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On critical values concerning the evolution of the long period families *

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Abstract In a previous paper, we proposed another special critical value concerning the evolution of the long period family around the equilateral equilibrium points, besides the two values given by Henrard. Are there any other special critical values? After studying the stability curves of the long period family carefully, we gave a negative answer. During the study, we found an interesting family of periodic orbits which we called the homo family. We studied the evolution of this family following the increase of μ. With these findings, we were able to explain the origin of the four branches of periodic families emanating from L_4 and the stability results of the equilateral equilibrium points.

Key words: celestial mechanics

1 INTRODUCTION

The equilateral equilibrium points of the circular restricted three-body problem are points of the elliptic type for $\mu < \mu_1 = 0.0385\cdots$. There are two fundamental families of periodic orbits emanating from them. They are the long period family and the short period family. Among the $p-$ bifurcation long period orbits and $q-$ bifurcation short period orbits, there are periodic families $B(pL, qS)$, $B(qS, qS')$ and $B(qS, (q+1)S)$ connecting them. These periodic families are often called bridges (Deprit & Henrard 1968; Henrard 2002). One special bridge is the long period family. Generally, it terminates onto a bifurcation short period orbit traveling q times, so we can denote the long period family as bridge $B(L, qS)$.

For some values of μ, the two fundamental frequencies of the equilateral equilibrium points are in commensurability k. These values are (Szebehely 1967)

$$\mu_k = [1 - (k^4 + 38k^2/27 + 1)^{1/2}/(k^2+1)]/2\,. \tag{1}$$

They are called critical values because the long period family undergoes changes when μ passes through these values. Generally, when $\mu \in (\mu_{k+1}, \mu_k)$, the long period family $B(L, (k+1)S)$ exists. When $\mu \in (\mu_k, \mu_{k-1})$, the original long period family $B(L, (k+1)S)$ turns into the new long period family $B(L, kS)$. However, for smaller values of k, there exists a special critical value $\mu_c \in (\mu_{k+1}, \mu_k)$. The bridge $B(L, kS)$ emerges when μ passes through μ_c but before it reaches μ_k. The existence of these special critical mass ratios may lead to the appearance of some special periodic families connecting two long periodic orbits (Hou & Liu 2008b). In previous papers (Hou & Liu 2008a; Henrard 1970), three

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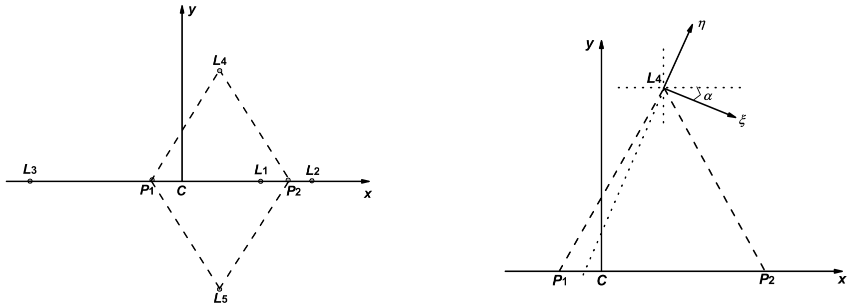

Fig. 1 Synodic coordinate and the coordinate chosen in our work.

special critical values were given out, but the question of whether other special critical values exist was not answered. In this paper, by carefully studying the stability curve of the long period family, we answered this question by providing a negative answer.

The evolutionary details of the long period family around the special critical value $\overline{\mu}$ used in Hou & Liu (2008a) are different from those around the two special critical values μ^* and μ^{**} used in Henrard (1970). This difference leads to the four branches of periodic families emanating from the equilibrium point L_4 when $\mu = \mu_4$. When μ passes through μ_4, an interesting periodic family emerges. Different from all the bridges $B(pL, qS)$, $B(qS, (q+1)S)$ and $B(qS, qS')$, this family terminates onto a member of itself. We call this family the "homo family." Following the increase of μ, we studied the evolutionary details of this family.

In the last part of the paper, we explained some stability results of the equilateral equilibrium points (Deprit & Deprit-Bartholome 1967; Zhao & Liu 1992) with the results found in this paper.

2 METHODOLOGY

The usual synodic coordinate (Szebehely 1967) for the restricted three-body problem is shown in the left figure of Figure 1. The equations of motion are Equation (2). There is an integral of Equation (2), which we call the Jacobi constant, as shown in Equation (3). The coordinate that we choose to analyze with the periodic families is shown in the right figure of Figure 1. It has its origin at L_4 and has a fixed angle α with the synodic coordinate. The angle α is a function of μ. About the details of this coordinate, readers can refer to the book of Szebehely (Szebehely 1967).

$$\begin{cases} \ddot{\boldsymbol{r}} + 2(-\dot{y}, \dot{x})^T = (\partial\Omega/\partial\boldsymbol{r})^T \\ \Omega(x, y) = (\mu(1-\mu) + x^2 + y^2)/2 + (1-\mu)/r_1 + \mu/r_2, \end{cases} \tag{2}$$

$$2\Omega - v^2 = 2\Omega - (\dot{x}^2 + \dot{y}^2) = C. \tag{3}$$

The ways to compute the periodic orbits, to continue the whole periodic family and to identify the $n-$ bifurcation orbits are the same as those in the previous paper (Hou & Liu 2008a), so we will omit the details here. We still use $trace - 2$ as the stability index of a periodic orbit, where $trace$ indicates the trace of the monodromy matrix (Arnold 1999).

The integrator we used is the traditional RKF78 integrator. The accuracy of the refinement is guaranteed to be around 10^{-8}.

3 RESULTS

3.1 Non-existence of Other Special Critical Values

In order to clearly tell the difference between the special critical values μ_c and ordinary critical values μ_k, we first briefly recall the results of the previous paper (Hou & Liu 2008a).

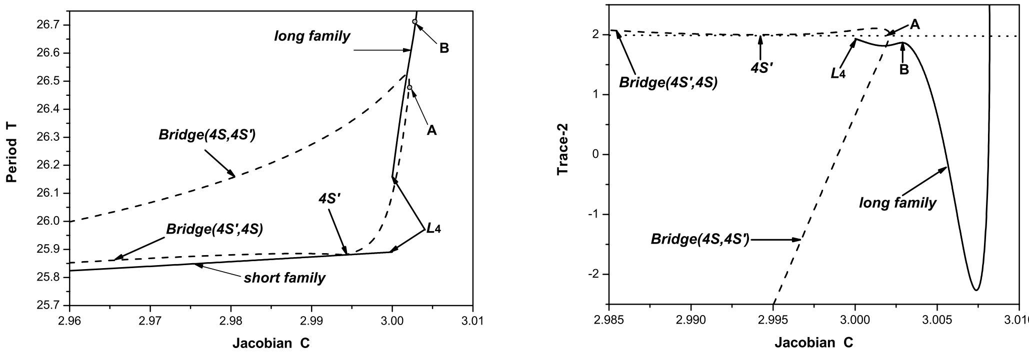

Fig. 2 $T - C$ curve and stability curve for the mass ratio 0.00812.

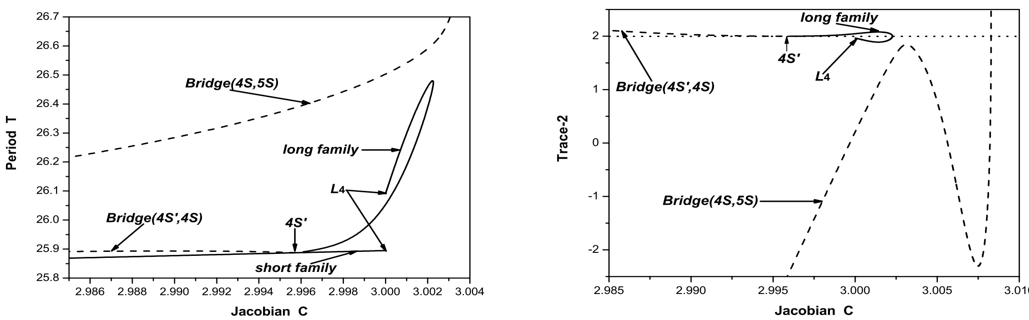

Fig. 3 $T - C$ curve and stability curve for the mass ratio 0.00816.

In that paper, a special critical value $\overline{\mu}$ around 0.00814172 was found. The bridges $B(L, 5S)$ and $B(4S, 4S')$ break up when μ passes through $\overline{\mu}$ but before it reaches μ_4. Shown in Figure 2 are the $T - C$ curve and the stability curve for the mass ratio $0.00812 < \overline{\mu}$. Displayed in Figure 3 are the $T - C$ curve and the stability curve for the mass ratio $0.00816 > \overline{\mu}$. When μ approaches μ_4 but is smaller than $\overline{\mu}$, a hump B in the stability curve of the long period family appears. When $\mu = \overline{\mu}$, the stability index of hump B equals 2 and it indicates a bifurcation long period orbit. At this particular mass ratio, hump B coincides with a bifurcation orbit in one lane of the bridge $B(4S, 4S')$ (denoted as point A in Fig. 2). When $\mu > \overline{\mu}$, the original bridges $B(L, 5S)$ and $B(4S, 4S')$ break into a new long period family $B(L, 4S')$ and a new bridge $B(4S, 5S)$. For the special critical values μ^* and μ^{**}, similar phenomena happen. A hump B in the stability curve of the long period family appears before μ reaches μ_k $(k = 2, 3)$, and breaking up and recombination of bridges happen at the hump B when μ reaches these special critical values. Readers can refer to the paper (Henrard 1970) for more details.

Now, we study the case of $\mu \in (\mu_6, \mu_5)$ where no special critical value exists. Shown in Figure 4 are the $T - C$ curve and stability curve for the mass ratio $0.0055 < \mu_5$. Exibited in Figure 5 are the $T - C$ curve and the stability curve for the mass ratio $0.00552 > \mu_5$. Before μ reaches μ_5, no bifurcation long period orbit appears. When $\mu = \mu_5$, the origin of the long period family at L_4 becomes a bifurcation orbit, and breaking up and recombination happen at the point of L_4 when $\mu = \mu_5$.

Comparing Figures 2 and 3 with Figures 4 and 5, we can see that the hump B in the long period family is crucial for the existence of special critical values. The stability index of the hump B can reach 2 before μ reaches μ_k and becomes a bifurcation orbit instead of the point L_4. Then break up of the bridges $B(L, (k+1)S)$ and $B(kS, kS')$ happens at this hump instead of at the point L_4. Careful studies of the

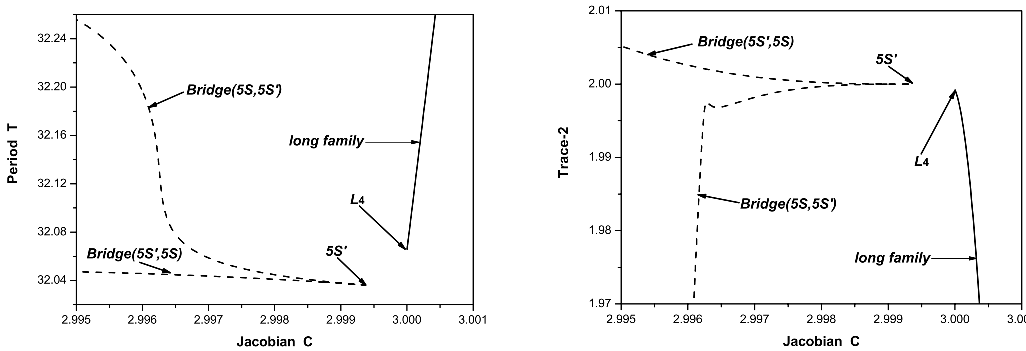

Fig. 4 $T - C$ curve and stability curve for the mass ratio 0.0055.

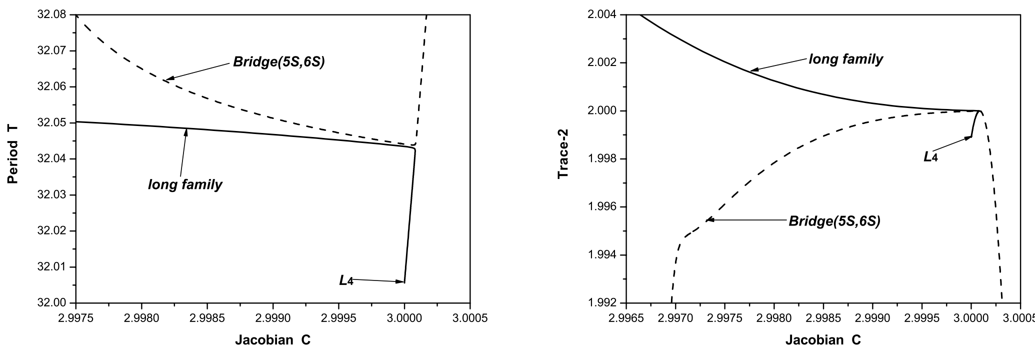

Fig. 5 $T - C$ curve and stability curve for the mass ratio 0.00552.

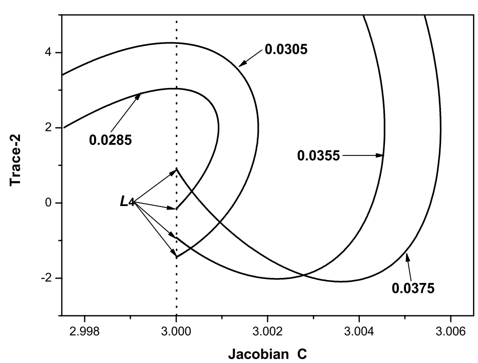

Fig. 6 Stability curves of the long period family for four typical values between μ_2 and μ_1.

stability curve of the long period family show that hump B does not appear for $\mu \in (\mu_{k+1}, \mu_k), k \geq 5$. So, it is impossible for special critical values to exist for $\mu \in (\mu_{k+1}, \mu_k), k \geq 5$. Interestingly, for $\mu \in (\mu_2, \mu_1)$, there is no hump B in the stability curve either. Shown in Figure 6 are the stability curves of the long period family for four typical mass ratios between μ_2 and μ_1. That's the reason why Henrard hasn't been able to find special critical values between μ_2 and μ_1 (Henrard 1970).

3.2 Homo Family

Continuing the periodic families in Figure 3 to the mass ratio μ_4, an interesting phenomenon happens. The point L_4 coincides with the orbit $4S'$, but the long period family does not disappear. This is unique for this special critical value. For the other two critical values, when $\mu = \mu_2$ or $\mu = \mu_3$, the orbit $2S'$ or $3S'$ (see Fig. 9) shrinks to the point L_4. The long period family also shrinks to the point L_4 (we can say the long period family "disappears"). There are four branches of periodic families emanating from the point L_4 now (including the short period family). Although there are also four branches of periodic families emanating from the point L_4 for other critical values $\mu_k(k \geq 5)$ (see Figs. 4 and 5), the difference is that two branches of the long periodic families continuously join when $\mu = \mu_4$ (Deprit & Henrard 1969).

Continuing the periodic families to mass ratios even higher, the two branches of families which join together detach themselves from the equilibrium point L_4 and form a single periodic family, which terminates onto one member of itself, as shown in Figure 7. The mass ratio is 0.008275, which is a little larger than μ_4. Comparing Figures 7 with 3, we can see how this family emanates from the long period family when $\overline{\mu} < \mu < \mu_4$. Since this family terminates onto itself, we call it a homo family.

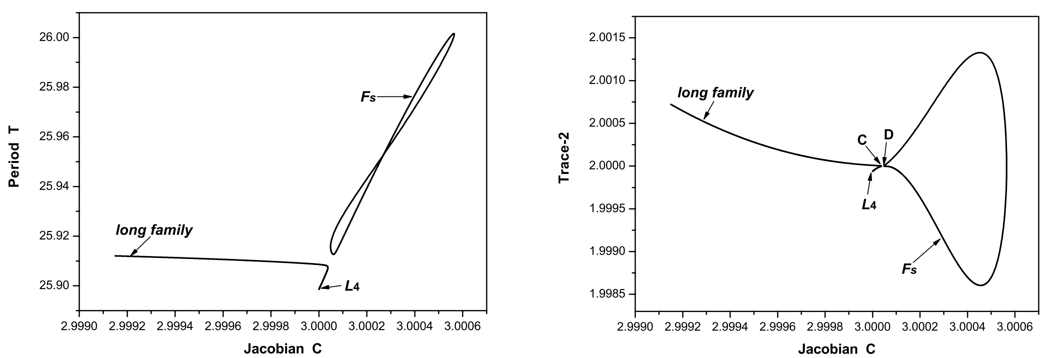

Fig. 7 Periodic families for the mass ratio 0.008275.

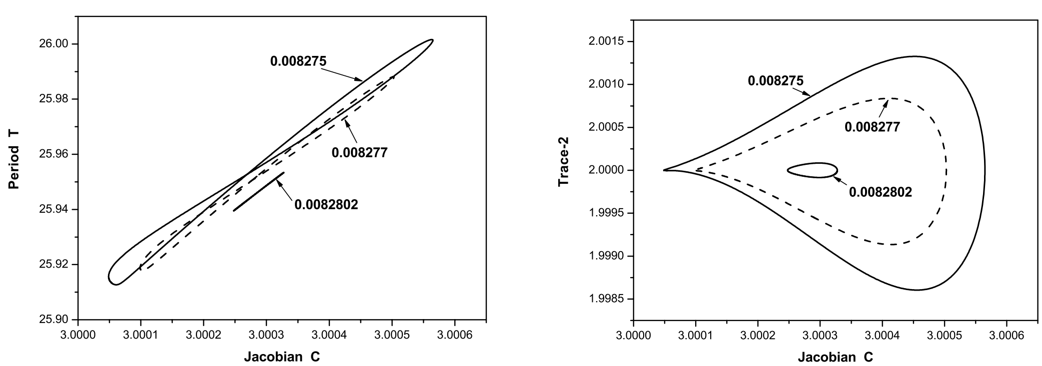

Fig. 8 Homo families for different mass ratios 0.008275, 0.008277 and 0.0082802.

Following the increase of μ, we studied the evolution of the homo family. We find the family shrinks rapidly with increasing μ. Shown in Figure 8 are the $T-C$ curves and the stability curves for mass ratios 0.008275, 0.008277 and 0.0082802. We speculate that the homo family will disappear at one particular

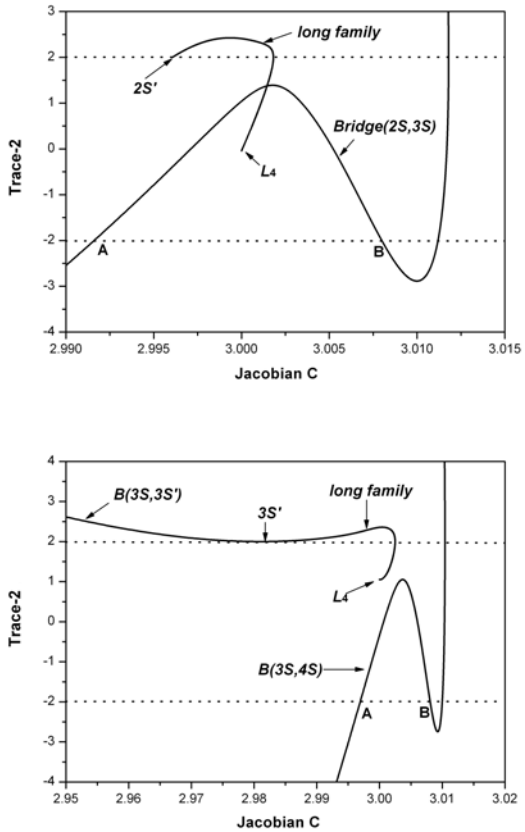

Fig. 9 Stability curves for the mass ratios 0.0208 (top) and 0.0124.

mass ratio which is very close to 0.0082802. However, we are not able to calculate this particular value precisely. When the family shrinks near to a point, it is difficult to continue analyzing it numerically.

The homo family here is not a unique phenomenon in the Circular Restricted Three-Body Problem. For example, many symmetric horseshoe periodic families are also homo families which terminate onto themselves (Hou & Liu 2008c).

3.3 Stability Analysis

As known to everyone, the equilateral equilibrium points are stable for $\mu < \mu_1 = 0.0385\cdots$. The stable region increases with the increase of μ (Zhao & Liu 1992). However, for μ_2 and μ_3, the stable region of the equilateral equilibrium points is zero (Deprit & Deprit-Bartholome 1967). For μ around μ_2 or μ_3, the stable region increases with the increase of $|\mu - \mu_2|$ or $|\mu - \mu_3|$.

For all $\mu < \mu_1$, the stability curve of the long period family consists of a stable part and an unstable part. The stable part starts from the equilibrium point L_4 and extends to a long period orbit with an amplitude A. From this orbit, the long period family triggers the unstable part (with increasing amplitude). For the stable long period orbits, motion around them is stable, and for the unstable ones, motion around them is unstable (Arnold 1999). Generally, the orbits around the equilateral equilibrium points consist of two components of motion: the short period motion and the long period motion. The short period motion is always stable (Deprit & Henrard 1968), so the long period orbit with amplitude A can be taken as an approximation of the stable region. Generally, A increases with the increase of μ. This explains the fact that the stable region grows with increasing μ. As for the zero stable region for μ_2 and μ_3, it is associated with special critical values μ^* and μ^{**}.

For mass ratios $\mu \in (\mu^{**}, \mu_2)$, the bridges $B(L, 3S)$ and $B(2S, 2S')$ break into bridges $B(L, 2S')$ and $B(2S, 3S)$. Shown in the left panel of Figure 9 (x axis indicates the Jacobi constant, while y axis indicates the stability index "$trace - 2$") are stability curves of these bridges for mass ratio $\mu^{**} < 0.0208 < \mu_2$. With μ approaching μ_2 from μ^{**}, the point L_4 is approaching the orbit $2S'$ and the stable part of the long period family gets smaller. The stable region also reduces in size. When $\mu = \mu_2$, L_4 coincides with the orbit $2S'$ and the long period family disappears. At this mass ratio, the AB part of the bridge $B(2S, 3S)$ will also become unstable. Hence, there are no stable periodic orbits around the point L_4 when $\mu = \mu_2$ (with the exception of the short period orbits). This explains the region with zero stability when $\mu = \mu_2$. When $\mu > \mu_2$, the new long period family appears. The stable part of the long period family grows with the increase of μ, and so does the region of stability. For mass ratios $\mu \in (\mu^*, \mu_3)$, similar phenomena happen. Shown in the right panel of Figure 9 are the stability curves for the mass ratio $\mu^* < 0.0124 < \mu_3$. Although there is one lane of the bridge $B(3S, 3S)$ compared with the left figure, it is always unstable.

For the special critical value $\overline{\mu}$, the case is different from the previous two values. As stated before, when μ reaches μ_4 from $\overline{\mu}$, the long period family doesn't appear and still has a stable part. When μ grows larger than μ_4, the homo family, which is very close to the point L_4, also has a stable part. In addition, the new long period family with stable members appears. So, for μ growing from smaller than μ_4 to larger than μ_4, the stable region is never zero. For other critical mass ratios $\mu_k (k \geq 5)$, the stable region is also not zero when μ grows from smaller than μ_k to larger than μ_k, which can be seen from the evolutionary details shown in Figures 4 and 5.

4 CONCLUSIONS

In this paper, we prove the non-existence of other special critical values concerning the evolution of the long period family except the three ones already given in literature. The unique evolution of the periodic families for μ larger than $\overline{\mu}$ leads to the special phenomenon of periodic families for $\mu = \mu_4$ and the interesting family when $\mu > \mu_4$. With these findings, we explained some stability results of the equilateral equilibrium points. Although we based our conclusions on limited computed examples, the results computed are representative. For mass ratios smaller or larger than the critical mass ratios, similar characteristic curves of periodic families as the computed examples can be found. Therefore, the conclusions we drew are proper.

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