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A new interpretation of zero Lyapunov exponents in BKL time for Mixmaster cosmology *

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Abstract A global relationship between cosmological time and Belinskii-Khalatnikov -Lifshitz (BKL) time during the entire evolution of the Mixmaster Bianchi IX universe is used to explain why all the Lyapunov exponents are zero at the BKL time. The actual reason is that the domain of the cosmological time is finite as the BKL time runs from minus infinity to infinity.

Key words: cosmology: early universe — relativity

1 INTRODUCTION

Recently there have been a large number of papers written on the subject of chaos in Relativity and Cosmology (Levin 2006; Wu & Zhang 2006; Wu & Xie 2007; Ma et al. 2009). The former deals mainly with the orbital dynamics of test particles moving in a given gravitational field, and the latter relates to the evolution of a metric itself. Generally speaking, standard indicators of deterministic chaos (like Lyapunov exponents) that were constructed in classical physics and are not invariant under space-time diffeomorphisms can be effectively applied to the first field. Of course, it is better to select coordinate independent manifestations of chaos, such as the invariant Lyapunov exponents and fast Lyapunov indicators of two nearby trajectories developed by Wu & Huang (2003) and Wu et al. (2006). However, these standard indicators used in a certain system about the second field may give rise to a fatal risk in declaring that the system is chaotic. In particular, there was a long history of debating whether the evolution of the Mixmaster Bianchi IX cosmological model becomes really chaotic because different chaotic indicators provided distinct answers to its dynamical nature (Barrow 1982; Berger 1991; Ferraz & Francisco 1992; Burd & Tavakol 1993).

Initially, Barrow (1982) obtained positive Lyapunov exponents that indicate the presence of chaotic behavior [1] from the analysis of the one-dimensional Gaussian map, as discretized approximations of the continuous flow of the Mixmaster cosmology. However, it was uncovered after years of study that there were apparent contradictions between the result and those of some numerical studies. For instance, earlier numerical experiments (Hobill et al. 1992) confirmed the lack of chaos by

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[1] This is not true in all cases. A positive Lyapunov exponent means the onset of chaos for a compact manifold, but not necessarily for an unbounded system.

finding that all Lyapunov exponents are zero [2]. The choice of time variable was regarded as a reason for causing the conflict. From then on it was widely accepted that Lyapunov exponents depend on the choice of time parameter as well as one of distance measure. In this case, it becomes important to evaluate the relationship between the original time parameter (namely, cosmological time t) and a new time variable (e.g. BKL (1970) time τ) in the Mixmaster dynamics. It should be mentioned that the relationship between t and τ in the Mixmaster universe has been viewed as one in the Bianchi I model, namely, $\tau = \ln t$ (Belinskii et al. 1970). However, Szydlowski (1997) and Contopoulos et al. (1999) suggested that the relationship should be $\tau = -\ln t$. Szydlowski also investigated the dependence of Lyapunov exponents on time reparametrizations. In addition, by observing the transformation laws of the Lyapunov exponent flows under space-time reparametrizations, Motter (2003) found that chaos remains invariant with respect to any other time parameter that satisfies the four required conditions. These conditions are sufficient but not necessary for the occurrence of invariant chaos. Usually time transformations in relativistic cosmology cause the violation of these hypotheses for a proper definition of the Lyapunov exponents, so the noninvariance of chaos might appear under these transformations. These facts imply that it is very significant to look for coordinate independent methods of quantifying chaos. As two of the most promising proposals about this point in the Mixmaster dynamics, one is fractal techniques made by Cornish & Levin (1997a,b), and the other deals with a geometrical criterion for local instability or chaos in terms of negative curvature of the Jacobi metric introduced by Szydlowski and coworkers (Szydlowski & Szczesny 1994; Szydlowski & Krawiec 1996). Following the geometrical criterion, Imponente & Montani (2001) found positive Lyapunov exponents of the Mixmaster cosmology by projecting a geodesic deviation vector on an orthogonal tetradic basis. In a word, all these invariant indicators demonstrated that the Mixmaster dynamics is in fact chaotic. It is worth emphasizing that the two methods for identifying the existence of invariant chaos are not perfect. The fractal method is somewhat problematic (Motter & Letelier 2001), and has its limitations in application (Wu & Xie 2008). As to the geometrical criterion, it is neither necessary nor sufficient for the prediction of chaos in some circumstances (Wu 2009).

Although whether the Mixmaster cosmology is chaotic, as an old problem over a decade ago, was solved, the reparametrization of the logarithmic time $\tau = \ln t$ leading to zero Lyapunov exponents in the BKL time τ has been commonly believed in the existing literature. This seems to be cloudy and confusing. In fact, the logarithmic relation $\tau = \ln t$ arises only when the evolution is close to the initial or final singularity. In such cases the Bianchi-IX can be well approximated by a sequence of Kasner transitions. In the language of numerical relativity, the relation between τ and t is a transformation from a unit lapse function to a singularity avoiding a lapse that "freezes the dynamics" as the singularity is approached. That is to say, the so-called logarithmic relation is a local one at the approach of the singularity. However, the computation of the Lyapunov exponents is demanded in order to consider the average over the whole orbit during the entire evolution. If the logarithmic relation between t and τ is a global one applicable to the entire dynamical evolution, one cannot truly explain why all the Lyapunov exponents vanish in time τ (as will be proved in Sects. 2 and 3). In addition, the relationship $\tau = -\ln t$ can explain this, but it is in disagreement with qualitative properties of the function t vs τ. In view of this, the aim of this paper is to provide some insights into a global relationship between the two categories of time so as to truly uncover an actual and direct source leading to chaos hidden behind time τ during the entire evolution of the Mixmaster universe. For this purpose, the effect of Lyapunov exponents on time reparametrizations is discussed in detail in Section 2. This discussion is viewed as an extension to the work of Szydlowski (1997) and Motter (2003). Then, the relationship between the two categories of time is

[2] The statement that a zero Lyapunov exponent indicates a lack of chaos is not true in all cases. It does not distinguish the case of transient chaos at all. On the other hand, if a dynamical system can lie on the center-manifold, the linear analysis that leads to the definition of a Lyapunov exponent is not sufficient to determine the stability or instability of the system. In spite of those, by neglecting the rigor of this concept this paper adopts the same meaning as many references in the literature.

described in Section 3. Meanwhile, the reason for the lack of chaos in time τ is mentioned. Finally, Section 4 draws my conclusions.

2 DEPENDENCE OF LYAPUNOV EXPONENTS ON TIME REPARAMETRIZATIONS

Lyapunov exponents are an important criterion for measuring the rate of exponential divergence between a given trajectory and its neighboring one in the phase space [3] of dynamical systems. The largest of the Lyapunov exponents is defined as the limit

$$\lambda_t = \lim_{t\to\infty} \frac{1}{t} \ln \frac{d(t)}{d_0}, \tag{1}$$

where d_0 [4] and $d(t)$ are the distances of the two trajectories at times 0 and t respectively. Then the compact system considered turns out to be ordered if $\lambda_t = 0$ and chaotic if $\lambda_t > 0$, as stated in Footnotes 1 and 2.

Now let me perform a time transformation $t \to \tau$ such that $t = t(\tau)$, where $t(\tau)$ stands for a strictly positive, continuously differentiable function with respect to new time variable τ. Provided that $d(t)$ is an invariant quantity independent of the choice of space-time coordinates [5], I obtain the maximal Lyapunov exponent

$$\lambda_\tau = \lim_{\tau\to\infty} \frac{1}{\tau} \ln \frac{d(t(\tau))}{d_0} \tag{2}$$

at time τ. Equations (1) and (2) imply that

$$\lambda_\tau = \sigma\lambda_t \tag{3}$$

with

$$\sigma = \lim_{\tau\to\infty} \frac{t(\tau)}{\tau}. \tag{4}$$

The following conclusions can immediately be inferred from Equation (3).

(I) If $\sigma = \infty$, then λ_τ and λ_t may have rather distinct values. In other words, there may be a completely different evaluation of dynamical features for a deterministic system. For instance, the power-law divergence of initially close trajectories with $d(t) = d_0 t^\lambda$ can be transformed into the exponential divergence with $d(\tau) = d_0 \exp(\lambda\tau)$ by means of a logarithmic time reparametrization $\tau = \ln t$. That is to say, this reparametrization converts a zero Lyapunov exponent into a positive Lyapunov exponent such that the original regular system is wrongly interpreted as a chaotic one. As an illustration, the statement that "Chaos can seemingly be removed by a coordinate transformation: simply let $\tau' = \log\tau$ and the chaos disappears" [6] is not true.

(II) If $1 < \sigma < \infty$ and $\lambda_t > 0$, then $\lambda_\tau > \lambda_t$. This leads to an overestimation of Lyapunov exponents observed in time τ such that the strength of chaos seems to increase. Of course, both λ_τ and λ_t remain of the same sign. In this sense, the dynamical information achieved from the two different categories of time should be the same.

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[3] It is better to use the configuration space instead of the phase space in certain cases, as mentioned in the article of Wu & Huang (2003).

[4] The initial separation d_0 should have been infinitesimal, but it is demanded to have a suitable choice from practical calculations. See the article of Tancredi & Sánchez (2001) for more details.

[5] This hypothesis is possible. For example, the proper distance between the observer and the neighbor adopted by Wu & Huang (2003) does satisfy this requirement.

[6] Time parameters τ and τ' in the article of Hartl (2003) refer to time variables t and τ respectively in the present paper.

(III) If $\sigma = 1$, then $\lambda_\tau = \lambda_t$. In order to remove the singularity at $r = 2GM$ of the Schwarzschild space-time, Lemaître (1933) introduced new space-time variables $(\bar{t}, \bar{r}, \bar{\theta}, \bar{\phi})$ in place of Schwarzschild space-time coordinates (t, r, θ, ϕ) such that $\bar{\theta} = \theta$, $\bar{\phi} = \phi$, and

$$\bar{t} = t + 2\sqrt{2GMr} + 2GM \ln \left| \frac{\sqrt{r} - \sqrt{2GM}}{\sqrt{r} + \sqrt{2GM}} \right|, \tag{5}$$

$$r = (2GM)^{1/3} \left[\frac{3}{2} (\bar{r} - \bar{t}) \right]^{2/3}. \tag{6}$$

When the system is compact, r should be bounded. Thus, Lyapunov exponents are not different for the two time parameters. Similarly, Eddington-Finkelstein time (Finkelstein 1958) and Schwarzschild time have this property.

(IV) If $0 < \sigma < 1$ and $\lambda_t > 0$, then $\lambda_\tau < \lambda_t$. The extent of chaos seems to become weaker. For example, through a time transformation (Wu & Huang 2003)

$$t \to \tau : \tau = 10t + \frac{1}{2} r^2, \tag{7}$$

a positive Lyapunov exponent in time τ is about 10 times smaller than that in time t.

(V) If $\sigma = 0$, then a positive Lyapunov exponent in the original time parameter t can be converted into a zero Lyapunov exponent for a new time variable τ. This is so-called chaos hidden behind the new time parameter. For example, positive Lyapunov exponents at old time t would all become zero in any of the following new time parameters: (V.1) $t = \sqrt{\tau}$, (V.2) $t = \ln \tau$ (Cornish 1996), (V.3) $t = e^{-\tau}$ (Szydlowski 1997; Contopoulos et al. 1999), (V.4) Kruskal time (Weinberg 1972), and (V.5) $t = t(\tau)$ but $t(\tau) \in [0, T]$ (T being a finite number) as $|\tau| \to \infty$.

It is shown clearly in the above discussions that Lyapunov exponents are time coordinate dependent. In cases (II)–(IV), the dynamical properties of chaos and order remain invariant under time reparametrizations, but in cases (I) and (V), they may not all be true. In brief, Lyapunov exponents are not very reliable indicators of chaos. Therefore, it is necessary to choose a reference coordinate system in which space-time coordinates are physical. The dynamics of a system should be determined by this reference system. Since there is a gauge freedom of choosing time and space coordinates for a particle moving along a gravitational field in general relativity, the above time t is merely the proper time as a reference. In addition, the proper distances are required. In this way, Wu & Huang (2003) used case (III) as the construction of the definition of invariant Lyapunov exponents independent of space-time transformations. On the other hand, in universal dynamics the universal time can nearly be regarded as a reference because it is just the proper time of a stationary observer in a synchronous reference system, while the proper distances like those in the article of Wu & Huang (2003) are not very easy to obtain.

As is well known, the Mixmaster cosmology is chaotic in the universal time t for the existence of positive Lyapunov exponents, but it seems to be integrable in the BKL time variable τ because all the Lyapunov exponents vanish. In particular, the relation between the two categories of time variables was commonly expressed as $\tau = \ln t$, so time τ was called logarithmic time. As claimed in the Introduction, the relation is not a global one. Otherwise, this statement is completely in conflict with case (I). Next, let me visit the global relation so as to clearly understand why chaos can be hidden behind time τ.

3 WHY LYAPUNOV EXPONENTS VANISH AT TIME τ IN THE BIANCHI IX COSMOLOGY

At first, I introduce the Mixmaster cosmology. Then I discuss the global relationship between the related time parameters according to the evolution of the three-space volume of the universe with time τ, so that the cause of zero Lyapunov exponents at time τ can be found.

3.1 Mixmaster Universe

The Mixmaster cosmology is a homogeneous but anisotropic model of an early closed universe. In a synchronous reference system, the Bianchi type IX metric reads as

$$ds^2 = -dt^2 + \eta_{ij}\omega^i\omega^j, \tag{8}$$

where t is named the cosmological time, $\boldsymbol{\omega}$ is a differential one-form, and the matrix $(\eta_{ij}) = diag(a^2, b^2, c^2)$. The three-volume of the universe is $16\pi^2 abc$ (Belinskii et al. 1970; Landau & Lifshitz 1971). Hereafter, I take $V = abc$. The vacuum Einstein field equations provide the evolution equations governing the behavior of these scale factors a, b and c for the three spatial axes in the forms (Belinskii et al. 1970; Landau & Lifshitz 1971)

$$\frac{(\dot{a}bc)^{\cdot}}{abc} = \frac{1}{2a^2b^2c^2}[(b^2-c^2)^2 - a^4], \tag{9}$$

$$\frac{(a\dot{b}c)^{\cdot}}{abc} = \frac{1}{2a^2b^2c^2}[(a^2-c^2)^2 - b^4], \tag{10}$$

$$\frac{(ab\dot{c})^{\cdot}}{abc} = \frac{1}{2a^2b^2c^2}[(a^2-b^2)^2 - c^4], \tag{11}$$

$$\frac{\ddot{a}}{a} + \frac{\ddot{b}}{b} + \frac{\ddot{c}}{c} = 0. \tag{12}$$

BKL (1970) introduced a new set of integration variables (τ, α, β, γ) such that

$$dt = V d\tau,\ \alpha = \ln a,\ \beta = \ln b,\ \gamma = \ln c. \tag{13}$$

Equations (9)–(12) are simplified to the forms

$$2\alpha'' = (e^{2\beta} - e^{2\gamma})^2 - e^{4\alpha}, \tag{14}$$

$$2\beta'' = (e^{2\alpha} - e^{2\gamma})^2 - e^{4\beta}, \tag{15}$$

$$2\gamma'' = (e^{2\alpha} - e^{2\beta})^2 - e^{4\gamma}, \tag{16}$$

$$\alpha'' + \beta'' + \gamma'' = 2(\alpha'\beta' + \beta'\gamma' + \gamma'\alpha'). \tag{17}$$

Obviously, it is more advantageous to integrate Equations (14)–(17) than Equations (9)–(12) in numerical calculations because there is no singularity in Equations (14)–(17) as τ runs from $-\infty$ to ∞. However, when $\tau \to -\infty$, $t \to 0$, which means the starting time of the universe and its singularity. In addition, it can be inferred from Equations (14)–(17) that

$$0 = -4(\alpha'\beta' + \beta'\gamma' + \gamma'\alpha') + e^{4\alpha} + e^{4\beta} + e^{4\gamma} - 2e^{2(\alpha+\beta)} - 2e^{2(\beta+\gamma)} - 2e^{2(\gamma+\alpha)}. \tag{18}$$

Assume $p_\alpha = -4(\beta' + \gamma')$, $p_\beta = -4(\gamma' + \alpha')$ and $p_\gamma = -4(\alpha' + \beta')$, then Equation (18) is reduced to a zero Hamiltonian (Contopoulos et al. 1999)

$$H = \frac{1}{16}(p_\alpha^2 + p_\beta^2 + p_\gamma^2 - 2p_\alpha p_\beta - 2p_\beta p_\gamma - 2p_\gamma p_\alpha) + e^{4\alpha} + e^{4\beta} + e^{4\gamma} - 2e^{2(\alpha+\beta)} - 2e^{2(\beta+\gamma)} - 2e^{2(\gamma+\alpha)} \equiv 0. \tag{19}$$

Its Hamiltonian canonical equations are

$$\alpha' = (p_\alpha - p_\beta - p_\gamma)/8, \quad p'_\alpha = -4[e^{4\alpha} - e^{2(\alpha+\beta)} - e^{2(\gamma+\alpha)}], \tag{20}$$

$$\beta' = (p_\beta - p_\alpha - p_\gamma)/8, \quad p'_\beta = -4[e^{4\beta} - e^{2(\alpha+\beta)} - e^{2(\gamma+\beta)}], \tag{21}$$

$$\gamma' = (p_\gamma - p_\alpha - p_\beta)/8, \quad p'_\gamma = -4[e^{4\gamma} - e^{2(\alpha+\gamma)} - e^{2(\gamma+\beta)}]. \tag{22}$$

It should be pointed out that Equations (14)–(17) are completely equivalent to Equations (19)–(22).

3.2 Relationship between the Two Categories of Time

For a long time, crucial to the understanding of the relationship between time variables t and τ in the Mixmaster dynamics, studies had been based on the Kasner solution which is related to the closed form solution of the Bianchi I model for which the right-hand sides of Equations (9)–(11) or (14)–(16) are absent. Without a doubt, the Bianchi I dynamics has an analytic solution, in which

$$a \sim t^{p_l}, \;\; b \sim t^{p_m}, \;\; c \sim t^{p_n} \tag{23}$$

with

$$p_l + p_m + p_n = p_l^2 + p_m^2 + p_n^2 = 1. \tag{24}$$

It is easy to get $V = t$ and $\tau = \ln t + const$ (Belinskii et al. 1970). This is why τ in the Mixmaster dynamics was called the logarithmic time. It is worth emphasizing that the treatment is valid only at the approach to singularity. Without a doubt, it is not correct far from the singularity because $\langle 1 \rangle$ the right-hand sides of Equations (9)–(11) or (14)–(16) are always present, and $\langle 2 \rangle$ the logarithmic relation is inconsistent with the analysis of case (I), as stated above. The following lists my investigations about this problem.

My analysis is based on the auxiliary quantity $\Omega = V^{-2}$ given by Cushman & Sniatycki (1995). Then I arrive at

$$\Omega' \; = \; -\frac{2\Omega}{V}V' = \frac{\Omega}{4}(p_\alpha + p_\beta + p_\gamma), \tag{25}$$

$$\Omega'' \; = \; \Omega(p_\alpha^2 + p_\beta^2 + p_\gamma^2)/8. \tag{26}$$

Since $\Omega'' \geq 0$, Ω' is a monotonically increasing function of time τ. This means that there is only a certain time $\tau = \tau_m$ such that $p_\alpha + p_\beta + p_\gamma = 0$ but $p_\alpha^2 + p_\beta^2 + p_\gamma^2 \neq 0$. In addition, $\Omega' < 0$ when $\tau < \tau_m$, and $\Omega' > 0$ when $\tau > \tau_m$. Therefore, Ω reaches one minimum at time τ_m, and decreases monotonically as $\tau < \tau_m$, but increases monotonically as $\tau > \tau_m$. In other words, the volume V has only one maximum V_m at this time, and becomes monotonically increasing if $\tau < \tau_m$, but decreasing if $\tau > \tau_m$. In sum, V is monotonically decreasing to zero as $|\tau|$ goes to infinity, and it varies in the domain $(0, V_m]$.

In the case $\tau = -\infty$, $V = 0$ results in the occurrence of the singularity from which the universe begins to evolve. Meanwhile, this corresponds to cosmological time $t = 0$. Since then, the lapse of time t with time τ obeys the relations

$$t' = V > 0, \quad t'' = -\Omega' \frac{V}{2\Omega}. \tag{27}$$

Several points I conclude from the above equations are as follows: (i) t is a strictly increasing function with respect to time τ; (ii) τ_m is a turning point of the function t; (iii) t is a convex function that has a positive second derivative as $\tau < \tau_m$, but it is a concave function when $\tau > \tau_m$; (iv) t is monotonically decreasing to a constant T_1 (for example, I specify $T_1 = 0$ at the beginning of the universe) for smaller τ, while it is monotonically increasing to another constant T_2 for larger τ. Namely, T_1 is the infimum of t, and T_2 is the supremum. $T = T_2 - T_1$ is called as the universal age similar to that of the uniform, isotropic, standard universal model given by the Robertson-Walker metric with curvature $k = +1$ (Weinberg 1972). These are basic qualitative properties of the function t versus τ. More detailed quantitative descriptions of the function are arranged in the following numerical simulations.

I use a Runge-Kutta-Fehlberg 8(9) integrator with automatical choices of step-sizes to integrate Equations (20)–(22) along the orbit A. Seen from curve A in Figure 1 that draws the evolution of the volume V with time τ, V remains almost invariant in the neighborhood of zero although it increases

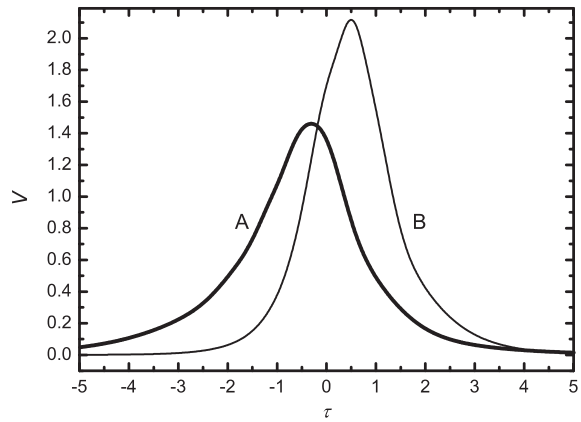

Fig. 1 Dependence of volume V on the BKL time τ. Orbit A corresponds to initial conditions ($\tau = 0$) of $\alpha = 0.3$, $\beta = \gamma = 0$, $\alpha' = -1$ and $\beta' = -0.5$, and orbit B relates to initial conditions of $\alpha = 0.1$, $\beta = \gamma = 0$, $\alpha' = 0$ and $\beta' = -0.71$.

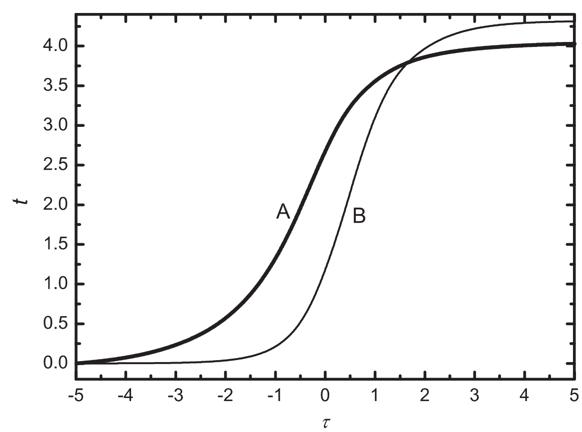

Fig. 2 Same as Fig. 1 but with the cosmological time t in place of V.

slowly as τ goes from $-\infty$ to –5. To my surprise, V expands exponentially for $\tau \in [-5, -0.34]$. Note that there is the largest volume $V_m = 1.459$ at time $\tau_m = -0.34$. Then V decays exponentially and tends nearly to zero when $\tau = 5$. Finally, it becomes zero within an infinite time span. On the other hand, the exponential expansion of t occurs mainly in the τ-time domain [–5, 5], as shown in Figure 2(A). In addition, I can observe that t stabilizes almost to zero for $\tau < -5$, but to $T = 4.029$ for $\tau > 5$.

In light of both the analytic properties of the t-function and the numerical results, the relationship between t and τ seems to be expressed as

$$t = \begin{cases} \nu_1 e^{\mu_1 \tau} + T_1 \,, & (\tau < \tau_m) \,, \\ -\nu_2 e^{-\mu_2 \tau} + T_2 \,, & (\tau > \tau_m) \,, \end{cases} \tag{28}$$

with fit parameters $\nu_1 > 0$, $\nu_2 > 0$, $\mu_1 > 0$, $\mu_2 > 0$, $T_1 \geq 0$ and $T_2 > 0$. These parameters also satisfy

$$\nu_1 e^{\mu_1 \tau_m} + T_1 = -\nu_2 e^{-\mu_2 \tau_m} + T_2 , \quad (29)$$

since the t-function is continuous at time τ_m. Note that Equation (28) is still suitable for other orbits, but τ_m, V_m and T are generally different. In fact, I obtain $\tau_m = 0.5$, $V_m = 2.012$ and $T = 4.312$ from orbit B in Figures 1 and 2.

It is clear that t is only limited to the finite interval (T_1, T_2) as τ ranges from the infinite interval $(-\infty, +\infty)$. As claimed in case (V) of Section 2, all the Lyapunov exponents must be zero at time τ. It should also be emphasized that the so-called logarithmic relationship $\tau = \ln t$ (Belinskii et al. 1970) just matches with the upper part of Equation (28) ($t \to 0$ as $\tau \to -\infty$). In this sense, the relationship seems reasonable from the local point of view. However, it is wrong from the global domain of τ, as mentioned above. In addition, although the relationship $\tau = -\ln t$ (Szydlowski 1997) can explain why all the Lyapunov exponents vanish in time τ, it is completely contrary to the above statement that t increases strictly monotonically with time τ. So, the latter relationship should not be admitted from the global point of view.

4 CONCLUSIONS

In this paper, I discuss the dependence of Lyapunov exponents on time reparametrizations in detail. I analyze the dependence of volume V and cosmic time t on the BKL time τ in the Mixmaster universe so as to truly describe the relationship between these two categories of time from the entire dynamical evolution as well as one from the local dynamical evolution as it approaches singularity. It is worth pointing out that the local relationship is $\tau = \ln t$, but the global one is neither $\tau = \ln t$ nor $\tau = -\ln t$. One should distinguish the two cases when understanding why all the Lyapunov exponents are zero in time τ. The global relationship where the cosmic time t runs only in a finite domain for an infinite span of time τ is suggested to explain the reason.

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