant and $N(t)$ is an arbitrary function of time. Then from Equation (31) we get

$$\begin{aligned}&\exp\,(-\,h)M''+\,\exp\,(-\,h)\,\left(\frac{4}{r}\;-\;\frac{h'}{2}\right)\,M'\;=\;a_0\left[1\;-\;\dot{N}(t)\right]\\ &\times\;\left[\frac{h'}{2r}\,\exp(-\,h)\;+\;r^{-2}\;-\;r^{-2}\exp(-\,h)\;-\;\exp(k)\ddot{k}\;-\;8\pi Ga^2\exp(-\,2k)\right].\end{aligned} \tag{35}$$

Since the left-hand side of this equation is only a function of r, the right-hand side must be a function of either only r or only t. Here we assume that

$$\frac{h'}{2r}\,\exp\,(-h)\;+\;r^{-2}\;-\;r^{-2}\exp(-\,h)\;=\;-\,2m,$$

where m is an arbitrary constant. This gives

$$h\;=\;-\,\log\,(1\;+\;mr^2). \tag{36}$$

If we make use of relation (36) in Equation (35) we get

$$\begin{aligned}&\left(1\;+\;mr^2\right)M''\;+\;\left(\frac{4}{r}\;+\;5mr\right)\,M'\\ &=\;a_0\,\left[1\;-\;\dot{N}(t)\right]\,\left[-\,2m\;-\;\exp(k)\,\ddot{k}\;-\;8\pi Ga^2\exp(-\,2k)\right],\end{aligned}$$

where we see that the left-hand side is only a function of r whereas the right-hand side is only a function of t. Therefore, we can separate this equation into the relations

$$\left(1\;+\;mr^2\right)\;M''\;+\;\left(\frac{4}{2}\;+\;5mr\right)\;M'\;=\;b_0, \tag{37}$$

and

$$\Big[-2m - \exp(k)\,\ddot{k} - 8\pi G a^2 \exp(-2k)\Big] = \frac{b_0}{a_0}\Big[1 - \dot{N}(t)\Big]^{-1}, \tag{38}$$

where b_0 is a separation constant. Here we see that a solution of Equation (38) is

$$\left.\begin{aligned} N(t) &= \left(1 + \tfrac{b_0}{2ma_0}\right) t + m_0, \\ k &= \tfrac{2}{3}\log\Big[(12\pi G)^{\frac{1}{2}} at + m_1\Big], \end{aligned}\right\} \tag{39}$$

where m_0 and m_1 are arbitrary constants. In this case we have

$$\sigma = abr^2\Big[(12G)^{\frac{1}{2}} at + m_1\Big]^{\frac{4}{3}}, \tag{40}$$

$$\begin{aligned} P_r &= \Big\{(12\pi G^{\frac{1}{2}} at + m_1\Big\}^{-\frac{2}{3}} \\ &\quad\times\left[\frac{m}{8\pi G} + b^2 r^{-4} + \frac{a^2}{2}\Big\{(12\pi G)^{\frac{1}{2}} at + m_1\Big\}^{-\frac{4}{3}} - abr^{-2}\Big\{(12\pi G)^{\frac{1}{2}} at + m_1\Big\}^{-\frac{2}{3}}\right], \end{aligned} \tag{41}$$

$$P_\perp = \frac{m}{8\pi G}\Big\{(12\pi G)^{\frac{1}{2}} at + m_1\Big\}^{-\frac{2}{3}} + \frac{a^2}{2}\Big\{(12\pi G)^{\frac{1}{2}} at + m_1\Big\}^{-2}, \tag{42}$$

$$\begin{aligned} \rho &= a^2\Big\{(12\pi G)^{\frac{1}{2}} at + m_1\Big\}^{-2}\left[\frac{1}{2} + \left(1 + abr^{-2}\Big\{(12\pi G)^{\frac{1}{2}} at + m_1\Big\}^{\frac{2}{3}}\right)\right] \\ &\quad - \frac{3m}{8\pi G}\Big\{(12\pi G)^{\frac{1}{2}} at + m_1\Big\}^{\frac{2}{3}}, \end{aligned} \tag{43}$$

and

$$\phi = \left(\frac{\sqrt{1+r^2} - 1}{r}\right)^{b\sqrt{m}} \Big\{(12\pi G)^{\frac{1}{2}} at + m_1\Big\}^{\frac{1}{(12\pi G)^{\frac{1}{2}}}}. \tag{44}$$

Now taking m = 1, m = 0, and $m = -1$, which respectively correspond to open, flat and closed models, from Equation (37), we obtain three general solutions for $M(r)$ and the corresponding solutions for $\Omega(r,t)$ and $\omega(r,t)$ as: when m = 1, we get

$$M(r) = \frac{b_0}{16}\left(2r^2 + 1\right) - \frac{1}{3}\left(\frac{3}{8} b_0 \sinh^{-1} r + a_1\right)\left(r^{-3} - 2r^{-1}\right)\left(1 + r^2\right)^{\frac{1}{2}} + a_2,$$

where a_1 and a_2 are arbitrary constants. Thus

$$\left.\begin{aligned} \Omega(r,t) &= \Big\{\tfrac{b_0}{16}\left(2r^2+1\right) - \tfrac{1}{3}\left(\tfrac{3}{8} b_0 \sinh^{-1} r + a_1\right)\left(r^{-3} - 2r^{-1}\right)\left(1+r^2\right)^{\frac{1}{2}} + a_2\Big\} \\ &\quad\times\Big\{(12\pi G)^{\frac{1}{2}} at + m_1\Big\}^{-1} + \Big\{\left(1 + \tfrac{b_0}{2a_0}\right) t + m_0\Big\}, \\ \omega(r,t) &= \Big\{(12\pi G)^{\frac{1}{2}} at + m_1\Big\}^{-1} \\ &\quad\times\Big[\tfrac{b_0}{2} + \Big\{\tfrac{b_0}{16}\left(2r^{-2}+1\right) - \tfrac{1}{3}\left(\tfrac{3}{8} b_0 \sinh^{-1} r + a_1\right)\left(r^{-3} - 2r^{-1}\right)\left(1+r^2\right)^{\frac{1}{2}} + a_2\Big\}\Big] \\ &\quad + \Big\{\left(1 + \tfrac{b_0}{2a_0}\right) t + m_0\Big\}. \end{aligned}\right\} \tag{45}$$

For $m = 0$, we get

$$M(r) = \frac{b_0}{10} r^2 - a_3 r^{-3} + a_4,$$

where a_3 and a_4 are arbitrary constants. Therefore here

$$\left.\begin{aligned} \Omega(r,t) &= \left\{\tfrac{b_0}{10}r^2 - a_3r^{-3} + a_4\right\}\left\{(12\pi G)^{\frac{1}{2}}at + m_1\right\}^{-1} + (t + m_0), \\ \omega(r,t) &= \left\{(12\pi G)^{\frac{1}{2}}at + m_1\right\}^{-1}\left[\left\{\tfrac{b_0}{10}r^2 - a_3r^{-3} + a_4\right\} - b_0\right] + (t + m_0). \end{aligned}\right\} \tag{46}$$

Also when $m = -1$ we have

$$M(r) = \frac{b_0}{16}\left(2r^{-2} + 1\right) - \frac{1}{3}\left(\frac{3}{8}b_0\sin^{-1}r + a_5\right)\left(2r^{-1} + r^{-3}\right)\left(1 - r^2\right)^{\frac{1}{2}} + a_6,$$

where a_5 and a_6 are arbitrary constants. Thus

$$\left.\begin{aligned} \Omega(r,t) &= \left\{\tfrac{b_0}{16}(2r^{-2}+1) - \tfrac{1}{3}\left(\tfrac{3}{8}b_0\sin^{-1}r + a_5\right)\left(2r^{-1}+r^{-3}\right)\left(1-r^2\right)^{\frac{1}{2}} + a_6\right\} \\ &\quad\times\left\{(12\pi G)^{\frac{1}{2}}at + m_1\right\}^{-1}\left\{\left(1 - \tfrac{b_0}{2a_0}\right)t + m_0\right\}, \\ \omega(r,t) &= \left\{(12\pi G)^{\frac{1}{2}}at + m_1\right\}^{-1} \\ &\quad\times\left[\left\{\tfrac{b_0}{16}\left(2r^{-2}+1\right) - \tfrac{1}{3}\left(\tfrac{3}{8}b_0\sin^{-1}r + a_5\right)\left(2r^{-1}+r^{-3}\right)\left(1-r^2\right)^{\frac{1}{2}} + a_6\right\} - \tfrac{b_0}{2}\right] \\ &\quad+\left\{\left(1 - \tfrac{b_0}{2a_0}\right)t + m_0\right\}. \end{aligned}\right\} \tag{47}$$

3.2 Case II

Here we assume that

$$\omega = N(t), \tag{48}$$

where $N(t)$ is an arbitrary function of time. Then Equation (33) becomes

$$\begin{aligned} &\frac{1}{2}\left[\frac{M''}{M} + \left(\frac{4}{2} - \frac{h'}{2}\right)\frac{M'}{M}\right]\exp(-h) \\ &= \left[\left(\frac{h'}{2r}\right)\exp(-h) + r^{-2} - r^{-2}\exp(-h) - \exp(k)\,\ddot{k} - 8\pi Ga^2\exp(-2k)\right]. \end{aligned} \tag{49}$$

Now since the left-hand side of Equation (49) is a function of only r, the right-hand side must be a function of either only r or only t. Thus we assume that

$$r^{-2}\exp(-h) - r^{-2} - \frac{1}{2}r^{-1}\exp(-h)\,h' = n_0,$$

where n_0 is an arbitrary constant. This gives

$$h = -\log\left(1 + \frac{n_0r^2}{2}\right). \tag{50}$$

Now making use of relation (50) in Equation (49) we have

$$\left[\frac{M''}{M} + \left(\frac{4}{r} - \frac{h'}{2}\right)\frac{M'}{M}\right]\left(1 + \frac{n_0r^2}{2}\right) = 2\left[-n_0 - \exp(k)\ddot{k} - 8\pi Ga^2\exp(-2k)\right],$$

which can be separated into

$$\left(1 + \frac{n_0r^2}{2}\right)\frac{M''}{M} + \left(\frac{4}{r} + \frac{5n_0}{2}r\right)\frac{M'}{M} = -S_0, \tag{51}$$

and

$$n_0 + \exp(k)\ddot{k} + 8\pi G a^2 \exp(-2k) = \frac{S_0}{2}, \tag{52}$$

where S_0 is a separation constant. If we solve Equation (52), we only obtain an approximate series solution for k. Therefore, to find an exact solution we assume some relation between the constants, say

$$n_0 = \frac{S_0}{2}.$$

Then in this case we get from Equation (52)

$$\dot{k}^2 = \frac{16\pi G a^2}{3} \exp(-3k) + S_1,$$

where S_1 is a positive arbitrary constant, and further we get

$$k = \frac{2}{3} \log\left[k_1 \sinh(k_2 t + k_3)\right], \tag{53}$$

where $k_1 = 4a\left(\frac{\pi G}{3S_1}\right)^{\frac{1}{2}}$, $k_2 = \frac{3}{2} S_1^{\frac{1}{2}}$ and k_3 is an arbitrary constant. If we use the substitution $y = -\frac{n_0 r^2}{2}$ in Equation (51), we obtain

$$y(1-y) M_{yy} + \left(\frac{5}{2} - 3y\right) M_y - \frac{z}{2n_0} M = 0, \tag{54}$$

where M_{yy} and M_y respectively mean differentiation of M with respect to y twice and once. We see that Equation (54) is similar to the hypergeometric equation

$$y(1-y) F_{yy} + [\gamma - (1+\alpha+\beta)y] F_y - \alpha\beta F = 0,$$

for which the general solution is given by

$$F = B_0 F(\alpha,\beta;\gamma;y) + B_1 y^{1-\gamma} F(1-\gamma+\alpha, 1-\gamma+\beta; 2-\gamma; y), \tag{55}$$

where B_0 and B_1 are arbitrary constants and

$$F(\alpha,\beta;\gamma;y) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! \gamma_n} y^n.$$

Therefore, we derive the general solution of Equation (54) to be

$$M(r) = B_0 \sum_{n=0}^{\alpha} \frac{(\alpha)_n (\beta)_n}{n! \left(\frac{5}{2}\right)_n} y^n + B_1 y^{\frac{3}{2}} \sum_{n=0}^{\infty} \frac{\left(\alpha - \frac{3}{2}\right)_n \left(\beta - \frac{3}{2}\right)_n}{n! \left(-\frac{1}{2}\right)_n} y^n. \tag{56}$$

Here since the second term is not regular at $y = 0$, we take $B_1 = 0$. Then we obtain

$$M(r) = B_0 \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{n! \left(\frac{5}{2}\right)_n} y^n,$$

or

$$M(r) = B_0 (1-y)^{\frac{5}{2}-\alpha-\beta} \sum_{n=0}^{\infty} \frac{\left(\frac{5}{2}-\alpha\right)_n \left(\frac{5}{2}-\beta\right)_n}{n! \left(\frac{5}{2}\right)_n} y^n. \tag{57}$$

Thus, according to the different values of α and β, we find different values of $M(r)$ and thereby different values of Ω. For example,

(i) If $\alpha = -1, \beta = 3$, then

$$F\left(-1,\, 3\,;\, \frac{5}{2}\,; y\right) \;=\; B_0\,(1\,-\,y)^{\frac{1}{2}}\,\left(1\,-\,\frac{6}{5}\,y\right).$$

In this case we obtain

$$\Omega(r,t) \;=\; B_0\left(1\,+\,\frac{1}{2}\,n_0 r^2\right)^{\frac{1}{2}}\,\left(1\,+\,\frac{3}{5}\,n_0 r^2\right)\,\{k_1\sinh(k_2 t\,+\,k_3)\}^{-1} +\; N(t)\,.$$

(ii) For $\alpha = 4, \beta = -2$, we have

$$F\left(4,\,-2\,;\,\frac{5}{2}\,;\,y\right) \;=\; B_0\,(1\,-\,y)^{\frac{1}{2}}\,\left(1\,-\,\frac{16}{5}\,y\,+\,\frac{16}{7}\,y^2\right).$$

Thus here we find

$$\Omega(r,t) \;=\; B_0\left(1\,+\,\frac{1}{2}\,n_0 r^2\right)^{\frac{1}{2}}\,\left(1\,+\,\frac{8}{5}\,n_0 r^2 - \frac{4}{7}\,n_0^2 r^4\right)\,\{k_1\sinh(k_2 t\,+\,k_3)\}^{-1} + N(t).$$

(iii) $F\left(\frac{9}{2}, -\frac{5}{2}; \frac{5}{2}\,;\,y\right) \;=\; B_0(1\,-\,y)^{\frac{1}{2}}\left(1\,-\,\frac{9}{2}\,y\,+\,\frac{57}{35}\,y^2\right)$, where $\alpha \;=\; \frac{9}{2}$, $\beta \;=-\frac{5}{2}$ and here

$$\Omega(r,t) \;=\; B_0\left(1\,+\,\frac{1}{2}n_0 r^2\right)^{\frac{1}{2}}\left(1\,+\,\frac{9}{4}\,n_0 r^2\,-\,\frac{57}{140}\,n_0^2 r^4\right)\,\{k_1\sinh(k_2 t\,+\,k_3)\}^{-1} + N(t).$$

(iv) If $\alpha \;=\; -\frac{1}{2}, \beta \;=\; \frac{5}{2}$ then

$$F\left(-\frac{1}{2},\,\frac{5}{2}\,;\,\frac{5}{2}\,;\,y\right) \;=\; B_0\,(1\,-\,y)^{\frac{1}{2}}\,\left(1\,-\,\frac{1}{2}y\,-\,\frac{1}{8}y^2\right).$$

In this case

$$\Omega(r,t) \;=\; B_0\left(1\,+\,\frac{1}{2}\,n_0 r^2\right)^{\frac{1}{2}}\,\left(1\,+\,\frac{1}{4}\,n_0 r^2 + \frac{1}{32}\,n_0^2 r^4\right)\,\{k_1\sinh(k_2 t\,+\,k_3)\}^{-1} + N(t)\,.$$

Also relations (30) and (53) give

$$\sigma \;=\; abr^{-2}\,[k_1\sinh(k_2 t\,+\,k_3)]^{-\frac{4}{3}}\,. \tag{58}$$

In this case, we obtain

$$\begin{aligned} P_r \;&=\; \{k_1\sinh(k_2 t\,+\,k_3)\}^{-\frac{2}{3}}\,\left[\frac{n_0}{16\pi G}\,+\,b^2 r^{-4} - abr^{-2}\,\{k_1\sinh(k_2 t\,+\,k_3)\}^{-2}\right] \\ &\quad + \frac{k_2^2}{12\pi G}\,\left[\text{cosech}^2(k_2 t\,+\,k_3)\,-\,\frac{1}{2}\,\coth^2(k_2 t\,+\,k_3)\right], \end{aligned} \tag{59}$$

$$P_\perp = \frac{1}{8\pi G}\Big[\frac{n_0}{2}\,\{k_1\sinh\,(k_2 t + k_3)\}^{-\frac{2}{3}} + \frac{2}{3}k_2^2\text{cosech}^2\,(k_2 t + k_3) - \frac{k_2^2}{3}\coth^2\,(k_2 t + k_3)\Big], \tag{60}$$

$$\begin{aligned} \rho \;&=\; a^2\,\{k_1\sinh\,(k_2 t\,+\,k_3)\}^{-2}\left[1\,+\,a^{-1}br^{-2}\,\{k_1\sinh\,(k_2 t\,+\,k_3)\}^{\frac{2}{3}}\right] \\ &\quad + \frac{k_2^2}{24\pi G}\,\coth^2\,(k_2 t\,+\,k_3)\,-\,\frac{3n_0}{16\pi B}\,\{k_1\sinh\,(k_2 t\,+\,k_3)\}^{-\frac{2}{3}}, \end{aligned} \tag{61}$$

and

$$\phi \;=\; \tanh\,(k_2 t\,+\,k_3)^{\frac{a}{k_1}}\,\exp\,\left[-\,br^{-1}(1\,+\,\frac{n_0}{2}\,r^2)^{\frac{1}{2}}\right], \tag{62}$$

where cosech means cosecant hyperbolic.

3.3 Case III

Here we assume that

$$\omega \;=\; \Omega \,+\, (N(t) \,-\, \Omega)\; \frac{Q(t)}{M(r)}\,, \tag{63}$$

where $N(t)$ and $Q(t)$ are arbitrary functions of time and $M(r)$ is an arbitrary function of r. Then Equation (33) becomes

$$\begin{aligned} &\left[M'' \,+\, \left(\frac{4}{r} \,-\, \frac{h'}{2}\right)\, M'\right]\, \exp\left(-\,h\right) \\ &=\; 2\left[\frac{1}{2r}\exp(-h)h' + r^{-2} - r^{-2}\exp(-h) \,-\, \exp(k)\ddot{k} \,-\, 8\pi G a^2\exp(-2k)\right]\, Q(t)\,. \end{aligned} \tag{64}$$

Now, since the left-hand side of Equation (64) is a function of only r, the right-hand side must be a function of either only r or only t. Thus we assume here that

$$\frac{1}{2r}\,\exp\left(-h\right)h' \,+\, r^{-2} - r^{-2}\exp\left(-h\right) \;=\; -\,d\,,$$

where d is an arbitrary constant. This gives

$$h \;=\; -\,\log\left(1 \,+\, \frac{dr^2}{2}\right)\,. \tag{65}$$

Now using relation (65) in Equation (64) we obtain

$$\begin{aligned} &\left(1 \,+\, \frac{dr^2}{2}\right)\, M'' \,+\, \left(\frac{5}{2}\,dr \,+\, \frac{4}{r}\right)\, M' \\ &=\; 2\,\left[-\,d \,-\, \exp(k)\ddot{k} \,-\, 8\pi G a^2\exp(-\,2k)\right]\, Q(t)\,. \end{aligned} \tag{66}$$

We see that the left-hand side is a function of only r whereas the right-hand side is a function of only t, therefore, we can equate both of them to a constant. Thus

$$\left(1 \,+\, \frac{dr^2}{2}\right)\, M'' + \left(\frac{5}{2}\,dr \,+\, \frac{4}{r}\right)\, M' \;=\; z_0, \tag{67}$$

and

$$2\,\left[-\,d \,-\, \exp(k)\ddot{k} \,-\, 8\pi G a^2\exp(-\,2k)\right]\, Q(t) \;=\; z_0, \tag{68}$$

where z_0 is a separation constant. Here we see that a solution of Equation (68) is

$$\left.\begin{aligned} Q(t) \;&=\; \tfrac{z_0}{2}\,\left[\pi G a^2\left\{\left(\tfrac{27}{2}\,\pi G\right)^{\frac{1}{2}} at \,+\, \sqrt{\tfrac{3}{4}}\,d_0\right\} \,-\, d\right]^{-1}\,, \\ k \;&=\; \tfrac{2}{3}\,\log\,\left[\left(\tfrac{27}{2}\,\pi G\right)^{\frac{1}{2}}\, at \,+\, \sqrt{\tfrac{3}{4}}\,d_0\right]\,, \end{aligned}\right\} \tag{69}$$

where d_0 is an arbitrary constant. By taking $d = 2$, $d = 0$ and $d = -2$, which correspond to open, flat and closed models respectively, from Equation (67), we obtain three general solutions of $M(r)$ and the corresponding solutions of $\Omega(r, t)$ and $\omega(r, t)$ as: when $d = 2$,

$$M(r) \;=\; \frac{z_0}{16}\,\left(1 \,+\, 2r^{-2}\right) \,-\, \frac{1}{3}\,\left(\frac{3}{8}\,z_0\sinh^{-1} r \,+\, d_1\right)\,\left(r^{-3} \,-\, 2r^{-1}\right)\,\left(1 \,+\, r^2\right)^{\frac{1}{2}} \,+\, d_2\,,$$

where d_1 and d_2 are arbitrary constants. Thus

$$\left.\begin{aligned}\Omega(r,t) &= \left[\tfrac{z_0}{16}\left(1+2r^{-2}\right)-\tfrac{1}{3}\left(\tfrac{3}{8}z_0\sinh^{-1}r+d_1\right)\left(r^3-2r^{-1}\right)\left(1+r^2\right)^{\frac{1}{2}}+d_2\right]\\ &\quad\times\left\{\left(\tfrac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\right\}^{-1}+N(t),\\ \omega(r,t) &= \left[\left\{\tfrac{z_0}{16}\left(1+2r^{-2}\right)-\tfrac{1}{3}\left(\tfrac{3}{8}z_0\sinh^{-1}r+d_1\right)\left(r^{-3}-2r^{-1}\right)\left(1+r^2\right)^{\frac{1}{2}}+d_2\right\}\right.\\ &\quad\left.-\tfrac{z_0}{2}\left\{\pi Ga^2\left[\left(\tfrac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\right]-2\right\}^{-1}\right]\left\{\left(\tfrac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\right\}^{-1}+N(t).\end{aligned}\right\} \tag{70}$$

For $d = 0$ we have

$$M(r) = \left[\frac{z_0}{10}r^2 - r^{-3}d_3 + d_4\right],$$

where d_3 and d_4 are arbitrary constants. Then

$$\left.\begin{aligned}\Omega(r,t) &= \left[\tfrac{z_0}{10}r^2-r^{-3}d_3+d_4\right]\left\{\left(\tfrac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\right\}^{-1}+N(t),\\ \omega(r,t) &= \left[\left(\tfrac{z_0}{10}r^2-r^{-3}d_3+d_4\right)-\tfrac{z_0}{2}\left\{\pi Ga^2\left[\left(\tfrac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\right]-2\right\}^{-1}\right]\\ &\quad\times\left\{\left(\tfrac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\right\}^{-1}+N(t).\end{aligned}\right\} \tag{71}$$

When $d = -2$, we have

$$M(r) = \frac{z_0}{16}\left(1+2r^{-2}\right)-\frac{1}{3}\left(\frac{3}{8}z_0\sin^{-1}r+d_3\right)\left(2r^{-1}+r^3\right)\left(1-r^2\right)^{\frac{1}{2}}+d_6,$$

where d_5 and d_6 are arbitrary constants. Thus

$$\left.\begin{aligned}\Omega(r,t) &= \left[\tfrac{z_0}{16}\left(1+2r^{-2}\right)-\tfrac{1}{3}\left(\tfrac{3}{8}z_0\sin^{-1}r+d_5\right)\left(2r^{-1}+r^{-3}\right)\left(1-r^2\right)^{\frac{1}{2}}+d_6\right]\\ &\quad\times\left\{\left(\tfrac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\right\}^{-1}+N(t),\\ \omega(r,t) &= \left[\left\{\tfrac{z_0}{16}\left(1+2r^{-2}\right)-\tfrac{1}{3}\left(\tfrac{3}{8}z_0\sin^{-1}r+d_5\right)\left(2r^{-1}+r^{-3}\right)\left(1-r^2\right)^{\frac{1}{2}}+d_6\right\}\right.\\ &\quad\left.-\tfrac{z_0}{2}\left\{\pi Ga^2\left[\left(\tfrac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\right]-2\right\}^{-1}\right.\left\{\left(\tfrac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\right\}^{-1}+N(t).\end{aligned}\right\} \tag{72}$$

In this case, we obtain

$$\left.\begin{aligned}P_r &= \{(\tfrac{27}{2}\pi G)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\}^{-\frac{2}{3}}\\ &\quad\times\left[\tfrac{9}{16}a^2\{(\tfrac{27}{2}\pi G)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\}^{-\frac{4}{3}}+\tfrac{d}{16\pi G}+\tfrac{b^2}{r^4}-\tfrac{ab}{r^2}\{(\tfrac{27}{2}\pi G)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\}^{\frac{2}{3}}\right],\\ P_\perp &= \tfrac{1}{16}\{(\tfrac{27}{2}\pi G)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\}^{-\frac{2}{3}}\left[\tfrac{d}{\pi G}+9a^2\{(\tfrac{27}{2}\pi G)^{\frac{1}{2}}at+\sqrt{\tfrac{3}{4}}d_0\}^{-\frac{4}{3}}\right].\end{aligned}\right\} \tag{73}$$

$$\begin{aligned}\rho &= \left[\frac{9}{16}+\left\{1+\frac{b}{ar^2}\left[\left(\frac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\frac{3}{4}}d_0\right]^{\frac{2}{3}}\right\}\right]\\ &\quad\times a^2\left\{\left(\frac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\frac{3}{4}}d_0\right\}^{-2}-\frac{3d}{16\pi G}\left\{\left(\frac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\frac{3}{4}}d_0\right\}^{-\frac{2}{3}}.\end{aligned} \tag{74}$$

Therefore from the relations (28), (29), (65) and (69) we have

$$\phi \;=\; \left\{\left(\frac{27}{2}\,\pi G\right)^{\frac{1}{2}} at \;+\; \left(\frac{3}{4}\right)^{\frac{1}{2}} d_0\right\}^{\left(\frac{27}{2}\pi G\right)^{-\frac{1}{2}}} \exp\left[-br^{-1}\left(1+\frac{d}{2}r^2\right)^{\frac{1}{2}}\right]. \qquad (75)$$

4 DISCUSSION OF THE RESULTS AND CONCLUSIONS

In all the cases we have studied in this problem, analyzing the perturbations in the form of differential rotations of spherically symmetric cosmologies, some physical restrictions are imposed on the metric rotation function. Studies are made to reveal the intrinsic nature of rotation and to elucidate the role of Ω; here it may be noted that Ω plays a role in the dragging of local inertial frames. It is observed that the field equations split into two parts, namely, the field equations for radiating fluid spheres with a scalar field and two equations that determine the metric rotation function $\Omega(r,t)$ and the matter rotation function $\omega(r,t)$. Here we also see that the pressure and the density are unperturbed to first order.

In the models obtained in Case I, we see that as we approach the center, both the radial pressure and the density begin to increase beyond bounds, however the tangential pressure is independent of position. For large distances from the origin, the second term in the density distribution becomes negligible and gives us an approximately homogeneous density. For these models, the anisotropy as a function of position and time is given by

$$\begin{aligned} F(r,t) \;&=\; \frac{P_r}{P_\perp} \\ &=\; \left[\frac{m}{8\pi G} + b^2 r^{-4} + \frac{a^2}{2}\left\{(12\pi G)^{\frac{1}{2}}\, at + m_1\right\}^{-\frac{4}{3}} - abr^{-2}\left\{(12\pi G)^{\frac{1}{2}}\, at + m_1\right\}^{-\frac{2}{3}}\right] \\ &\quad \times \left[\frac{m}{8\pi G} + \frac{a^2}{2}\left\{(12\pi G)^{\frac{1}{2}}\, at + m_1\right\}^{-\frac{4}{3}}\right]^{-1}. \end{aligned}$$

Here we observe that the anisotropy gradually decays along with the distance from the center of the model and vanishes at very large distance from the center. Also the anisotropy is found to be a decreasing function of time. In a perturbative analysis, we can show that anisotropies in the pressure distribution could grow faster than the expansion of the universe, and hence could be important in processes with time scales shorter than the age of universe. Taking up the models obtained in this case, we see that the rotational velocities (thereby, also the rotational perturbations) decay with both time and position if $N(t)$ is a decreasing function of time, except for the case of the flat universe in which the rotational velocities are found to increase with the increase of the radial distance from the center. For a closed universe, the solutions are found to be realistic only when the radial distance r lies within the limit $-1 \le r \le 1$. In all the three models obtained in this case, the rotational velocities are not defined at the point $r = 0$. For all the models, the expansion factor θ (of the fluid lines) is obtained as

$$\theta \;=\; a(12\pi G)^{\frac{1}{2}}\left[(12\pi G)^{\frac{1}{2}}\, at + m_1\right]^{-1}.$$

Here it is seen that these model universes are expanding, though their rate of expansion decreases with time. The scalar field as well as the radiation field are both found to be decreasing functions of time and distance.

In Case II, the metric rotation $\Omega(r,t)$ as well as the matter rotation $\omega(r,t)$ are found to decay with time if $N(t)$ happens to be a decreasing function of time. Here we see that the larger the value of 'a' is (that is, the stronger the scalar field), the smaller the values of $\Omega(r,t)$ and $\omega(r,t)$ are, which shows that the scalar field slows down the rotational velocity. On the other hand, as the value of 'a'

becomes smaller, the radiation field increases and the values of $\Omega(r,t)$ and $\omega(r,t)$ become larger, which shows that the radiation field tends to increase the rotation. For the models obtained here, the fluid pressure is seen to be a decreasing function of both time and radial distance and has a singularity at the center of the universe. Here the expansion factor is found to be

$$\theta = k_2 \coth\left(k_2 t + k_3\right),$$

which shows that the model universes in this case are expanding as well as rotating and thereby may be taken as models representing real astrophysical objects. In this case, the anisotropy (as a function of position and time) is given by

$$F(r,t) \;=\; \frac{8\pi G\{k_1\sinh(k_2t+k_3)\}^{-\frac{2}{3}}[\frac{n_0}{16\pi G}+ab^2r^{-4}-abr^{-2}\{k_1\sinh(k_2t+k_3)\}^{-2}]}{[\frac{n_0}{2}\{k_1\sinh\left(k_2t+k_3\right)\}^{-\frac{2}{3}}+\frac{2}{3}k_2^2\mathrm{cosech}^2\left(k_2t+k_3\right)-\frac{k_2^2}{3}\coth^2+(k_2t+k_3)]}$$
$$+\frac{\frac{2k_2^2}{3}\left[\mathrm{cosech}^2\left(k_2t+k_3\right)\right]-\frac{1}{2}\coth^2(k_2t+k_3)}{\left[\frac{n_0}{2}\left\{k_1\sinh\left(k_2t+k_3\right)\right\}^{-\frac{2}{3}}+\frac{2}{3}k_2^2\mathrm{cosech}^2\left(k_2t+k_3\right)-\frac{k_2^2}{3}\coth^2(k_2t+k_3)\right]}.$$

For all the models obtained in Case III, the rotational velocities (and also the rotational perturbations) decay with the increase of time if $N(t)$ is a decreasing function and $k(t)$ is an increasing function of time. In these models, the rotational velocities are arbitrary at the centers of the models since a singularity exists at such points. In the case when $d=-2$, the solution is realistic only for $-1 \le r \le +1$. In the course of evolution of these model universes, the matter rotation $\omega(r,t)$ as well as the rotational velocity $\Omega(r,t)$ will be affected by the radiation and the scalar fields during their life spans and within the regions where the solutions are valid. Here the fluid pressure and the fluid density of these models are found to be decreasing functions of both r and t. The radiation field decays with time in these models.

Also, we see that the smaller the value of 'a' is (that is, the stronger the radiation field), the smaller the values of $\Omega(r,t)$ and $\omega(r,t)$ are, which shows that the radiation field slows down the rotational velocities in this case. On the other hand, the stronger the scalar field is (the larger the value of 'a'), the larger the values of $\Omega(r,t)$ are, which means that the scalar field tends to increase the rotational velocity (thereby, also the rotational perturbations). In this case, the expansion factor of the fluid lines is found to be

$$\theta \;=\; \left(\frac{27}{2}\pi G\right)^{\frac{1}{2}} a\left[\left(\frac{27}{2}\pi G\right)^{\frac{1}{2}} at+\sqrt{\frac{3}{4}}d_0\right]^{-1}.$$

Thus if 'a' is positive, the models here are found to be expanding as well as rotating, which may be taken as realistic astrophysical representations. For this case, the anisotropy (as a function of position and time) is given by

$$F(r,t) \;=\; 16\left[\frac{d}{16\pi G}+b^2r^{-4}-abr^{-2}\left\{\left(\frac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\frac{3}{4}}d_0\right\}^{\frac{-2}{3}}\right.$$
$$\left.+\frac{9}{16}a^2\left\{\left(\frac{27}{2}\pi G\right)^{\frac{1}{2}}at+\sqrt{\frac{3}{4}}d_0\right\}^{\frac{-4}{3}}\right]$$
$$\times\left[\frac{d}{\pi G}+9a^2\left\{\left(\frac{27}{2}\pi G\right)^{\frac{1}{2}}+\sqrt{\frac{3}{4}}d_0\right\}^{\frac{-4}{3}}\right]^{-1}.$$

Here, the fluid pressure comes out to be a decreasing function both of time and distance. However, the pressure is infinitely large at the center of the model. In this universe, the scalar field is found to be an increasing function of both time and distance, whereas the radiation field is seen to be decreasing with both time and distance. The scalar field has a tendency to increase the rotation while the radiation field has a damping effect on the rotational velocities.

References

Alpher, R. A., & Herman, R. C. 1949, Phys. Rev., 75, 1089
Bayin, S. S. 1982, Phys. Rev. D, 26, 1262
Chernin, A. D. 1966, Soviet Ast., 9, 871
Cohen, J. M. 1967, Nature, 216, 249
Fabbri, R., Guidi, I., Melchiorri, F., & Natale, V. 1982, Second Marcel Grossmann Meeting on General Relativity, ed. R. Ruffini (Amsterdam: North Holland), 909
Herrera, L., Martin, J., & Ospino, J. 2002, Journal of Mathematical Physics, 43, 4889
Herrera, L., Prisco, A. D., Ospino, J., & Fuenmayor, E. 2001, Journal of Mathematical Physics, 42, 2129
Herrera, L., & Santosh, N. O. 1997, Phys. Rep., 286, 53
Herrera, L., di Prisco, A., Martin, J., Ospino, J., Santos, N. O., & Troconis, O. 2004, Phys. Rev. D, 69, 084026
Letelier, P. S. 1980, Phys. Rev. D, 22, 807
Lubin, P. M., Epstein, G. L., & Smoot, G. F. 1983, Phys. Rev. Lett., 50, 616
MacCallum, M. A. H. 1979, in General Relativity: An Einstein Centenary Survey, eds. S. W. Hawking, & W. Isreal (Cambridge Univ. Press), 533
Maniharsingh, K. 1990, Ap&SS, 172, 29
Maniharsingh, K. 1991, Ap&SS, 180, 253
Maniharsingh, K. 1993, Ap&SS, 205, 259
Maniharsingh, K. 1995, General Relativity and Gravitation, 27, 1145
McIntosh, C. B. G. 1968a, MNRAS, 138, 423
McIntosh, C. B. G. 1968b, MNRAS, 140, 461
Nowotny, E. 1976, ApJ, 206, 402
Payne, A. D. 1970, Australian Journal of Physics, 23, 177
Peccei, R. D., & Quinn, H. R. 1977, Phys. Rev. D, 16, 1791
Singh, K. M., & Bhamra, K. S. 1987, International Journal of Theoretical Physics, 26, 175
Singh, K. M. 1987, Ap&SS, 139, 183
Singh, K. M. 1988a, Ap&SS, 148, 149
Singh, K. M. 1988b, International Journal of Theoretical Physics, 27, 345
Singh, K. M. 1989a, Ap&SS, 161, 111
Singh, K. M. 1989b, Ap&SS, 162, 129
Singh, K. M., & Bhamra, K. S. 1990, International Journal of Theoretical Physics, 29, 1015
Singh, K. M. 2009, Ap&SS, 323, 297
Singh, K. M. 2010a, Ap&SS, 325, 127
Singh, K. M. 2010b, Ap&SS, 325, 293
Smoot, G. F., Gorenstein, M. V., & Muller, R. A. 1977, Phys. Rev. Lett., 39, 898
Tolman, R. C. 1934, Proceedings of the National Academy of Science, 20, 169
Vaidya, P. C. 1966, ApJ, 144, 943
Weinberg, S. 1978, Phys. Rev. Lett., 40, 223
Wilczek, F. 1978, Phys. Rev. Lett., 40, 279