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An Exact Anisotropic Quark Star Model

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Abstract We present an exact analytical solution of the gravitational field equations describing a static spherically symmetric anisotropic quark matter distribution. The radial pressure inside the star is assumed to obey a linear equation of state, while the tangential pressure is a complicated function of the radial coordinate. In order to obtain the general solution of the field equations a particular density profile inside the star is also assumed. The anisotropic pressure distribution leads to an increase in the maximum radius and mass of the quark star, which in the present model is around three solar masses.

Key words: Stars — quark: interior solution: anisotropy

1 INTRODUCTION

Since the pioneering work of Bowers and Liang (1974) there has been an extensive literature devoted to the study of anisotropic spherically symmetric static general relativistic configurations. The study of static anisotropic fluid spheres is important for relativistic astrophysics. The theoretical investigations of Ruderman (1972) about more realistic stellar models showed that nuclear matter may be anisotropic at least in certain very high density ranges ($\rho > 10^{15}\,\mathrm{g\,cm^{-3}}$), where the nuclear interactions must be treated relativistically. According to these views in such massive stellar objects the radial pressure may not be equal to the tangential one. No celestial body is composed of purely perfect fluid. Anisotropy in fluid pressure could be introduced by the existence of a solid core or by the presence of type 3A superfluid (Kippenhahn & Weigert 1990), different kinds of phase transitions (Sokolov 1980), pion condensation (Sawyer 1972) or by other physical phenomena. On the scale of galaxies, Binney and Tremaine (1987) considered anisotropies in spherical galaxies, from a purely Newtonian point of view. Other source of anisotropy, due to the effects of the slow rotation in a star, has been proposed recently by Herrera and Santos (1995). The mixture of two gases (e.g., monatomic hydrogen, or ionized hydrogen and electrons) can formally be also described as an anisotropic fluid (Letelier 1980).

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Observations of pulsars predict large surface magnetic fields of the order of $B \sim 10^{14}\,$G (Weber 1999). The interior magnetic fields are a few order of magnitude higher, with the virial theorem predicting fields of $\sim 10^{18}\,$G or more (Weber 1999). The magnetic field generates a force $\boldsymbol{F}$ which acts upon a medium through which electric current, with density $\boldsymbol{j}$, flows: $\boldsymbol{F} = \boldsymbol{j} \times \boldsymbol{B}/c$. With the use of the Maxwell equations, relating the current to the field, $\nabla \times \boldsymbol{B} = 4\pi \boldsymbol{j}/c$ and $\nabla \cdot \boldsymbol{B} = 0$, the expression for the force can be expressed in the form $F_i = -\partial T_{ij}/\partial x^j$, where $T_{ij} = -(1/4\pi)\left[B_i B_j - (1/2)\,\delta_{ij} B^2\right]$, a form which is similar to the effect of a pressure, $\boldsymbol{F} = -\nabla p$ (Weber 1999). Instead of a single scalar quantity (pressure), in the case of a magnetic field we deal with a stress tensor: in the direction of the magnetic field there is a tension which is equivalent to a force per unit area $B^2/8\pi$, and in the other two perpendicular directions there is a pressure, opposite in sign but of the same magnitude, $-B^2/8\pi$. For a magnetic field $\sim 10^{18}\,$G or higher, the magnetic pressure could be of the same order of magnitude to the matter pressure. Therefore strong magnetic fields could generate an anisotropic pressure distribution inside a compact astrophysical object.

The starting point in the study of fluid spheres is represented by the interior Schwarzschild solution from which all problems involving spherical symmetry can be modelled. Bowers and Liang (1974) investigated the possible importance of locally anisotropic equations of state for relativistic fluid spheres by generalizing the equations of hydrostatic equilibrium to include the effects of local anisotropy. Their study shows that anisotropy may have non-negligible effects on such parameters as maximum equilibrium mass and surface redshift. Heintzmann and Hillebrandt (1975) studied fully relativistic, anisotropic neutron star models at high densities by means of several simple assumptions and showed that for arbitrary large anisotropy there is no limiting mass for neutron stars, but the maximum mass of a neutron star still lies beyond $3-4M_{\odot}$. Hillebrandt and Steinmetz (1976) considered the problem of stability of fully relativistic anisotropic neutron star models. They derived the differential equation for radial pulsations and argued that there exists a static stability criterion similar to the one obtained for isotropic models. Anisotropic fluid sphere configurations have been analyzed, using various additional assmptions, in Bayin (1982) and Cosenza et al. (1981) (Krori et al. 1984; Maharaj & Maartens 1989; Stewart 1982; Singh et al. 1992; Magli & Kijowski 1992; Magli 1993; Bondi 1992; Chan et al. 1993; Herrera & Ponce de Leon 1985; Gokhroo & Mehra 1994; Durgapal & Bannerji 1983; Knutsen 1988; Patel & Mehra 1995; Harko & Mak 2000; Harko & Mak 2002).

For static spheres in which the tangential pressure differs from the radial one, Bondi (1992) examined the link between the surface value of the potential and the highest occurring ratio of the pressure tensor to the local density. Chan, Herrera and Santos (1993) studied in detail the role played by the local pressure anisotropy in the onset of instabilities and showed that small anisotropies might in principle drastically change the stability of the system. Herrera and Santos (1995) have extended the Jeans instability criterion in Newtonian gravity to systems with anisotropic pressures. Recent reviews on isotropic and anisotropic fluid spheres can be found in Delgaty & Lake (1998) and Herrera & Santos (1997). There are very few interior solutions (both isotropic and anisotropic) of the gravitational field equations satisfying the required general physical conditions inside the star. From 127 published solutions analyzed in Delgaty and Lake (1998) only 16 satisfy all the conditions.

It is widely believed today that strange quark matter consisting of the u, d and s quarks is the most energetically favorable state of baryon matter. Witten (1984) specified two ways of formation of the strange matter: the quark-hadron phase transition in the early universe and conversion of neutron stars into strange ones at ultrahigh densities. Quark bag models in the theories of strong interactions suppose that the breaking of physical vacuum takes place inside hadrons. As a result the vacuum energy densities inside and outside a hadron become essentially different and the vacuum pressure B on a bag wall equilibrates the pressure of quarks thus stabilizing the system.

There are several proposed mechanisms for the formation of quark stars. Quark stars are expected to form during the collapse of the core of a massive star after the supernova explosion as a result of a first or second order phase transition, resulting in deconfined quark matter (Dai, Peng & Lu 1995). The proto-neutron star core or the neutron star core is a favorable environment for the conversion of ordinary matter to strange quark matter (Cheng et al. 1998). Another possibility is that some neutron stars in low-mass X-ray binaries can accrete sufficient mass to undergo a phase transition to become strange stars (Cheng & Dai 1996). This mechanism has also been proposed as a source of radiation emission for cosmological γ-ray bursts (Cheng & Dai 1998).

The structure of a realistic strange star is quite complicated and can be described as follows (Cheng et al. 1998). Beta-equilibrated strange quark - star matter consists of an approximately equal mixture of up, down and strange quarks, with a slight deficit of the latter. The Fermi gas of $3A$ quarks constitutes a single color-singlet baryon with baryon number A. This structure of the quarks leads to a net positive charge inside the star. Since stars in their lowest energy state are supposed to be charge neutral, electrons must balance the net positive quark charge in strange matter stars. Being bound by the Coulomb, rather than the strong force, as is the case for quarks, the electrons extend several hundred fermis beyond the surface of the strange star. Associated with this electron displacement is a very strong electric dipole layer that can support, out of contact with the surface of a strange star, a crust of nuclear material, which it polarizes. The neutron drip density determines the maximal possible density at the base of the crust (the inner crust density) (Cheng et al. 1998). Being electrically charge neutral the neutrons do not feel the Coulomb force and hence would gravitate toward the quark core where they become converted into strange quark matter.

If the hypothesis of the quark matter is true, then some neutron stars could actually be strange stars, built entirely of strange matter (Alcock et al. 1986; Haensel et al. 1986). However, there are general arguments against the existence of strange stars (Caldwell & Friedman 1991).

The basis for the study of most of the static relativistic models of strange stars has been the bag model equation of state (BMEOS) $p = \left(\rho c^2 - 4B\right)/3$, where ρ is the energy density and B is the bag constant (Cheng et al. 1998). A complete description of static strange stars was obtained based on numerical integration of mass continuity and TOV (hydrostatic equilibrium) equations for different values of the bag constant (Witten 1984; Haensel et al. 1986). Using numerical methods the maximum gravitational mass $M_{\max}$, the maximum baryon mass $M_{B,\max} \equiv 1.66 \times 10^{-27}\,\mathrm{kg} \times N_B$ (N_B-the total baryon number of the stellar configuration) and the maximum radius $R_{\max}$ of the strange star, have been obtained, as a function of the bag constant, in the form (Witten 1984; Alcock et al. 1986; Haensel et al. 1986; Haensel & Zdunik 1989; Friedman et al. 1989; Gourgoulhon et al. 1999):

$$M_{\max} = \frac{1.9638 M_\odot}{\sqrt{B_{60}}}, \quad M_{B,\max} = \frac{2.6252\, M_\odot}{\sqrt{B_{60}}}, \quad R_{\max} = \frac{10.172\, d\mathrm{km}}{\sqrt{B_{60}}}, \tag{1}$$

where $B_{60} \equiv B/(60\,\mathrm{MeV}\, fm^{-3})$.

More sophisticated investigations of quark-gluon interactions clarified that BMEOS represents a limiting case of more general equations of state. For example MIT bag models with massive strange quarks and lowest order QCD interactions lead to some correction terms in the equation of state of quark matter. Models incorporating restoration of chiral quark masses at high densities and giving absolutely stable strange matter can no longer be accurately described by BMEOS. On the other hand in models in which quark interaction was described by an interquark potential originating from gluon exchange and by a density dependent scalar potential which restores the chiral symmetry at high densities (Dey et al. 1998), the equation of state $P = P(\rho)$ can be well approximated by a linear function in the energy density ρ (Gondek-Rosinska et al. 2000). It is interesting to note that already Frieman and Olinto (1989) and Haensel and Zdunik (1989) mentioned the approximation of the EOS by a linear function (see also (Prakash et al. 1990; Lattimer et al. 1990)). Recently Zdunik (2000) examined the linear approximation of the equation of state, obtaining all the parameters of the EOS as polynomial functions of strange quark mass, QCD coupling constant and bag constant. The scaling relations have been applied to the determination of the maximum frequency of a particle in stable circular orbit around strange stars.

It is the purpose of the present paper to consider an exact analytical model for an anisotropic quark matter distribution, with the radial pressure obeying a linear equation of state. Moreover, in order to obtain a non-singular stellar model, we shall also fix the density profile inside the star. With the use of these assumptions the general solution of the Einstein gravitational field equations can be obtained in an exact form, and it describes an anisotropic quark matter distribution, with unequal tangential and radial pressures.

The present paper is organized as follows. In Section 2 we obtain the general solution of the gravitational field equations for the anisotropic quark star. In Section 3 we discuss and conclude our results.

2 GENERAL SOLUTION OF THE FIELD EQUATIONS FOR AN ANISOTROPIC STRANGE STAR

In the following we shall adopt geometrized units such that $8\pi G = c = 1$. The sign conventions used are those of the Landau-Lifshitz timelike convention. Let us consider a spherically symmetric static distribution of strange quark matter. In Schwarzschild coordinates the line element takes the following form:

$$\mathrm{d}s^2 = A^2(r)\,\mathrm{d}t^2 - V^{-1}(r)\,\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\chi^2\right). \tag{2}$$

The energy-momentum tensor T_i^k inside the strange star is assumed to have the form

$$T_i^k = (\rho + p_\perp)\,u_i u^k - p_\perp \delta_i^k + (p_r - p_\perp)\,\chi_i \chi^k, \tag{3}$$

where u^i is the four-velocity $Au^i = \delta_0^i$, χ^i is the unit spacelike vector in the radial direction, $\chi^i = \sqrt{V}\delta_1^i$, ρ is the energy density, p_r the pressure in the direction of χ^i (normal pressure) and $p_\perp$ the pressure orthogonal to χ_i (transversal pressure). We assume $p_r \neq p_\perp$. The case $p_r = p_\perp$ corresponds to the isotropic fluid sphere. $\Delta = p_\perp - p_r$ is a measure of the anisotropy and is called the anisotropy factor. The Einstein field equations describing the interior of a strange star can be expressed as

$$R_i^k - \frac{1}{2}\delta_i^k R = T_i^k. \tag{4}$$

Using the line element (2) the field equations (4) and the conservation equations $T_{k;i}^i = 0$, (where a semicolon ";" denotes the covariant derivative with respect to the metric), take the form (we denote the derivative with respect to the radial coordinate r by a prime):

$$\rho = \frac{1-V}{r^2} - \frac{V^{'}}{r}, \tag{5}$$

$$p_r = \frac{2A'V}{Ar} + \frac{V-1}{r^2}, \tag{6}$$

and

$$p_r' + (p_r + \rho)\frac{A'}{A} = \frac{2}{r}\Delta. \tag{7}$$

In the conservation equation Eq. (7) a supplementary term of the form $\frac{2(p_\perp - p_r)}{r}$ appears, representing a force that is due to the anisotropic nature of the fluid. This force is directed outward when $p_\perp > p_r$ and inward when $p_\perp < p_r$. The existence of a repulsive force (in the case $p_\perp > p_r$) allows the construction of more compact objects when using anisotropic fluid than when using isotropic fluid.

In the standard static fluid stellar models Einstein's equation represents an under-determined system of nonlinear ordinary differential equations. In the present anisotropic stellar model the field equations of Einstein's theory can be reduced to a set of three coupled ordinary differential equations in five unknowns p_r, $p_\perp$, ρ, A and V. In order to obtain a realistic stellar model and to complete the system of the field equations, we assume that quark matter is described by a linear barotropic equation of state, which takes the form

$$p_r = \frac{1}{n}\left(\rho - \rho_0\right), \tag{8}$$

where n and ρ_0 are constants. For the MIT bag model equation of state $n = 3$ and $\rho_0 = 4\times B = 4\times 10^{14}\mathrm{g\,cm^{-3}}$. Rearranging Eq. (5) yields

$$(Vr)' = 1 - r^2\rho. \tag{9}$$

On integration we obtain the metric function V in the form

$$V = 1 - \frac{2m\left(r\right)}{r}, \tag{10}$$

where we have introduced the mass function of the star defined as $m\left(r\right) = \frac{1}{2}\int_0^r \rho\xi^2 \mathrm{d}\xi$. From Equations (6) and (7) we obtain the following differential equation for the radial pressure p_r:

$$\frac{2V}{r}\frac{\mathrm{d}p_r}{\mathrm{d}r} + (p_r+\rho)\left(p_r + \frac{1-V}{r^2}\right) = \frac{4V}{r^2}\Delta. \tag{11}$$

With the use of the Eqs. (8), (10) and (11) one can derive the following highly non-linear second order differential equation for the mass function of the star:

$$\begin{aligned}(r-2m)\frac{\mathrm{d}^2m}{\mathrm{d}r^2} + &\left[(5+n)\frac{m}{r} - \frac{\rho_0(n+2)}{2n}r^2 - 2\right]\frac{\mathrm{d}m}{\mathrm{d}r} + \frac{1+n}{n}\left(\frac{\mathrm{d}m}{\mathrm{d}r}\right)^2 \\ - &\frac{\rho_0}{2}rm + \frac{\rho_0^2}{4n}r^4 = nr\left(r-2m\right)\Delta,\end{aligned} \tag{12}$$

which leads to the expression of the anisotropy parameter inside the star:

$$\Delta(r) = \frac{2r^2\frac{\mathrm{d}\rho}{\mathrm{d}r} + \frac{1+n}{n}\rho^2r^3 - \frac{\rho_0(n+2)}{n}\rho r^3 + \frac{\rho_0^2}{n}r^3 - \left[\rho_0 - (1+n)\rho + 2r\frac{\mathrm{d}\rho}{\mathrm{d}r}\right]\int_0^r \xi^2\rho\mathrm{d}\xi}{4n\left(r - \int_0^r \xi^2\rho\mathrm{d}\xi\right)}. \tag{13}$$

In order to have a non-singular monotonic decreasing matter density inside the star, we assume for the density profile of the quark star the following functional form:

$$\rho(r) = \rho_c \left[1 - \left(1 - \frac{\rho_0}{\rho_c}\right) \frac{r^2}{R^2}\right], \tag{14}$$

where ρ_c is the central density of the star and R the radius of the sphere. We require that the condition $\rho_c \geq \rho_0$ holds. Obviously, the matter density has a maximum value ρ_c at center $r = 0$, and the value ρ_0 at the surface of the sphere. With the use of Eq. (14) we obtain immediately the mass distribution inside the anisotropic quark star:

$$m\,(r) = \frac{\rho_c}{6} r^3 - \frac{\rho_c - \rho_0}{10R^2} r^5. \tag{15}$$

Therefore an explicit exact solution describing the interior of an anisotropic strange quark star is given by

$$A^2(r) \;=\; A_0 \prod_{i=+,-} \left[R\,(C + 5\epsilon_i R\rho_c) + 6\epsilon_i\,(\rho_c - \rho_0)\, r^2\right]^{D_{\epsilon_i}}, \tag{16}$$

$$V(r) \;=\; 1 - \frac{\rho_c}{3} r^2 + \frac{\rho_c - \rho_0}{5R^2} r^4, \tag{17}$$

$$\Delta(r) = \frac{r^2}{4n^2R^2} \left\{ \frac{\Delta_0(r)}{3\,(\rho_c - \rho_0)\, r^4 - 5R^2\,(\rho_c r^2 - 3)} - 4n\,(\rho_c - \rho_0) \right\}, \tag{18}$$

and

$$p_r(r) \;=\; \frac{(\rho_c - \rho_0)}{n} \left(1 - \frac{r^2}{R^2}\right), \tag{19}$$

$$p_\perp(r) \;=\; \Delta(r) + \frac{(\rho_c - \rho_0)}{n} \left(1 - \frac{r^2}{R^2}\right), \tag{20}$$

where A_0 is an arbitrary constant of integration,

$$C = \sqrt{5\left[36\rho_0 + \rho_c\,(5\rho_c R^2 - 36)\right]},$$

$$D_\pm = -\frac{5+n}{4n} + \epsilon_\pm \frac{5R\left[6\rho_0 - (1+n)\rho_c\right]}{4nC},$$

$$\Delta_0(r) = \left[(1+n)\,(\rho_0 - \rho_c)\, r^2 + R^2\left((1+n)\,\rho_c - \rho_0\right)\right]\left[3(5+n)\,(\rho_0 - \rho_c)\, r^2 + 5R^2\left((3+n)\,\rho_c - 3\rho_0\right)\right]$$

and $\epsilon_\pm = \pm 1$.

In view of Eq. (19), the condition of the radial pressure vanishing at the surface of the star, $p_r\,(R) = 0$ holds. On the other hand, the tangential pressure is generally not zero at the surface of the quark star, a situation specific to anisotropic stellar models (Herrera & Santos 1997).

For $r > R$ the solution of the Einstein equations is given by the Schwarzschild metric as

$$\mathrm{d}s^2 = (1 - 2M/r)\,\mathrm{d}t^2 - (1 - 2M/r)^{-1}\,\mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\chi^2\right), \tag{21}$$

where M is the total mass the star. To match the line element (2) with the Schwarzschild metric across the boundary at $r = R$ we require the continuity of the gravitational potentials A^2 and V at $r = R$. The continuity of A^2 leads to the determination of the constant A_0:

$$A_0 = \frac{1 - \frac{2M}{R}}{\prod_{i=+,-}\left[R\,(C + 5\epsilon_i R\rho_c) + 6\epsilon_i\,(\rho_c - \rho_0)\, R^2\right]^{D_{\epsilon_i}}}. \tag{22}$$

At the centre of the star the line element is flat and is given by

$$\mathrm{d}s^2 = A_0 \prod_{i=+,-} \left[R\left(C + 5\epsilon_i R\rho_c\right)\right]^{D_{\epsilon_i}} \mathrm{d}t^2 - \mathrm{d}r^2 - r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\chi^2\right). \tag{23}$$

We have now obtained the complete solution of the gravitational field equations for an anisotropic strange quark star described by an arbitrary linear equation of state.

3 DISCUSSION AND FINAL REMARKS

The following conditions have been generally recognized to be crucial for anisotropic fluid spheres (Herrera & Santos 1997):

a) the density ρ and pressure p_r should be positive inside the star;

b) the gradients $\frac{\mathrm{d}\rho}{\mathrm{d}r}$, $\frac{\mathrm{d}p_r}{\mathrm{d}r}$ and $\frac{\mathrm{d}p_\perp}{\mathrm{d}r}$ should be negative;

c) inside the static configuration the speed of sound should be less than the speed of light, i.e. $0 \leq \frac{\mathrm{d}p_r}{\mathrm{d}\rho} \leq 1$ and $0 \leq \frac{\mathrm{d}p_\perp}{\mathrm{d}\rho} \leq 1$;

d) a physically reasonable energy-momentum tensor has to obey the conditions $\rho \geq p_r + 2p_\perp$ and $\rho + p_r + 2p_\perp \geq 0$;

e) the interior metric should be joined continuously with the exterior Schwarzschild metric, that is $A^2(R) = 1 - 2M/R$;

f) the radial pressure p_r must vanish but the tangential pressure $p_\perp$ may not vanish at the boundary of the sphere. However, the radial pressure should be equal to the tangential pressure at the center of the fluid sphere.

The solution to the gravitational field equations we have obtained satisfies most of these six criteria, since we have $\rho \geq 0$, $p_r \geq 0$, $p_\perp \geq 0$, $\frac{\mathrm{d}\rho}{\mathrm{d}r} = -2\rho_c\left(1 - \frac{\rho_0}{\rho_c}\right)\frac{r}{R^2} < 0$ and $\frac{\mathrm{d}p_r}{\mathrm{d}r} = -2\frac{\rho_c - \rho_0}{n}\frac{r}{R^2} < 0$. The radial speed of sound is given by $v_s = c/\sqrt{n}$. Moreover, all the physical (density, pressure and anisotropy parameter) and geometrical (metric tensor components) quantities are finite throughout the star and thus $\rho + p_r + 2p_\perp \geq 0$. At the center of the star the radial pressure equals the tangential pressure and the radial pressure vanishes at the surface. However, the condition $\frac{dp_\perp}{dr} < 0$ is not satisfied generally inside the star, showing that the $p_\perp$ is an increasing function of r. For some regions the speed of sound in the tangential direction $\frac{dp_\perp}{d\rho}$ is not defined or can exceed the speed of light. Now, Caporaso and Brecher (1979) claimed that $dp/d\rho$ does not represent the signal speed. If therefore this speed exceeds the speed of light, this does not necessarily mean that the fluid is non-causal. In fact, this argument is quite controversial and not all authors accept it (Glass 1983).

The variation of the anisotropy parameter Δ as a function of the radial coordinate r and for different values of the central density is presented in Fig. 1. At the center of the quark star the anisotropy is zero, $\Delta(0) = 0$. The anisotropy has a maximum value inside the star and it tends to zero at the vacuum boundary.

The radius of the star can be obtained from the requirement that the anisotropy vanishes at the surface of the quark star, $\Delta(R) = 0$. This condition gives

$$R = \sqrt{60}\sqrt{\frac{\rho_c - \rho_0}{8\rho_c^2 + 2(2+n)\rho_0\rho_c - 3(4-n)\rho_0^2}}. \tag{24}$$

Hence in this model both the radial and tangential pressures p_r and $p_\perp$ vanish at the surface of the star. However, a non-vanishing surface tangential pressure $p_\perp$ is also acceptable from a physical point of view (Herrera & Santos 1997). The radius of the anisotropic quark star can be expressed as a function of the central density ρ_c and of the parameters of the equation of state of the quark matter in an exact form. The total mass of the anisotropic quark star is

$$M = \frac{(2\rho_c + 3\rho_0)}{30} R^3. \tag{25}$$

The radius of the star has a maximum for the value of the central density $\rho_c^{(R_{\max})}$ satisfying the equation $\mathrm{d}R/\mathrm{d}\rho_c = 0$. The value of the central density for which the radius is maximum is given by

$$\rho_c^{(R_{\max})} = \left[1 + \sqrt{\frac{5n}{8}}\right]\rho_0. \tag{26}$$

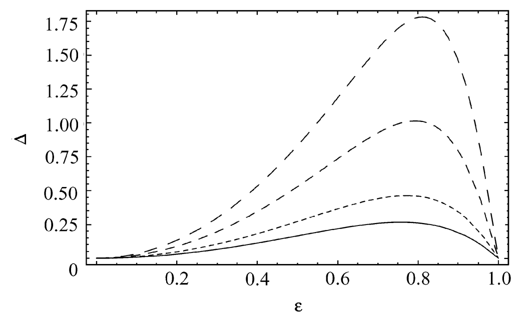

Fig. 1 Variation of the anisotropy parameter Δ inside the anisotropic quark star with the radial pressure obeying the bag model equation of state ($n = 3$ and $\rho_0 = 4 \times 10^{14}\mathrm{g\,cm^{-3}}$), as a function of $\epsilon = r/R$ for different values of the central density: $\rho_c = 1.0 \times 10^{15}\mathrm{g\,cm^{-3}}$ (solid curve), $\rho_c = 1.2 \times 10^{15}\mathrm{g\,cm^{-3}}$ (dotted curve), $\rho_c = 1.6 \times 10^{15}\mathrm{g\,cm^{-3}}$ (dashed curve) and $\rho_c = 2 \times 10^{15}\mathrm{g\,cm^{-3}}$ (long dashed curve).

For an anisotropic quark star with the radial pressure obeying the bag model equation of state with $n = 3$ and $\rho_0 = 4 \times B = 4 \times 10^{14}\mathrm{g\,cm^{-3}}$, the maximum radius is at the central density $\rho_c^{(R_{\max})} = 9.48 \times 10^{14}\mathrm{g\,cm^{-3}}$. The density corresponding to the maximum radius in the case of isotropic quark stars obeying the BMEOS is $\rho_c^{(R_{\max})} = 10.15 \times 10^{14}\mathrm{g\,cm^{-3}}$ (Cheng & Harko 2000). Hence the maximum radius of the anisotropic quark matter distribution is given by

$$R_{\max} = \frac{\sqrt{30}}{\sqrt{10 + n + 2\sqrt{10n}}} \frac{1}{\sqrt{\rho_0}}. \tag{27}$$

For $n = 3$ and $\rho_0 = 4 \times B = 4 \times 10^{14}\mathrm{g\,cm^{-3}}$ we obtain $R_{\max} = 1.29 \times 10^6$ cm. The value of the maximum radius in case of isotropic quark stars is $R_{\max} = 1.14 \times 10^6$ cm (Cheng & Harko 2000). Therefore the pressure anisotropy increases the maximum radius of quark stars.

With respect to a scaling of the parameter ρ_0 of the form $\rho_0 \to \alpha\rho_0, \alpha = constant$, the maximum radius scales as $R_{\max} \to \alpha^{-\frac{1}{2}} R_{\max}$, a scaling property also specific to isotropic quark stars. The variations of the radii of the anisotropic quark star, with the radial pressure obeying the bag model equation of state and of the isotropic quark star are presented in Fig. 2.

The mass of the anisotropic strange star corresponding to the maximum radius is given by

$$M\left(R_{\max}\right) = \frac{(2\rho_c + 3\rho_0)}{30} \left(\frac{\sqrt{30}}{\sqrt{10 + n + 2\sqrt{10n}}}\right)^3 \rho_0^{-3/2}. \tag{28}$$

For the bag model equation of state we obtain $M\left(R_{\max}\right) = 2.78 M_\odot$, while for the isotropic quark star this value is only $1.81 M_\odot$ (Cheng & Harko 2000). For the anisotropic quark star $M\left(R_{\max}\right)$ obeys the scaling property $M\left(R_{\max}\right) \to \alpha^{-3/2} M\left(R_{\max}\right)$ with respect to the scaling of the parameter ρ_0 of the form $\rho_0 \to \alpha\rho_0$.

The central density $\rho_c^{(M_{\max})}$, giving the maximum mass of the star, follows from the equation

$\mathrm{d}M/\mathrm{d}\rho_c = 0$ and is found to be

$$\rho_c^{(M_{\max})} = \frac{4 + n + \sqrt{n\,(n+30)}}{4}\rho_0. \tag{29}$$

For the radial pressure obeying the BMEOS the central density of the maximum mass anisotropic star is $\rho_c^{(M_{\max})} = 1.69 \times 10^{15}\mathrm{g\,cm^{-3}}$. For the isotropic quark stars this value is $\rho_c^{(M_{\max})} = 1.97 \times 10^{15}\mathrm{g\,cm^{-3}}$ (Cheng & Harko 2000). The maximum mass of the anisotropic quark star is given, as a function of the parameters of the equation of state, by the expression

$$M_{\max} = 9\sqrt{\frac{5}{2}}\frac{10 + n + \sqrt{n}\sqrt{n+30}}{\left[5(n+6) + 4\sqrt{n}\sqrt{n+30}\right]^{3/2}}\frac{1}{\sqrt{\rho_0}}. \tag{30}$$

For the maximum mass of the anisotropic quark star we obtain $M_{\max} = 3.26M_{\odot}$. For the isotropic quark star the maximum mass is $M_{\max} = 2M_{\odot}$ (Cheng & Harko 2000). Hence the inclusion of an anisotropic pressure distribution leads to a significant increase in the maximum mass of the quark star, which is of the same order of magnitude as the maximum mass of the isotropic neutron stars.

The maximum mass of the anisotropic quark star has the same scaling property with respect to the scale transformation of ρ_0, $\rho_0 \to \alpha\rho_0$, as the maximum radius of the star, $M_{\max} \to \alpha^{-1/2}M_{\max}$. This scaling property is similar to that of isotropic quark stars described by a linear equation of state (Zdunik 2000). The variation of the mass of the anisotropic and of the anisotropic star, respectively, as a function of the central density, is presented in Fig. 3.

An important observational parameter, which in principle could distinguish between anisotropic and isotropic quark stars or even anisotropic quark and neutron stars, is the surface red-shift parameter, defined as $z = (1 - 2GM/c^2R)^{-1/2} - 1$. The variation, as a function of the central density, of the surface red-shift for anisotropic and isotropic quark stars, respectively, is presented in Fig. 4.

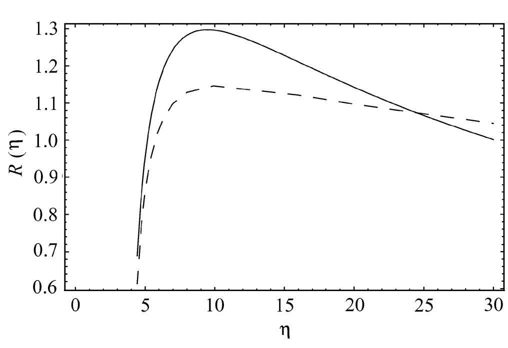

Fig. 2 Variation of the radius R (in units of 10^6 cm) of the anisotropic quark star with the radial pressure obeying the bag model equation of state ($n = 3$ and $\rho_0 = 4 \times 10^{14}\mathrm{g\,cm^{-3}}$) (solid curve) and of the isotropic star (dashed curve) as a function of the parameter $\eta = \rho_c/B$.

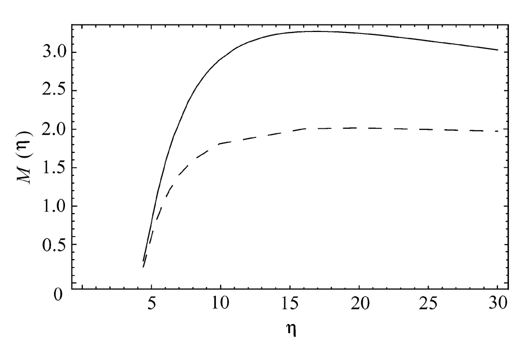

Fig. 3 Variation of the mass M (in solar mass units) of the anisotropic quark star with the radial pressure obeying the bag model equation of state ($n = 3$ and $\rho_0 = 4 \times 10^{14}\mathrm{g\,cm^{-3}}$) (solid curve) and of the isotropic star (dashed curve) as a function of the parameter $\eta = \rho_c/B$.

For values of the central density $\rho_c > 10^{15}\,\mathrm{g\,cm^{-3}}$, the z for anisotropic quark stars is much higher than for the isotropic quark or neutron stars. For a range of central densities $4\times10^{14}\,\mathrm{g\,cm^{-3}} < \rho_c < 10^{15}\,\mathrm{g\,cm^{-3}}$, the red-shift of the isotropic and anisotropic quark stars is approximately equal.

From the study of the data base obtained by the measurement of gravitationally red-shifted 511 keV $e^{\pm}$ pair annihilation lines from the surface of neutron stars, Liang (1986) suggested that the neutron star red-shift ranges over $0.2 \le z \le 0.5$, with the highest concentration in the range $0.25 \le z \le 0.35$. As can be seen from Fig. 4, the maximum redshift of isotropic quark stars also cannot exceed the value $z \approx 0.5$. Therefore detection of compact objects with red-shifts higher than $z = 0.5$ would be a strong observational evidence for relativistic stars with anisotropic pressure distribution.

An important problem is the origin of the anisotropy in quark stars. In the Introduction we have mentioned several mechanisms which could generate an anisotropic pressure distribution inside a neutron star. Since in the interior of a strange star a solid core should not appear, the existence of a superfluid core inside the star could generate different pressures along the radial and tangential directions. It has been suggested a long time ago that the quarks may eventually form Cooper pairs (Bailin & Love 1984). In fact pairing is unavoidable in a degenerate Fermi liquid if there is an attractive interaction. The resulting superfluidity (and in case of charged particles, superconductivity) has a major effect on the star's evolution and also induces an intrinsic anisotropic pressure distribution.

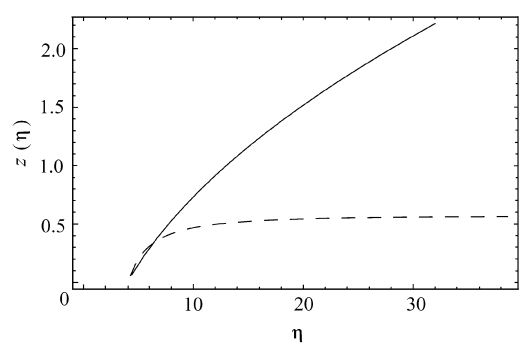

Fig. 4 Variation of the surface red-shift z of the anisotropic quark star with radial pressure obeying the bag model equation of state ($n = 3$ and $\rho_0 = 4\times10^{14}\,\mathrm{g\,cm^{-3}}$) (solid curve) and of the isotropic quark star (dashed curve) as a function of the parameter $\eta = \rho_c/B$.

To analyze this effect in more details let us consider a strange star consisting of two different components: a superfluid with energy density ρ_s, pressure p_s and four-velocity U^i and a normal matter component with ρ_n, pressure p_n and four-velocity W^i. Then the total energy-momentum tensor is

$$T^{ik} = (p_s+\rho_s)\,U^iU^k - p_s g^{ik} + (p_n+\rho_n)\,W^iW^k - p_n g^{ik}, \tag{31}$$

where $U_iU^i = 1$ and $W_iW^i = 1$. By means of the transformations (Bayin 1982)

$$U^{*i} = U^i\cos\alpha + \sqrt{\frac{P_2+\rho_2}{P_1+\rho_1}}W^i\sin\alpha, W^{*i} = -\sqrt{\frac{P_2+\rho_2}{P_1+\rho_1}}U^i\sin\alpha + W^i\cos\alpha, \tag{32}$$

the energy momentum tensor (31) can always be cast into the standard form for anisotropic fluids,

$$T^{ik} = (\rho + p_{\perp}) V^i V^k - p_{\perp} g^{ik} + (p_r - p_{\perp}) \chi^i \chi^k, \tag{33}$$

where $V^i = U^{*i}/\sqrt{U^{*i}U_i^*}$, $\chi^i = W^{*i}/\sqrt{-W^{*i}W_i^*}$, $\rho = T_{ik}V^iV^k$, $p_{\perp} = p_s + p_n$ and $p_r = -\frac{1}{2}(\rho_s - p_s + \rho_n - p_n) + \frac{1}{2}\left[(\rho_s + p_s - \rho_n - p_n)^2 + 4(\rho_s + p_s)(\rho_n + p_n)\left(U^iW_i\right)^2\right]^{1/2}$ (Herrera & Santos 1997). Therefore a quark star consisting of a mixture of two fluids will have the general structure of an anisotropic compact general relativistic object.

The behavior of the anisotropy parameter Δ, given by Eq. (18), can also be interpreted in the framework of the two-fluid model. Since one expects that the superfluid or superconducting components are located in the core of the star, the anisotropic effects are strongest in the interior of the star, for $r \in [0, R_c]$, where R_c is a critical distance below which anisotropic effects are important. Assuming that for $r > R_c$ the star consists of only the normal component (usual quark matter), then $\rho_s, p_s \to 0$, and hence in this region the energy momentum tensor Eq. (31) takes the usual form of a single perfect fluid, corresponding to equal radial and tangential pressures and with vanishing anisotropy.

Whether the anisotropic stellar model presented in this paper actually describes a well-determined stellar structure can only be decided once reliable knowledge about the mechanisms that characterize strange and neutron star formation becomes available from the underlying theory of stellar evolution and from observational data.

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