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Magnetic Energy of Force-Free Fields with Detached Field Lines *

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Abstract Using an axisymmetrical ideal MHD model in spherical coordinates, we present a numerical study of magnetic configurations characterized by a levitating flux rope embedded in a bipolar background field whose normal field at the solar surface is the same or very close to that of a central dipole. The characteristic plasma β (the ratio between gas pressure and magnetic pressure) is taken to be so small ($\beta = 10^{-4}$) that the magnetic field is close to being force-free. The system as a whole is then let evolve quasi-statically with a slow increase of either the annular magnetic flux or the axial magnetic flux of the rope, and the total magnetic energy of the system grows accordingly. It is found that there exists an energy threshold: the flux rope sticks to the solar surface in equilibrium if the magnetic energy of the system is below the threshold, whereas it loses equilibrium if the threshold is exceeded. The energy threshold is found to be larger than that of the corresponding fully-open magnetic field by a factor of nearly 1.08 irrespective as to whether the background field is completely closed or partly open, or whether the magnetic energy is enhanced by an increase of annular or axial flux of the rope. This gives an example showing that a force-free magnetic field may have an energy larger than the corresponding open field energy if part of the field lines is allowed to be detached from the solar surface. The implication of such a conclusion in coronal mass ejections is briefly discussed and some comments are made on the maximum energy of force-free magnetic fields.

Key words: Sun: magnetic fields — Sun: coronal mass ejections (CMEs) — methods: numerical

1 INTRODUCTION

As a typical magnetic structure in the solar corona, magnetic flux ropes are believed to be closely related to various solar active phenomena such as coronal mass ejections (CMEs). Many authors suggested that a catastrophic loss of mechanical equilibrium might lead to an

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eruption of the flux rope and the opening of the ambient magnetic field, and the latter was usually considered as a necessary condition for CMEs (Hundhausen 1988). Both analytical and numerical models have been proposed to examine possible catastrophic behaviors of an ideal magnetohydrodynamic (MHD) system containing magnetic flux ropes (Forbes & Isenberg 1991; Forbes & Priest 1995; Isenberg et al. 1993; Lin et al. 1998, 2001; Hu & Liu 2000; Hu 2001; Hu & Jiang 2001; Hu et al. 2001, 2003; Li & Hu 2001; Wang & Hu 2003). A common conclusion of these studies is that catastrophe exists under certain conditions.

In the analysis of catastrophe of magnetic configurations, one of the most important issues is the total magnetic energy of the system at the catastrophic point. The reason is straightforward, since, as a possible mechanism for the initiation of CMEs, the catastrophe is required to fully open up the background field so as to create a channel for the erupting flux rope, and in addition, to provide the associated plasma with sufficient energy so as to make it erupt into interplanetary space against gravity. However, Aly (1984) put forward a conjecture that in an infinite domain and for a given distribution of normal field at the lower boundary, the maximum energy of force-free fields with at least one end of each field line anchored to the lower boundary is the corresponding open field energy. This conjecture was supported by numerical (Yang et al. 1986; Mikić & Link 1994; Roumeliotis et al. 1994; Amari et al. 1996) and analytical (Lynden-Bell & Boily 1994; Aly 1994; Wolfson 1995) examples. Meanwhile, Aly (1991) and Sturrock (1991) separately addressed proofs of the conjecture, but Aly (1991) admitted that the validity of these proofs relies on some intuitive assumptions. Choe & Cheng (2002) pointed out that the validity of these assumptions cannot be taken for granted, and presented an example consisting of a pair of intertwined flux loops whose energy may exceed the corresponding open field energy.

While the Aly conjecture remains to be questionable, another interesting issue is what happens if a part of the field lines is completely detached from the solar surface. Low & Smith (1993) found magnetostatic equilibrium solutions with a fully detached magnetic bubble which have energy in excess of the corresponding open field energy. The bubble was maintained in equilibrium by gas pressure and gravity. Recently, Hu et al. (2003, referred to as Paper I hereinafter) used a 2.5-D ideal MHD numerical model to study the equilibrium and catastrophe associated with a fully detached coronal flux rope. They found that there exists an energy threshold across which a catastrophe of the system occurs. Although a definite value of the threshold was not determined, they insisted with some plausible arguments that the threshold should be slightly larger than the open field energy. Incidentally, in one set of the equilibrium solutions obtained, β was taken to be 0.001, and thus the magnetic field is close to being force-free. As pointed out in Paper I, the equilibrium solution was obtained by an abrupt emergence of a flux rope from below the solar surface so that the system underwent a process that is far from quasi-static. This was the main reason why a definite threshold could not be estimated based on the numerical solutions. To determine the threshold exactly, one must let the system evolve quasi-statically with slowly increasing magnetic energy. This paper will undertake such a task in order to determine the energy threshold. To this end, we start with an equilibrium magnetic configuration with a detached flux rope within a bipolar background field that can be either completely closed or partly open. The characteristic value of β is taken to be 10^{-4} so as to approximate force-free fields. Then we let the system evolve quasi-statically with slowly increasing magnetic energy by an artificial enhancement of either the annular or the axial magnetic flux of the rope. At a certain point, a catastrophe occurs, and the energy threshold is thus determined. The basic equations and the initial conditions are given in Section 2. In Section 3, we describe the procedure of determining the energy threshold. Numerical results are discussed in Section 4. We conclude our work in Section 5.

2 BASIC EQUATIONS AND INITIAL CONDITIONS

2.1 Basic Equations

Following Paper I, we take the spherical coordinate system (r, θ, φ) and consider the 2.5-D problems in the meridional plane. Introducing a magnetic flux function $\psi(t, r, \theta)$ related to the magnetic field by

$$\boldsymbol{B} = \nabla \times \left(\frac{\psi}{r\sin\theta}\hat{\boldsymbol{\varphi}}\right) + \boldsymbol{B}_\varphi, \quad \boldsymbol{B}_\varphi = B_\varphi \hat{\boldsymbol{\varphi}}, \tag{1}$$

the 2.5-D ideal MHD equations may be cast in the non-dimensional form

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0, \tag{2}$$

$$\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + \frac{\beta}{2}\nabla T + \frac{\beta T}{2\rho}\nabla\rho + \frac{1}{\rho}[\mathcal{L}\psi\nabla\psi + \boldsymbol{B}_\varphi \times (\nabla\times\boldsymbol{B}_\varphi)] + \frac{1}{\rho r\sin\theta}\nabla\psi\cdot(\nabla\times\boldsymbol{B}_\varphi)\hat{\boldsymbol{\varphi}} + \frac{g_s}{r^2}\hat{\boldsymbol{r}} = 0\,, \tag{3}$$

$$\frac{\partial \psi}{\partial t} + \boldsymbol{v}\cdot\nabla\psi = 0, \tag{4}$$

$$\frac{\partial B_\varphi}{\partial t} + r\sin\theta\nabla\cdot\left(\frac{B_\varphi \boldsymbol{v}}{r\sin\theta}\right) + \left[\nabla\psi \times \nabla\left(\frac{v_\varphi}{r\sin\theta}\right)\right]_\varphi = 0\,, \tag{5}$$

$$\frac{\partial T}{\partial t} + \boldsymbol{v}\cdot\nabla T + (\gamma - 1)T\nabla\cdot\boldsymbol{v} = 0\,, \tag{6}$$

where

$$\mathcal{L}\psi \equiv \frac{1}{r^2\sin^2\theta}\left(\frac{\partial^2\psi}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} - \frac{\cot\theta}{r^2}\frac{\partial\psi}{\partial\theta}\right)\,, \tag{7}$$

γ is the polytropic index, taken to be 1.05 in this study, and g_s is the gravitational acceleration at the solar surface (GM_s/R_0^2) normalized by v_0^2/R_0. Here G is the gravitational constant, M_s is the mass of the Sun, R_0 is the radius of the Sun, taken as the unit of length, and v_0 is the characteristic Alfvén wave speed, taken as the unit of velocity. As usual, β is the characteristic ratio of gas pressure to magnetic pressure,

$$\beta = 2\mu\rho_0 R_{\rm G} T_0 R_0^4/\psi_0^2, \tag{8}$$

where μ is the vacuum magnetic permeability, $R_{\rm G}$ is the gas constant, ρ_0 and T_0 are the density and temperature at the base, respectively, and ψ_0 is the magnetic flux function at the solar equator, i.e., the total magnetic flux emanating from unit radian of the northern hemisphere. All symbols with subscript 0 represent the units of the corresponding quantities. Other units include $B_0 = \psi_0 R_0^{-2}$ for field strength, $v_0 \equiv v_A = B_0/\sqrt{\mu\rho_0}$ for velocity, $t_0 \equiv \tau_A = R_0/v_0$ for time, $W_0 = B_0^2 R_0^3/\mu$ for energy, and so on. In the following numerical examples, we take

$$T_0 = 2\times 10^6\ \mathrm{K}, \quad \rho_0 = 1.67\times 10^{-13}\ \mathrm{kg\ m^{-3}}, \quad \beta = 10^{-4}, \tag{9}$$

in addition to $R_0 = 6.95\times 10^8$ m, so that $\psi_0 = 5.69\times 10^{15}$ Wb, $B_0 = 1.18\times 10^{-2}$ T, $v_0 = 2.57\times 10^4$ km s^{-1}, the Alfvénic transit time is $\tau_A = R_0/v_0 = 27$ s, and $W_0 = 3.71\times 10^{28}$ J (cf. Paper I). The computational domain is taken to be $1 \le r \le 30$ (in units of the solar radius R_0), $0 \le \theta \le \pi/2$. Here we take a denser mesh of 200×120 in this paper instead of the 130×90 in Paper I for better numerical accuracy. The grid spacing increases according to a geometrical series of common ratio 1.021 from 0.01 at the solar surface ($r = 1$) to 0.60 at the top ($r = 30$), while a uniform mesh is adopted in the θ direction. The multistep implicit scheme (Hu 1989) is used to solve Eqs. (2)–(6).

2.2 Initial Conditions

The initial state is an equilibrium solution obtained by a similar procedure to that in Paper I. It consists of a flux rope embedded in a bipolar background field that is either completely closed (Case A) or partly open (Case B). The magnetic configurations are shown in Figs. 1a and 1b for the two cases. For case A, the background field has exactly the same magnetic flux distribution at the solar surface as that of a central dipole, namely,

$$\psi(0, R_0, \theta) = \frac{\sin^2\theta}{r}, \tag{10}$$

while the distribution for Case B is very close to Eq. (10): the maximum relative deviation being less than 0.44%. The annular flux Φ_{p0} per radian and the axial magnetic flux $\Phi_{\varphi 0}$ (in units of ψ_0) are (0.426, 0.119) for Case A and (0.381, 0.090) for Case B, respectively.

3 DETERMINATION OF MAGNETIC ENERGY THRESHOLD

Now let us design a quasi-static evolution process for the system caused by a slow change of either the annular or the axial magnetic flux of the flux rope. Physically, this change may be produced by a slow magnetic reconnection between the flux rope and a newly emerged magnetic flux, a twist of the very flux rope, or perhaps some other sources. For the present purpose, however, the detailed mechanism for this change will not be addressed. Our concern is to trace the quasi-static evolution process of the whole system and to find the catastrophic point at which the system loses equilibrium. To this end, we may either increase the annular flux from Φ_{p0} to Φ_p while maintaining $\Phi_\varphi = \Phi_{\varphi 0}$, or increase the axial flux from $\Phi_{\varphi 0}$ to Φ_φ while maintaining $\Phi_p = \Phi_{p0}$. The change proceeds slowly and linearly with time with a growth rate denoted by α in units of τ_A^{-1}, so it lasts for a time interval that is exactly equal to $1/\alpha$. After the change is finished, the simulation continues until the flux rope either reaches an equilibrium or erupts, depending on the values of Φ_p or Φ_φ. This way, a critical value is found, denoted by Φ_{pc} or $\Phi_{\varphi c}$, that corresponds to the catastrophic point. Just above the catastrophic point, the system loses equilibrium and the flux rope leaves the solar surface and erupts, i.e., a catastrophe takes place. When an equilibrium is reached, we calculate the magnetic energy of the system, normalized by $4\pi W_0 = 4\pi B_0^2 R_0^3/\mu$, with the use of

$$W_m = \frac{1}{2}\int_1^{30} dr \int_0^{\pi/2} B^2 r^2 \sin\theta d\theta + \frac{30^3}{2}\int_0^{\pi/2} (B_r^2 - B_\theta^2)_{r=30} \sin\theta d\theta, \tag{11}$$

where the first term on the right hand side is the magnetic energy in the numerical box ($1 \leq r \leq 30$), and the second is that above the box, having been transformed into a surface integral over the top (cf. Low & Smith 1993). The magnetic energy thus obtained at the catastrophic point, i.e., the energy threshold W_c, will be used to mark the catastrophic point instead of either Φ_{pc} or $\Phi_{\varphi c}$. Incidentally, with the same flux distribution at the solar surface, the open field energy is 0.554 for case A (1.662 times of the dipole field energy, see Low & Smith 1993) and 0.553 for Case B. Meanwhile, with the use of Eq. (11), the magnetic energy of the initial system, showing in Fig. 1, is found to be 0.562 for both Case A and Case B, which is already larger than the open field limit.

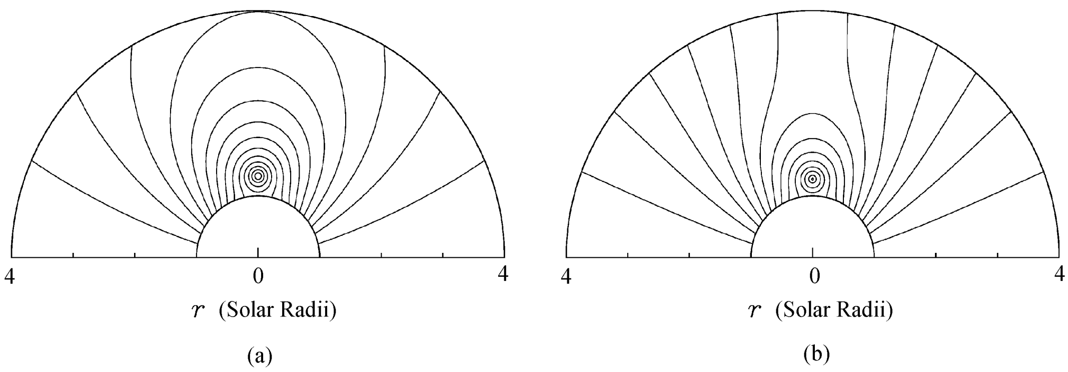

Fig. 1 Magnetic configurations of the initial state for a completely closed background field (Case A) and a partly open background field (Case B).

A sufficiently low growth rate α must be chosen in order to make the evolution process of the system quasi-static so that the energy threshold W_c can be accurately determined. Tentative simulations were made for Case A for a transition from $(\Phi_{p0}, \Phi_{\varphi 0})$ to $(1.14\Phi_{p0}, \Phi_{\varphi 0})$ that is very close to the catastrophic point. Two growth rates were tested: $\alpha = 5\times10^{-3}$ and 1.25×10^{-3}; the corresponding time intervals for accomplishing the transition are 28 and 112 τ_A, respectively. Figure 2 shows the heliocentric distance of the flux rope axis (h) as a function of time for the two cases (thick curves). After the transition is complete, h reaches to within 1% of $1.44\,R_0$. The corresponding profiles for $\Phi_p = 1.15\ \Phi_{p0}$ are also shown in Fig. 2 (thin curves). The flux rope erupts in this case, showing that the catastrophic point lies between $\Phi_p/\Phi_{p0} = 1.14$ and 1.15. The energy threshold obtained turns out to be the same within the numerical accuracy, i.e., between 0.595 and 0.596. This indicates that 5×10^{-3} is a proper value for α in the sense that it makes the evolution process of the system almost quasi-static without consuming too much computer time. Therefore, this value of α will be taken in the following numerical simulations.

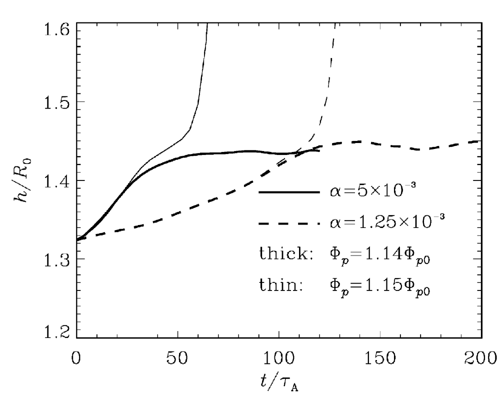

Fig. 2 Heliocentric distance of the flux rope axis h as a function of time for case A for different values of α and Φ_p.

4 NUMERICAL RESULTS

In comparison with Paper I, a much smaller value of β is taken for both cases A and B, $\beta = 10^{-4}$, so that the magnetic field obtained is very close to being force-free. As a result, the magnetic energy dominates, and other forms of energy are practically negligible, being less than one thousandth of the magnetic energy for all the numerical examples.

For Case A when the background is completely closed, we first increase the annular magnetic flux Φ_p of the rope while keeping the axial magnetic flux constant ($\Phi_\varphi = \Phi_{\varphi 0}$). The catastrophic point is found to be between 1.14 and 1.15 Φ_{p0}. The rope remains attached to the solar surface in equilibrium at $\Phi_p = 1.14\Phi_{p0}$, whereas it erupts at $\Phi_p = 1.15\Phi_{p0}$. The magnetic energy of the system at $t = 1/\alpha$ for the two values of Φ_p are 0.595 and 0.596, respectively. Next, the annular flux of the rope is fixed at $\Phi_p = \Phi_{p0}$ and the axial flux is increased. The catastrophic point thus obtained lies between 1.37 and 1.38 $\Phi_{\varphi 0}$. In other words, the flux rope sticks to the solar surface in equilibrium for $\Phi_\varphi = 1.37\Phi_{\varphi 0}$ and erupts for $\Phi_\varphi = 1.38\Phi_{\varphi 0}$. The magnetic energy for the two values of Φ_φ is 0.600 and 0.601. Although the catastrophic point is located at different sets of (Φ_p, Φ_φ) for the two situations, the energy threshold turns out to be about the same, $W_c \approx 0.60$, the deviations being within the numerical errors.

The calculations were repeated for Case B, and the results are listed in the last four rows of Table 1, including the values of the magnetic flux and corresponding magnetic energy on the two sides of the catastrophic point. The energy threshold is also around 0.60, as seen from this table. In fact, we also treated cases of $\beta = 0.001$ and $\beta = 0.01$. The magnetic field can still be safely considered as force free for $\beta = 0.001$, so the same conclusion holds, namely, the energy threshold is around 0.60. For $\beta = 0.01$, on the other hand, the gravity associated with the embedded prominence can no longer be neglected. As pointed out in Paper I, it raises the energy threshold by an amount that is approximately equal to the excess gravitational energy of the prominence. The threshold changes from 0.60 to 0.61 accordingly for the $\beta = 0.01$ case.

Table 1 Magnetic Energy Near the Catastrophic Point

Cases	Φ_p/Φ_{p0}	$\Phi_\varphi/\Phi_{\varphi 0}$	W_m
A	1.14	1	0.595
A	1.15	1	0.596
A	1	1.37	0.600
A	1	1.38	0.601
B	1.27	1	0.597
B	1.28	1	0.599
B	1	1.71	0.601
B	1	1.72	0.603

In our model, the top of the computational domain is taken to be $30\,R_0$, above which the magnetic energy is less than a few thousandth of the total magnetic energy. In addition, this amount of energy has been included in the magnetic energy of the system in Eq. (11). Therefore, the boundary conditions at the top do not much affect the energy threshold predicted above. To further make sure that this expectation is correct, we extended the computational domain to $50\,R_0$ and $80\,R_0$, and the energy threshold turned out to be the same. The deviations were less than 1%, within the numerical accuracy.

5 CONCLUDING REMARKS

Through present numerical analysis, we obtain an energy threshold of 0.60 for force-free magnetic configurations with a levitating flux rope. Below this threshold, the flux rope remains to be attached to the solar surface and the whole system stays in equilibrium. This provides an example showing that a force-free field with a part of the field lines detached from the solar surface may have an energy larger than the open field energy, 0.553–0.554 for the present case. The excess energy is then around $0.60 - 0.554 = 0.046$. For a flux rope stretching one radian along its axial direction, the total excess energy is estimated to be $0.046 \times 2W_0 = 3.4 \times 10^{34}$ erg for $\beta = 10^{-4}$ and $3.4{\times}10^{32}$ erg for $\beta = 0.01$, more than 10^{32} erg necessary for a CME event (Forbes 2000; Low 2001).

A realistic coronal flux rope should have both its ends anchored to the solar surface, and this must affect the maximum energy of the resulting 3-dimensional force-free fields and the conclusion reached above. We used a plane tangential to the solar surface right at the equator so that the magnetic flux rope becomes fully anchored to the plane. In terms of the numerical solutions obtained above, we calculated the magnetic energy in the domain above that plane. At the same time, based on the same normal component on the plane, we used the Green function to calculate the corresponding potential field and fully-open field above the plane, and their energies as well. It was found that the magnetic energy of the flux rope system above the tangential plane lies in between the potential field energy and the open field energy so the Aly conjecture is not contradicted. However, this does not mean that the present model of force-free fields is bound to be constrained by the Aly conjecture. The flux rope may be further twisted so as to increase the magnetic energy of the system before a catastrophe occurs. Whether the resultant maximum energy exceeds the open field energy remains open and to be answered by 3-dimensional MHD simulations. As a matter of fact, as pointed out by Choe & Cheng (2002), force-free fields with complex topology may have an energy in excess of the open field energy so that the Aly conjecture does not hold even if all field lines are anchored to the solar surface. The same conclusion probably holds for the present type of force-free fields. One of the key assumptions in the proofs of the Aly conjecture provided by Aly (1991) and Sturrock (1991) is that in the set of admissible force-free fields with all field lines unknotted and anchored to the lower boundary, an energy-maximizing sequence of force-free fields converges to a field $B+$ which belongs to the set. When a system like that discussed in this paper has a catastrophic behavior, the maximum energy state might be infinitely approximated but not ever reached, and thus the field $B+$ does not belong to the set of admissible force-free fields. Therefore, we conjecture that a magnetic configuration of force-free fields with a catastrophic behavior may have a maximum energy larger than the Aly-Sturrock limit, even if all the field lines are anchored to the solar surface.

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References

Aly J. J., 1984, ApJ, 283, 349
Aly J. J., 1991, ApJ, 375, L61
Aly J. J., 1994, A&A, 288, 1012
Amari T., Luciani J. F., Aly J. J., Tagger M., 1996, ApJ, 466, L39
Choe G. S., Cheng C. Z., 2002, ApJ, 574, L179
Forbes T. G., 2000, J. Geophys. Res., 105, 23153
Forbes T. G., Isenberg P. A., 1991, ApJ, 373, 294
Forbes T. G., Priest E. R., 1995, ApJ, 446, 377
Hu Y. Q., 1989, J. Comput. Phys., 84, 441
Hu Y. Q., 2001, Solar Phys., 200, 115
Hu Y. Q., Jiang Y. W., 2001, Solar Phys., 203, 309
Hu Y. Q., Liu W., 2000, ApJ, 540, 1119
Hu Y. Q., Jiang Y. W., Liu W., 2001, Chin. J. Astron. Astrophys., 1, 77
Hu Y. Q., Li G. Q., Xing X. Y., 2003, J. Geophys. Res., 108(A2), 1072, doi:10.1029/2002JA009419 (Paper I)
Hundhausen A. J., 1988, In: V. Pizzo et al., eds., Solar Wind Six, Tech. Note NCAR/TN-306, NCAR, Boulder, Colo., 181
Isenberg P. A., Forbes T. G., Démoulin P., 1993, ApJ, 417, 368
Li G. Q., Hu Y. Q., 2002, Chinese Science (Series A), 45, Supplement, 65
Lin J., Forbes T. G., Démoulin P., 1998, ApJ, 504, 1006
Lin J., Forbes T. G., Isenberg P. A., 2001, J. Geophys. Res., 106, 25053
Low B. C., 2001, J. Geophys. Res., 106, 25341
Low B. C., Smith D. F., 1993, ApJ, 410, 412
Lynden-Bell D., Boily C., 1994, MNRAS, 267, 146
Mikić Z., Linker J. A., 1994, ApJ, 430, 898
Roumeliotis G., Sturrock P. A., Antiochos S. K., 1994, ApJ, 423, 847
Sturrock P. A., 1991, ApJ, 380, 655
Wang Z., Hu Y. Q., 2003, Chin. J. Astron. Astrophys., 3, 241
Wolfson R., 1995, ApJ, 443, 810
Yang W. H., Sturrock P. A., Antiochos S. K., 1986, ApJ, 309, 383