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Global Axisymmetric Stability Analysis for a Composite System of Two Gravitationally Coupled Scale-Free Discs *

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Abstract For a composite system of gravitationally coupled stellar and gaseous discs, we have carried out a linear stability analysis for axisymmetric coplanar perturbations using the two-fluid formalism. The background stellar and gaseous discs are taken to be scale-free with all physical variables varying as powers of the cylindrical radius r with compatible exponents. The unstable modes set in as neutral modes or stationary perturbation configurations with angular frequency $\omega = 0$. The axisymmetric stable range is bounded by two marginal stability curves derived from stationary perturbation configurations. Because of the gravitational coupling between the stellar and the gaseous discs, one only needs to consider the parameter regime of the stellar disc. There exist two unstable regimes in general: a collapse regime corresponding to large-scale perturbations and a ring-fragmentation regime corresponding to short-wavelength perturbations. The composite system will collapse if it rotates too slowly and will succumb to ring-fragmentation instabilities if it rotates sufficiently fast. The overall stable range against axisymmetric perturbations is determined by a necessary $D-$criterion involving the effective Mach number squared D_s^2 (the squared ratio of the stellar disc rotation speed to the stellar velocity dispersion up to a numerical factor). Different mass ratio δ and sound speed ratio η of the gaseous and stellar disc components will alter the overall stability. For spiral galaxies or circumnuclear discs, we further include the dynamical effect of a massive dark matter halo. Astrophysical applications to disc galaxies, proto-stellar discs and circumnuclear discs are given as examples.

Key words: hydrodynamics — ISM: general — galaxies: kinematics and dynamics — galaxies: spiral — galaxies: structure — waves

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1 INTRODUCTION

Axisymmetric instabilities in models of disc galaxies have been investigated extensively in the last century (e.g., Safronov 1960; Toomre 1964; Binney & Tremaine 1987; Bertin & Lin 1996). For a single disc of either gaseous or stellar content, Safronov (1960) and Toomre (1964) originally introduced a dimensionless parameter Q such that $Q > 1$ means local stability against axisymmetric ring-like disturbances in the usual Wentzel-Kramers-Brillouin-Jeffreys (WKBJ) or tight-winding approximation. A more realistic model of a disc galaxy would involve both gas and stars as well as an unseen massive dark matter halo, all interacting gravitationally among themselves.[1] Many theoretical investigations have been conducted along this line with a composite system of two coupled discs (Lin & Shu 1966; Kato 1972; Jog & Solomon 1984a, b; Bertin & Romeo 1988; Romeo 1992; Elmegreen 1995; Jog 1996; Lou & Fan 1998b). These earlier treatments use either a combined approach of distribution function and fluid or the formalism of two fluids in a WKBJ model analysis. While the results of these models were initially derived in various galactic contexts, they can also be applied, with proper qualifications, to self-gravitating disc systems including accretion discs, circumnuclear, protostellar, planetary discs and so forth.

The local WKBJ or tight-winding approximation has been proven to be a powerful technique in analysing the dynamics of waves in discs. Meanwhile, theorists have long been keenly interested in a class of relatively simple disc models referred to as scale-free discs (Mestel 1963; Zang 1976; Lemos et al. 1991; Lynden-Bell & Lemos 1993; Syer & Tremaine 1996; Evans & Read 1998; Goodman & Evans 1999; Shu et al. 2000; Lou 2002; Lou & Fan 2002; Lou & Shen 2003; Shen & Lou 2003; Lou & Zou 2004, 2005; Lou & Wu 2004; Shen, Liu & Lou 2004). Scale-free discs, where all pertinent physical variables (e.g., disc rotation speed, surface mass density, angular speed, etc.) scale as powers of the cylindrical radius r, have become an effective and simple means to explore disc dynamics. Perhaps the most familiar case is the so-called singular isothermal disc (SID) or Mestel disc with an isothermal equation of state and a flat rotation curve (Mestel 1963; Zang 1976; Goodman & Evans 1999; Shu et al. 2000; Lou 2002; Lou & Shen 2003; Lou & Zou 2004, 2005; Lou & Wu 2004). In contrast to the usual WKBJ approximation for perturbations, perturbations in axisymmetric scale-free discs in some cases can be treated globally and exactly without the local restriction to the short-wavelength regime. It is then possible to derive global properties of perturbations. Using scale-free disc models, Lemos et al. (1991) and Syer & Tremaine (1996) both studied the axisymmetric stability problem for a single disc and found that instabilities first set in as neutral modes or stationary configurations with angular frequency $\omega = 0$.

The main motivation of this paper is to examine the global axisymmetric stability problem for composite systems of two gravitationally coupled scale-free discs. As a more general extension to the previous two-SID analysis (Lou & Shen 2003; Shen & Lou 2003), we further consider a much broader class of rotation curves as well as the equation of state. We shall give an explicit proof that stationary configurations ($\omega = 0$) do mark marginal stability in the two-fluid system, a cogent supplement to our recent investigation on stationary perturbation configurations (Shen & Lou 2004).

[1] Magnetic field and cosmic-ray gas component are dynamically important on large scales (Fan & Lou 1996; Lou & Fan 1998a, 2003; Lou & Zou 2004 and references therein) in the galactic gas disc of interstellar medium (ISM) but are not considered here for simplicity.

2 TWO-FLUID FORMALISM

As an expedient approximation, we treat both discs as razor-thin discs and use superscripts and subscripts s and g to refer to the stellar and gaseous disc, respectively. Large-scale coupling between the two discs is primarily caused by mutual gravitational interaction. In the present formulation of large-scale perturbations, we ignore non-ideal diffusive effects such as viscosity, resistivity and thermal conduction, etc. It is then straightforward to write down the set of basic coplanar fluid equations for the stellar disc in cylindrical coordinates $(r,\ \theta,\ z)$ in the $z = 0$ plane, namely

$$\frac{\partial \Sigma^s}{\partial t}+\frac{1}{r}\frac{\partial}{\partial r}(r\Sigma^s u^s)+\frac{1}{r^2}\frac{\partial}{\partial \theta}(\Sigma^s j^s)=0\ , \tag{1}$$

$$\frac{\partial u^s}{\partial t}+u^s\frac{\partial u^s}{\partial r}+\frac{j^s}{r^2}\frac{\partial u^s}{\partial \theta}-\frac{j^{s2}}{r^3}=-\frac{1}{\Sigma^s}\frac{\partial \Pi^s}{\partial r}-\frac{\partial \phi}{\partial r}\ , \tag{2}$$

$$\frac{\partial j^s}{\partial t}+u^s\frac{\partial j^s}{\partial r}+\frac{j^s}{r^2}\frac{\partial j^s}{\partial \theta}=-\frac{1}{\Sigma^s}\frac{\partial \Pi^s}{\partial \theta}-\frac{\partial \phi}{\partial \theta}\ , \tag{3}$$

where Σ^s is the surface mass density, u^s is the radial bulk flow velocity, j^s is the specific angular momentum about the rotation axis along the $z-$direction, Π^s is the vertically integrated effective (two-dimensional) pressure due to stellar velocity dispersion and ϕ is the *total* gravitational potential. For the gaseous disc, we simply replace superscript or subscript s by g in the above three equations. The coupling of the two sets of fluid equations is due to the gravitational potential through the Poisson integral,

$$\phi(r,\theta,t)=\oint d\psi\int_0^\infty \frac{-G\Sigma(r',\psi,t)r'dr'}{[r'^2+r^2-2rr'\cos(\psi-\theta)]^{1/2}}\ , \tag{4}$$

where $\Sigma = \Sigma^s + \Sigma^g$ is the total surface mass density of the composite disc system.

The barotropic equation of state assumes a relation between the two-dimensional pressure and surface mass density of the form of

$$\Pi = K\Sigma^n, \tag{5}$$

where the coefficients $K \geq 0$ and $n > 0$ are constant. This directly leads to the sound speed [2] a defined by

$$a^2=\frac{d\Pi_0}{d\Sigma_0}=nK\Sigma_0^{n-1}, \tag{6}$$

which scales as $\propto \Sigma_0^{n-1}$. The case of $n = 1$ corresponds to an isothermal sound speed a.

From the above basic fluid equations for the stellar and gaseous discs (the latter are not written out explicitly), we may derive the axisymmetric equilibrium background properties (Shen & Lou 2004). We presume that in the equilibrium background with axisymmetry, the rotation curves of the two discs both scale as $\propto r^{-\beta}$ and both their surface mass densities scale as $\propto r^{-\alpha}$, where α and β are two constant exponents and the coefficients of proportionality are allowed to be different in general. By assuming the same power-law index for the stellar and gaseous disc rotation curves (or equivalently the surface mass density profiles), it is possible to consistently construct a global axisymmetric background equilibrium for the composite disc system that meets the requirement that the radial forces balance out at all radii (see Eq. (2) and the corresponding equation with the superscript s replaced by g) and that simultaneously satisfies the Poisson integral (4) (see Shen & Lou (2004), Eq. 10).

[2] In a stellar disc, the velocity dispersion mimics the sound speed to some extent.

The scale-free condition requires the following relationship among α, β and n (see Syer & Tremaine 1996 and Shen & Lou 2004):

$$\alpha = 1 + 2\beta \qquad \text{and} \qquad n = \frac{1+4\beta}{1+2\beta} \ . \tag{7}$$

Once the rotation curve is specified, all the other physical variables are determined.

With the knowledge of computing the gravitational potential arising from an axisymmetric power-law surface mass density in a background in rotational equilibrium (Kalnajs 1971; Qian 1992; Syer & Tremaine 1996), we can derive a self-consistent axisymmetric background equilibrium surface mass density as

$$\Sigma_0^s = \frac{A_s^2(D_s^2+1)}{2\pi G(2\beta\mathcal{P}_0)r^{1+2\beta}(1+\delta)} \qquad \text{and} \qquad \Sigma_0^g = \frac{A_g^2(D_g^2+1)\delta}{2\pi G(2\beta\mathcal{P}_0)r^{1+2\beta}(1+\delta)} \ , \tag{8}$$

where the coefficient $\mathcal{P}_0$ is a function of β through the $\Gamma-$function,

$$\mathcal{P}_0 \equiv \frac{\Gamma(-\beta+1/2)\Gamma(\beta)}{2\Gamma(-\beta+1)\Gamma(\beta+1/2)} \, , \tag{9}$$

with $\delta \equiv \Sigma_0^g/\Sigma_0^s$ the ratio of the surface mass density of the gaseous disc to that of the stellar disc, and A and D, two dimensionless parameters. We note that the value of $2\beta\mathcal{P}_0$ falls within $(0,\infty)$ for the prescribed range[3] of $\beta \in (-1/4, 1/2)$ and is equal to 1 when $\beta \to 0$ (i.e., the case of two gravitationally coupled SIDs).

From the radial force balance of the background equilibrium, there also exists the following relation:

$$A_s^2(D_s^2+1) = A_g^2(D_g^2+1) \ , \tag{10}$$

whence $\eta \equiv A_s^2/A_g^2 = a_s^2/a_g^2$ is another handy dimensionless parameter for sound speed ratio squared. We note that A is actually the reduced[4] effective sound speed (scaled by a factor $(1+2\beta)^{1/2}$) and the parameter $D \equiv \mathcal{V}/A$ where $\mathcal{V}$ is the reduced disc rotation speed, is the effective Mach number (see Eqs. (11)–(12) later). Condition (10) is very important in our analysis because the rotations of the two discs are not independent of each other but are dynamically coupled. It suffices to examine the parameter regime of either the stellar or gaseous disc. In disc galaxies, the typical velocity dispersion in a stellar disc exceeds the sound speed in a gaseous disc, implying $\eta > 1$ so that the inequality $D_g^2 > D_s^2$ holds. Therefore, the physical requirement $D_s^2 > 0$ absolutely guarantees $D_g^2 > 0$ and it suffices to consider the stability problem in terms of $D_s^2 > 0$ together with different values of the parameters δ and η (note that $D_g^2 = \eta(D_s^2+1) - 1$). With these explanations, we shall express the other equilibrium physical variables in terms of the two parameters A and D.

The specific angular momenta j_0^s and j_0^g about the $z-$axis and the sound speeds a_s and a_g in the two coupled equilibrium discs are expressed by

$$j_0^s = A_s D_s r^{1-\beta} \ , \qquad j_0^g = A_g D_g r^{1-\beta} \ , \tag{11}$$

[3] The valid range of $\beta \in (-1/4, 1/2)$ is determined by (1) the barotropic index $n > 0$ for warm discs, (2) the surface mass density exponent $\alpha < 2$ such that the central point mass does not diverge, and (3) this β range is contained within a wider range of $\beta \in (-1/2, 1/2)$ (for a cold disc system) when the computed force arising from the background potential remains finite (Syer & Tremaine 1996).

[4] By "reduced", we refer to the part of a physical variable after removing the power-law radial dependence. For example, in the disc rotation speed $v = \mathcal{V}r^{-\beta}$, the quantity $\mathcal{V}$ is referred to as the reduced disc rotation speed.

$$a_s^2 = nK_s(\Sigma_0^s)^{n-1} = A_s^2/[(1+2\beta)r^{2\beta}]\ , \qquad a_g^2 = nK_g(\Sigma_0^g)^{n-1} = A_g^2/[(1+2\beta)r^{2\beta}]\ . \tag{12}$$

The disc angular rotation speed $\Omega \equiv j_0/r^2$ and the epicyclic frequency $\kappa \equiv [(2\Omega/r)d(r^2\Omega)/dr]^{1/2}$ are similarly expressed in terms of A and D as

$$\begin{aligned} \Omega_s &= A_s D_s r^{-1-\beta}\ , \qquad & \kappa_s &= [2(1-\beta)]^{1/2}\Omega_s\ , \\ \Omega_g &= A_g D_g r^{-1-\beta}\ , \qquad & \kappa_g &= [2(1-\beta)]^{1/2}\Omega_g\ , \end{aligned} \tag{13}$$

with $dj_0/dr = r\kappa^2/(2\Omega)$ to simplify later derivations.

2.1 Linear Perturbation Equations

For a composite system of two gravitationally coupled, axisymmetric discs in rotational equilibrium , we introduce small coplanar perturbations indicated by subscript 1. The linearized perturbation equations can be derived from the basic nonlinear Eqs. (1)–(4) to be

$$\begin{aligned} &\frac{\partial \Sigma_1^s}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}(r\Sigma_0^s u_1^s) + \Omega_s \frac{\partial \Sigma_1^s}{\partial \theta} + \frac{\Sigma_0^s}{r^2}\frac{\partial j_1^s}{\partial \theta} = 0\ , \\ &\frac{\partial u_1^s}{\partial t} + \Omega_s \frac{\partial u_1^s}{\partial \theta} - 2\Omega_s \frac{j_1^s}{r} = -\frac{\partial}{\partial r}\left(a_s^2 \frac{\Sigma_1^s}{\Sigma_0^s} + \phi_1\right)\ , \\ &\frac{\partial j_1^s}{\partial t} + \frac{r\kappa_s^2}{2\Omega_s} u_1^s + \Omega_s \frac{\partial j_1^s}{\partial \theta} = -\frac{\partial}{\partial \theta}\left(a_s^2 \frac{\Sigma_1^s}{\Sigma_0^s} + \phi_1\right)\ , \end{aligned} \tag{14}$$

for the stellar disc as well as their counterparts for the gaseous disc, together with the Poisson integral

$$\phi_1(r,\theta,t) = \oint d\psi \int_0^\infty \frac{-G(\Sigma_1^s + \Sigma_1^g)r'dr'}{[r'^2 + r^2 - 2rr'\cos(\psi-\theta)]^{1/2}}\ , \tag{15}$$

relating the total gravitational potential perturbation ϕ_1 and the total surface mass density perturbation $\Sigma_1 \equiv \Sigma_1^s + \Sigma_1^g$.

Given a Fourier periodic component of the form of $\exp[i(\omega t - m\theta)]$ for a small perturbation in general (after taking the real part, without loss of generality we can assume $m \geq 0$), we write the coplanar perturbations in the stellar disc in the form of,

$$\begin{aligned} \Sigma_1^s &= \mu^s(r)\exp[i(\omega t - m\theta)]\ , \\ u_1^s &= U^s(r)\exp[i(\omega t - m\theta)]\ , \\ j_1^s &= J^s(r)\exp[i(\omega t - m\theta)]\ , \end{aligned} \tag{16}$$

as well as their counterparts for the gaseous disc, together with the total gravitational potential perturbation

$$\phi_1 = V(r)\exp[i(\omega t - m\theta)]\ , \tag{17}$$

where the integer m is taken to be non-negative. For axisymmetric $m = 0$ perturbations, we introduce Fourier decompositions in the Eqs. (14)–(15) for the stellar disc to derive

$$\begin{aligned} &i\omega\mu^s + \frac{1}{r}\frac{d}{dr}(r\Sigma_0^s U^s) = 0\ , \\ &i\omega U^s - 2\Omega_s \frac{J^s}{r} = -\frac{d}{dr}\left(a_s^2 \frac{\mu^s}{\Sigma_0^s} + V\right)\ , \\ &i\omega J^s + \frac{r\kappa_s^2}{2\Omega_s} U^s = 0\ . \end{aligned} \tag{18}$$

We do the same for the gaseous disc to derive

$$
\begin{aligned}
&i\omega\mu^g + \frac{1}{r}\frac{d}{dr}(r\Sigma_0^g U^g) = 0\ ,\\
&i\omega U^g - 2\Omega_g \frac{J^g}{r} = -\frac{d}{dr}\left(a_g^2 \frac{\mu^g}{\Sigma_0^g} + V\right)\ ,\\
&i\omega J^g + \frac{r\kappa_g^2}{2\Omega_g}U^g = 0\ .
\end{aligned}
\tag{19}
$$

For the total gravitational potential perturbation, we simply have

$$
V(r) = \oint d\psi \int_0^\infty \frac{-G(\mu^s+\mu^g)\cos(m\psi) r' dr'}{(r'^2 + r^2 - 2rr'\cos\psi)^{1/2}}\ . \tag{20}
$$

Equations (18)−(20) are the basic coplanar perturbation equations used in our axisymmetric stability analysis.

2.2 Axisymmetric Stability Analysis

For axisymmetric stability analysis with radial oscillations, we chose the Kalnajs potential-density pairs below because the perturbations can be generally expanded in terms of these complete basis functions (Kalnajs 1971; Binney & Tremaine 1987; Lemos et al. 1991; Lou & Shen 2003). Specifically, we take

$$
\begin{aligned}
&\mu^s = \sigma^s r^{-3/2}\exp(i\xi\ln r)\ , \qquad \mu^g = \sigma^g r^{-3/2}\exp(i\xi\ln r)\ ,\\
&V = -2\pi G r(\mu^s+\mu^g)\mathcal{N}_m(\xi)\ ,
\end{aligned}
\tag{21}
$$

where ξ is a "wavenumber" characterizing the scale of the radial variation, σ^s and σ^g are two small real coefficients and the parameter function

$$
\mathcal{N}_m(\xi) = \frac{\Gamma(m/2+i\xi/2+1/4)\Gamma(m/2-i\xi/2+1/4)}{2\Gamma(m/2+i\xi/2+3/4)\Gamma(m/2-i\xi/2+3/4)} \tag{22}
$$

is the Kalnajs function (Kalnajs 1971) involving $\Gamma-$functions of complex arguments. Note that $\mathcal{N}_m$ is even in ξ. It then suffices to consider only $\xi \geq 0$.

Using the first mass conservations in Eqs. (18) and (19), we infer $U \propto i\omega r^{1/2+2\beta+i\xi}$ (see also Lou & Zou 2004). Using the potential-density pair (21), Eqs. (18) and (19) are reduced to

$$
\begin{aligned}
&(\omega^2 - H_1)U^s = -G_2 U^g\ ,\\
&(\omega^2 - H_2)U^g = -G_1 U^s\ ,
\end{aligned}
\tag{23}
$$

in the limit of $\omega \to 0$, where the parameter functions H_1, H_2, G_1 and G_2 are explicitly defined by

$$
\begin{aligned}
&H_1 \equiv \kappa_s^2 + \left(\frac{a_s^2}{r} - 2\pi G\mathcal{N}_0\Sigma_0^s\right)\frac{(\xi^2+1/4)}{r}\ ,\\
&H_2 \equiv \kappa_g^2 + \left(\frac{a_g^2}{r} - 2\pi G\mathcal{N}_0\Sigma_0^g\right)\frac{(\xi^2+1/4)}{r}\ ,\\
&G_1 \equiv 2\pi G\mathcal{N}_0\Sigma_0^s\frac{(\xi^2+1/4)}{r} > 0\ ,\\
&G_2 \equiv 2\pi G\mathcal{N}_0\Sigma_0^g\frac{(\xi^2+1/4)}{r} > 0\ .
\end{aligned}
\tag{24}
$$

The axisymmetric dispersion relation in the composite disk system follows from Eq. (23)

$$
\omega^4 - (H_1+H_2)\omega^2 + (H_1H_2 - G_1G_2) = 0 \tag{25}
$$

in the limit of $\omega \to 0$. This is identical in form with the earlier results obtained in the WKBJ regime (Jog & Solomon 1984a; Shen & Lou 2003). It may be of interest to note that the conditions for the stellar disc and the gaseous disc to be separately stable are $H_1 > 0$ and $H_2 > 0$, respectively. We may also recall here that in the familiar WKBJ regime, the dispersion relation in a single disc is

$$\omega^2 = \kappa^2 + k^2a^2 - 2\pi G|k|\Sigma_0 \ , \tag{26}$$

where k is the radial wavenumber, whereas in the present global analysis as applied to a single disc, we have in the limit of $\omega \to 0$,

$$\omega^2 = \kappa^2 + \frac{(\xi^2+1/4)}{r^2}a_s^2 - 2\pi G\frac{(\xi^2+1/4)\mathcal{N}_0}{r}\Sigma_0 \ . \tag{27}$$

If we replace $\mathcal{N}_0(\xi)$ approximately by $(\xi^2+1/4)^{-1/2}$ in the asymptotic regime of $\xi \gg 1$, we readily identify the correspondence between the effective wavenumber $(\xi^2+1/4)^{1/2}$ and the $|k|r$ in the WKBJ limit of $|k|r \gg 1$ by directly comparing the dispersion relations (26) and (27). Physically, the dispersion relation (27) is more generally applicable beyond the WKBJ regime and is globally accurate only in the limit of $\omega \to 0$ (Shu et al. 2000).

Returning to the composite system, Eq. (25) has two real roots[5] of ω^2:

$$\omega_\pm^2 = \frac{1}{2}\{(H_1+H_2) \pm [(H_1+H_2)^2 - 4(H_1H_2 - G_1G_2)]^{1/2}\} \ , \tag{28}$$

with ω_+^2 being always positive[6] (Shen & Lou 2003). For the purpose of axisymmetric stability analysis, we only need to examine the root ω_-^2, namely

$$\omega_-^2 = \frac{1}{2}\{(H_1+H_2) - [(H_1+H_2)^2 - 4(H_1H_2 - G_1G_2)]^{1/2}\} \ . \tag{29}$$

As the right-hand side of the above Eq. (29) is always real, axisymmetric instability sets in as stationary perturbation configurations with $\omega_-^2 = 0$ and this leads to the marginal stability condition

$$H_1H_2 = G_1G_2 \ , \tag{30}$$

which requires the inequality $H_1 + H_2 \geq 0$. This inequality can be shown in a straightforward manner to be automatically satisfied if Eq. (30) holds true. Let us first write

$$H_1 = F_1 - G_1 \qquad \text{and} \qquad H_2 = F_2 - G_2 \ , \tag{31}$$

where F_1 and F_2 are explicitly defined by

$$F_1 \equiv \kappa_s^2 + \frac{(\xi^2+1/4)}{r^2}a_s^2 > 0 \qquad \text{and} \qquad F_2 \equiv \kappa_g^2 + \frac{(\xi^2+1/4)}{r^2}a_g^2 > 0 \ . \tag{32}$$

It then follows from the condition (30) $H_1H_2 = G_1G_2$ that

$$F_1F_2 = F_1G_2 + F_2G_1 \ . \tag{33}$$

As F_1, F_2, G_1 and G_2 are all positive, we immediately conclude that

$$F_1 - G_1 > 0 \qquad \text{and} \qquad F_2 - G_2 > 0 \ , \tag{34}$$

[5] One can show that the determinant of equation (25) $\Delta \equiv (H_1 - H_2)^2 + 4G_1G_2 > 0$ is always true.

[6] If $H_1 + H_2 \geq 0$, then $\omega_+^2 > 0$; otherwise if $H_1 + H_2 < 0$, then at least one of H_1 and H_2 is negative. It therefore follows that $H_1H_2 - G_1G_2 < 0$ and hence $\omega_+^2 > 0$.

which finally lead to

$$H_1 + H_2 > 0 \ . \tag{35}$$

The composite disc system becomes inevitably unstable for $H_1 + H_2 < 0$ because $\omega_-^2 < 0$. Else if $H_1 + H_2 \geq 0$ but with $H_1H_2 - G_1G_2 < 0$, we again have $\omega_-^2 < 0$ for instability. Only when $H_1 + H_2 \geq 0$ and $H_1H_2 - G_1G_2 > 0$ at the same time can the composite disc system be stable against axisymmetric coplanar perturbations. This is an important necessary stability criterion for a composite system of two gravitationally coupled discs. In other words, once one disc is unstable by itself (i.e., $H_1 < 0$ or $H_2 < 0$ or both), the two-disc system must be unstable; even if the two discs are both separately stable, the composite disc system can still become unstable (i.e., $H_1 > 0$ and $H_2 > 0$ but $H_1H_2 - G_1G_2 < 0$).

By inserting the expressions of H_1, H_2, G_1 and G_2 into the marginal stability condition (30) together with the requirement of background rotational equilibrium, we readily derive a quadratic equation in $y \equiv D_s^2$, namely

$$C_2y^2 + C_1y + C_0 = 0 \ , \tag{36}$$

where the coefficients are explicitly defined by

$$\begin{aligned} C_2 &\equiv \mathcal{B}_0\mathcal{H}_0\eta \ , \\ C_1 &\equiv \left[(\mathcal{B}_0 - \mathcal{A}_0)\mathcal{H}_0 + \tfrac{(\mathcal{A}_0+\mathcal{B}_0)(\mathcal{H}_0-\mathcal{B}_0)}{(1+\delta)}\right]\eta - \tfrac{(\mathcal{A}_0+\mathcal{B}_0)(\mathcal{H}_0+\mathcal{B}_0\delta)}{(1+\delta)} \ , \\ C_0 &\equiv \left[-\mathcal{A}_0\mathcal{H}_0 + \tfrac{(\mathcal{A}_0+\mathcal{B}_0)(\mathcal{H}_0-\mathcal{B}_0)}{(1+\delta)}\right]\eta + (\mathcal{A}_0 + \mathcal{B}_0)^2 - \tfrac{(\mathcal{A}_0+\mathcal{B}_0)(\mathcal{H}_0+\mathcal{B}_0\delta)}{(1+\delta)} \ , \end{aligned} \tag{37}$$

and

$$\begin{aligned} \mathcal{A}_0(\xi) &\equiv \xi^2 + 1/4 \ , \\ \mathcal{B}_0(\beta) &\equiv (1 + 2\beta)(-2 + 2\beta) \ , \\ \mathcal{C}(\beta) &\equiv (1 + 2\beta)/(2\beta\mathcal{P}_0) \ , \\ \mathcal{H}_0(\beta, \xi) &\equiv \mathcal{C}\mathcal{N}_0\mathcal{A}_0 + \mathcal{B}_0 \ . \end{aligned} \tag{38}$$

This quadratic equation (36) can be readily proven to always have two real solutions (Shen & Lou 2004). Only positive solutions can be regarded as physically acceptable. Typically, there exist two different regimes bounded by the marginal D_s^2 stability curves that are unstable against axisymmetric perturbations, one is the collapse regime for long-wavelength perturbations, the other is the ring-fragmentation regime for short-wavelength perturbations. In contrast to the short-wavelength WKBJ approximation, the collapse regime is novel and exact. Systems with too fast a rotation parameter D_s^2 will fall into the ring-fragmentation regime (Safronov 1960; Toomre 1964; Syer & Tremaine 1996; Lou & Fan 1998a, b; Shu et al. 2000; Lou 2002; Lou & Shen 2003, 2004; Shen & Lou 2003), while those with too slow a D_s^2 parameter will fall into the collapse regime. Shown in Fig. 1 is an illustrative example with $\beta = 1/4$, $\eta = 1$ and an unconstrained δ. We note that cases with $\eta = 1$ are essentially the same as those of a single disc (Shen & Lou 2004). The boundaries of the two unstable regimes shown in Fig. 1 vary with different parameters $\eta > 1$ and δ for a chosen value of β which fixes the entire scale-free radial profile of the composite system (i.e., the rotation curves, surface mass densities and barotropic equation of state). Qualitatively, an increase of either η and δ will aggravate the ring-fragmentation instability and suppress the large-scale collapse instability (Shen & Lou 2003). While this can be directly seen from Eq. (36) in the relevant parameter regime, it can also be understood physically in terms of the dynamical coupling between the two discs through the condition (10) imposed upon by the scale-free conditions. For a larger η, the reduced gas disc rotation speed $\mathcal{V}_g$ will exceed the reduced stellar disc rotation speed $\mathcal{V}_s$ by a larger margin and this will tend to prevent an overall collapse of the composite system. In other words, a gaseous disc component with a relatively lower sound speed apparently inhibits collapse (Shen & Lou 2003). More details and examples can be found in Shen & Lou (2004).

It is well known that a Q parameter can be defined to determine local axisymmetric stability of a single-disc system (Safronov 1960; Toomre 1964; Binney & Tremaine 1987). For a composite disc system, it has been attempted to introduce an effective $Q_{\rm eff}$ parameter (Elmegreen 1995; Jog 1996; Lou & Fan 1998b). As a result of a straightforward numerical computation, we have recently introduced a powerful $D-$criterion for axisymmetric stability of a composite system of two coupled SIDs (Shen & Lou 2003). It is natural to generalize this $D-$criterion for axisymmetric stability of a composite system of two coupled barotropic discs by straightforward numerical computations as shown in Fig. 1. In the present case, it is more practical and simpler to use the D_s^2 parameter.

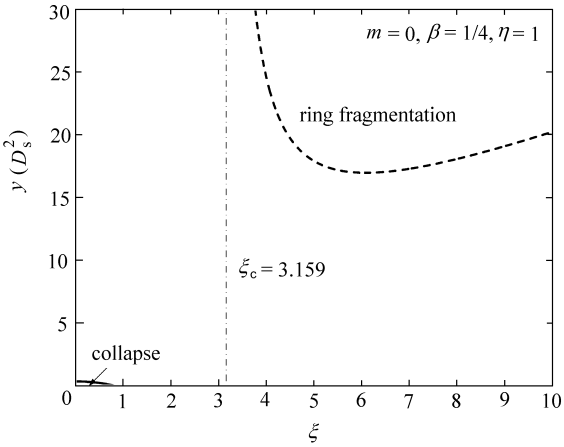

Fig. 1 Two unstable regimes in the case of $m = 0$, $\beta = 1/4$ and $\eta = 1$. The collapse regime is at the lower-left corner, while the ring-fragmentation regime is at the upper-right corner. In this special case of $\eta = 1$, the parameter δ can be arbitrary as can be seen from equation (37), that is, when $\eta = 1$ the coefficients C_2, C_1 and C_0 turn out to be independent of δ. The vertical dash-dotted line is the location of ξ_c where $\mathcal{H}_0 = 0$ and C_2 vanishes. The solid line and the dashed line bound the collapse regime and the ring fragmentation regime, respectively. Only when D_s^2 falls within the range between the top of the collapse regime and the bottom of the ring fragmentation regime can a composite disc system become stable against all axisymmetric coplanar perturbations.

2.3 Partial Disc Systems and Applications to Disc Galaxies

From observations of more or less flat rotation curves of most disc galaxies, massive dark matter halos have been inferred to exist ubiquitously as long as Newtonian gravity remains valid on galactic scales. If we naively attempt to relate the theoretical results obtained in Section 2.2 to a typical disc galaxy, we may take the simple isothermal equation of state as an illustrative example. The relevant parameters for a composite SID system are then given as $\beta = 0$ and $a_s = 50$ km s^{-1}, $\mathcal{V}_s = 220$ km s^{-1}, $\delta = 0.1$ and $\eta = 50$. Unfortunately, such a composite system of two coupled SIDs is inevitably unstable against ring-fragmentation perturbations because we have a $D_s^2 \simeq 20$ far exceeding the maximum value for stability against ring fragmentation. This dilemma can be resolved by attributing to an additional gravitational potential associated with an unseen massive dark matter halo. We refer to a composite disc system in association with an axisymmetric dark matter halo as a composite system of *partial discs* (e.g. Syer & Tremaine 1996; Shu et al. 2000; Lou 2002; Lou & Fan 2002; Shen & Lou 2003, 2004; Lou & Zou 2004, 2005; Lou & Wu 2004). In a simple treatment, the dynamical effect of a dark matter halo is assumed only to contribute an axisymmetric gravitational potential Φ in the background rotational equilibrium but not to respond to coplanar perturbations in the composite disk system. With $\phi_T \equiv \Phi + \phi$, we conveniently introduce a dimensionless parameter $\mathcal{F} \equiv \phi/\phi_T$, the ratio of the potential arising from the composite disc system to that of the entire system including the presumed axisymmetric dark matter halo. A full-disc system corresponds to $\mathcal{F} = 1$ (i.e., $\Phi = 0$) and a partial-disc system corresponds to $0 < \mathcal{F} < 1$ (i.e., $\Phi \neq 0$). For a composite system of two gravitationally coupled partial discs, we follow the same procedure of analysing coplanar perturbations in full discs to derive a similar quadratic equation of D_s^2 as the stationary dispersion relation. In a nutshell, we can simply replace all $\mathcal{N}_0(\xi)$ in our theoretical results by $\mathcal{F}\mathcal{N}_0(\xi)$ to effect this generalization or extension. The introduction of the ratio parameter $\mathcal{F}$ will significantly reduce both the ring-fragmentation regime and the collapse regime, as can already be seen from a comparative study of the WKBJ and global approaches (Shen & Lou 2003).

For the purpose of illustrating the stabilizing effect of a partial-disc system, we simply take $\mathcal{F} = 0.1$ with other parameters used earlier in this section. The minimum value of D_s^2 for unstable ring fragmentations now becomes ~ 650, far beyond the actual value of $D_s^2 \simeq 20$ in a disc galaxy. Meanwhile, the collapse regime disappears completely. Therefore, a typical composite system of two coupled partial discs is fairly stable against axisymmetric coplanar perturbations.

3 DISCUSSION AND SUMMARY

The main thrust of this investigation is to model linear coplanar axisymmetric ($m = 0$) perturbations in a composite system of two-fluid scale-free discs, one stellar and one gaseous. The two discs are dynamically coupled through mutual gravitational interaction. In order to include the dynamical effect of a massive axisymmetric dark matter halo, we further describe a composite system of two coupled partial discs (e.g. Syer & Tremaine 1996; Shu et al. 2000; Lou 2002; Lou & Shen 2003; Lou & Zou 2004, 2005; Lou & Wu 2004; Shen, Liu & Lou 2004). In a global perturbation analysis, we show that axisymmetric instabilities set in as stationary perturbation configurations with $\omega = 0$. The marginal D_s^2 stability curve (characterized by the stationary configurations) delineates two different unstable regimes, namely, the collapse regime for large-scale perturbations and the ring-fragmentation regime for short-wavelength perturbations. Apparently, the composite disc system becomes less stable than a single-disc system and can be unstable while the two discs are separately stable (Lou & Fan 1998b). In our analysis, stationary perturbation configurations turn out to be more than just an alternative equilibrium state, especially in view of the stability properties.

The basic results of this paper are generally applicable to self-gravitating disc systems with or without axisymmetric dark matter halos. The two-fluid treatment contains more realistic elements than a single-disc formulation in the context of disc galaxies. In addition to astrophysical applications to disc galaxies, the studies presented here can be valuable for exploring the dynamical evolution of protostellar discs and circumnuclear discs.

In the context of proto-stellar disc, it is usual to ignore the self-gravity effect. By considering the self-gravity of a composite disc system, our analysis indicates several qualitative yet interesting results. For example, if the initial disc system rotates sufficiently fast, the ring fragmentation (see the upper-right part of Fig. 1) can occur at relatively small radial scales. By further non-axisymmetric fragmentations, these condensed rings of materials may eventually become birthplaces of planets. On the other hand, if the initial disc system rotates sufficiently slowly, then gravitational collapse can be induced by perturbations of relatively large radial scales (see the lower left corner of Fig. 1). Once such a perturbation develops in the background equilibrium disc, it grows rapidly and destabilizes the disc. Subsequently, the system undergoes global Jeans collapse to form a central young stellar object. Finally, if the initial disc system rotates in a regime that is stable against all axisymmetric perturbations (see Fig. 1), then there may be two possibilities: (1) the composite disc system may become unstable from non-axisymmetric perturbations (not analyzed here) and (2) the disc rotation may be gradually slowed down by some braking mechanisms (e.g. magnetic field not included here and outflows or winds) and the disc eventually succumbs to a central collapse induced by large-scale perturbations.

Likewise, in the context of a circumnuclear disc around the center of a galaxy, we can readily conceive similar physical processes taking place. One important distinction is that a dark-matter halo should play an important dynamical role so that a formulation of a partial composite disc system would be more appropriate. Here, ring fragmentation can be induced by relatively small-scale perturbations in a disc system of sufficiently fast rotation. Such a ring of relatively dense materials around the galactic center would be a natural birthplace for circumnuclear starburst activities (e.g. Lou et al. 2001). Depending on the evolution history of a circumnuclear disc system, it may be stable initially and may gradually lose angular momentum through the generation and damping of spiral magnetohydrodynamic (MHD) density waves (Lou et al. 2001). When the disc rotation becomes sufficiently slow, Jeans collapse induced by large-scale perturbations can set in to form a bulge or a super massive black hole.

In summary, our global analysis shows the possible presence of an evolutionary stage for a composite disc system against all axisymmetric coplanar perturbations. More importantly, we reveal the parameter regime of ring fragmentation and that of large-scale collapse. Astrophysical applications are discussed in the contexts of disc galaxies, proto-stellar discs and circumnuclear discs.

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