

A Note on the Mixing Length Theory and Massive Star Evolution

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Abstract In this paper, we investigate the problem of supersonic convection caused by the application of the usual Mixing Length Theory (MLT), and give a modification to the original expression of the MLT. In the case of the usual MLT, a zone of supersonic convection exists when the stellar model goes into yellow–red supergiant phase. When the modification is applied, the models of late type supergiants possess shallower convective envelopes for a given temperature compared with the normal formalism of the MLT. Therefore the stellar models made with our new formalism move to lower effective temperature by 100–400 K depending on the luminosity of the star. Such a modification does not affect the convective envelope structure of lower luminosity late type stars, as both expressions of the MLT give the same results in these stars.

Key words: convection – stars: supergiants – stars: massive stars – stars: evolution

1 INTRODUCTION

The mixing length theory (MLT) for stellar convection originally developed by Vitense (1953, 1958) has been the most popularly used local convection theory in the studies of stellar structure and evolution. The theory was later modified and revised by many investigators, who suggested some different expressions. In fact, MLT is not a real hydrodynamic theory, rather, it is a simple “ballistic” theory which traces the motion of imaginary convective elements. In reality, stellar convective motion is highly turbulent and requires a theory based on hydrodynamic equations and the theory of turbulence. There are several theories on more sound hydrodynamic bases than the usual mixing-length theory (e.g. Unno 1967, 1969; Gough 1977, Spiegel 1963, Ulrich 1970, Kufuss 1986, Xiong 1977, Canuto 1993, Grossman et al. 1993). However, the basic equations related to the original idea are not fundamentally changed, and are still widely applied.

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For stars with effective temperatures lower than about 5000–7000 K (depending on the luminosity), a very extended convective envelope develop. Different ways of dealing with convection lead to very different results on the properties of the stellar model as regards structure, pulsation and evolution. When the classical MLT is applied in the case of late type supergiants (yellow and red), it is found that the convective motion becomes supersonic, and the relative temperature fluctuation becomes much larger than unity in the hydrogen ionization zone. Statistical theory for turbulent convection (Xiong 1989; 1996) is a nonlocal theory of convection. When all the third order correlation functions are omitted, it returns to the local theory (Xiong, 1977) which gives the same formulae as does the MLT. Such a treatment gives the MLT a theoretical support from hydrodynamics, and defines the region where the local theory such as the classical MLT can be used. Section 2 of this paper is devoted to a detailed analysis of the local MLT and its regime of validity. Based on such an analysis, a modified set of MLT formulae is presented. In section 3, the difference between the original and modified formalisms is expressed briefly in terms of numerical models. The evolution patterns of the red supergiants in the new formalism is also given.

2 THE MODIFIED MIXING LENGTH THEORY

Theoretically, using the original form of the MLT to deal with supersonic convection is not correct, before considering the existence of the supersonic motion, for the following reasons:

1) As is well known, the sound speed is a critical point in hydrodynamics. Subsonic and supersonic motions have distinct properties.

The statistical theory for turbulent convection has a set of strict working assumptions: the convective motion is subsonic ($|V_t/C_s| \ll 1$); the fluctuations in both temperature and density are much smaller than unity ($|T'/T| \ll 1$ and $|\rho'/\rho| \ll 1$). Since the MLT obeys these constraints it should only be applied to subsonic convection.

2) It is clear from a careful study of the formalism of the MLT, that the derivation of the MLT equations implicitly introduces an assumption on the relative temperature fluctuation ($|T'/T| \ll 1$). By eliminating such an assumption, we obtain the following more general expressions for the relative fluctuations in temperature and convective velocity:

$$\frac{T'}{T} = e^{\pm \frac{\alpha\gamma}{2(1+\gamma)}(\nabla - \nabla_{ad})} - 1, \quad (1)$$

$$V_t^2 = \frac{QP}{\rho} \left\{ \frac{\gamma}{(1+\gamma)} \frac{1}{(\nabla - \nabla_{ad})} \left[e^{\pm \frac{\alpha\gamma}{2(1+\gamma)}(\nabla - \nabla_{ad})} - 1 \right] \mp \frac{\alpha}{2} \right\}, \quad (2)$$

where $\alpha = l/H_P$ is the MLT parameter, $Q = -(\partial \ln \rho / \partial \ln T)_P$ is the gas expansion coefficient, $\gamma = \rho V_t C_P \kappa l / 6acT^3$ is a factor describing the heat exchange between the convective element and the surroundings, which is proportional to the Peclet number. The positive and negative signs in Eqs. (1) and (2) correspond to upward and downward motions of the convective elements. When $\frac{\alpha\gamma}{2(1+\gamma)}(\nabla - \nabla_{ad}) \ll 1$, Eqs. (1) and (2) return to the usual formulae of MLT:

$$\frac{T'}{T} = \pm \frac{\alpha\gamma}{2(1+\gamma)}(\nabla - \nabla_{ad}), \quad (3)$$

$$V_t^2 = \frac{QP}{8\rho} \alpha^2 \frac{\gamma}{1+\gamma} (\nabla - \nabla_{ad}). \quad (4)$$

It is evident from the above discussion, that the original MLT is valid only when the relative temperature fluctuations are far smaller than unity. Even if we put aside the new physical properties of supersonic convection, the MLT formalism is not generally correct.

(3) The physics of the supersonic- and subsonic- turbulence are very different. When the convection becomes supersonic, the turbulent kinetic energy dissipation increases greatly. According to the theory of turbulence, this effect corresponds to a rapid decrease of the characteristic length, i.e., the mixing length as defined by the MLT is shortened. On the other hand, the increasing turbulent fluctuations in temperature lead to violent interactions among the convective elements and make them break up faster. This again makes the characteristic length of the elements smaller. It is not possible to set up a general theory that covers both regimes of subsonic and supersonic convections at the moment. We introduce the following working assumption based on the above analysis to simulate the decrease of the characteristic length of turbulent elements (mixing length) with increase of the relative temperature fluctuations:

$$\alpha = \frac{l}{H_P} = \frac{l_0}{H_P} e^{-c\left(\frac{T'}{T}\right)^2} = \alpha_0 e^{-c\left(\frac{T'}{T}\right)^2}, \quad (5)$$

where l_0 and α_0 are the mixing length and the MLT parameter obtained when the turbulent fluctuations in temperature are negligible, and c is an adjustable parameter of order unity. It is clear from Eq. (5) that when $(T'/T) \ll 1$, $\alpha \approx \alpha_0$, i.e. the new formulae go back to the MLT. Only when $|T'/T|$ becomes comparable to unity, does the mixing length decrease rapidly with increasing $|T'/T|$.

3 RESULTS

To show the consequences of the modification of the MLT, we give here a series of envelope models for low effective temperature supergiants based on the modified and original MLT. We also present evolutionary sequences of a massive star and a low mass star, to show the effect of the modification of the MLT on the evolutionary tracks on the Hertzsprung-Russel (HR) Diagram .

Figure 1 displays (in dotted lines) 3 envelope models of a $20 M_\odot$ with effective temperatures: 3600 K (top panels), 3900 K (middle panels) and 4200 K (lower panels). The dotted lines represent the results obtained with the convective envelope for late type supergiants following the original MLT (Vitense 1958). The quantities plotted are the fractional convective energy flux L_c/L (panel a), the ratio between turbulent velocity V_t and the sound speed C_s (panel b) and the relative temperature fluctuations T'/T (panel c), as a function of $\log P$ (which measures the depth inside the star). It is clear that the convective motion can become supersonic in the hydrogen ionization zone, where the relative temperature fluctuations are much larger than unity. This is due to the fact that while the density is very low in the atmosphere of giants and supergiants and therefore convection is not efficient, in the hydrogen ionization zone, the opacity is extremely high. Both factors lead to the existence of a region of ultra-high superadiabatic gradient ($\nabla - \nabla_{ad} \gg 1$). This gives rise to the so-called ‘‘supersonic convection’’ as displayed by the dotted lines in Figure 1, near $\log P = 3 - 5$.

The solid lines in Figure 1 represent the results obtained with the same models, using the modified expression for the MLT. We can see that these new computations predict shallower convective envelopes for late type supergiants. Our results of envelope structure computations also imply a shift of about 100–400 K (depending upon the luminosity of the star) towards lower effective temperature relative to the old model (Figure 2). It has long been a problem in the

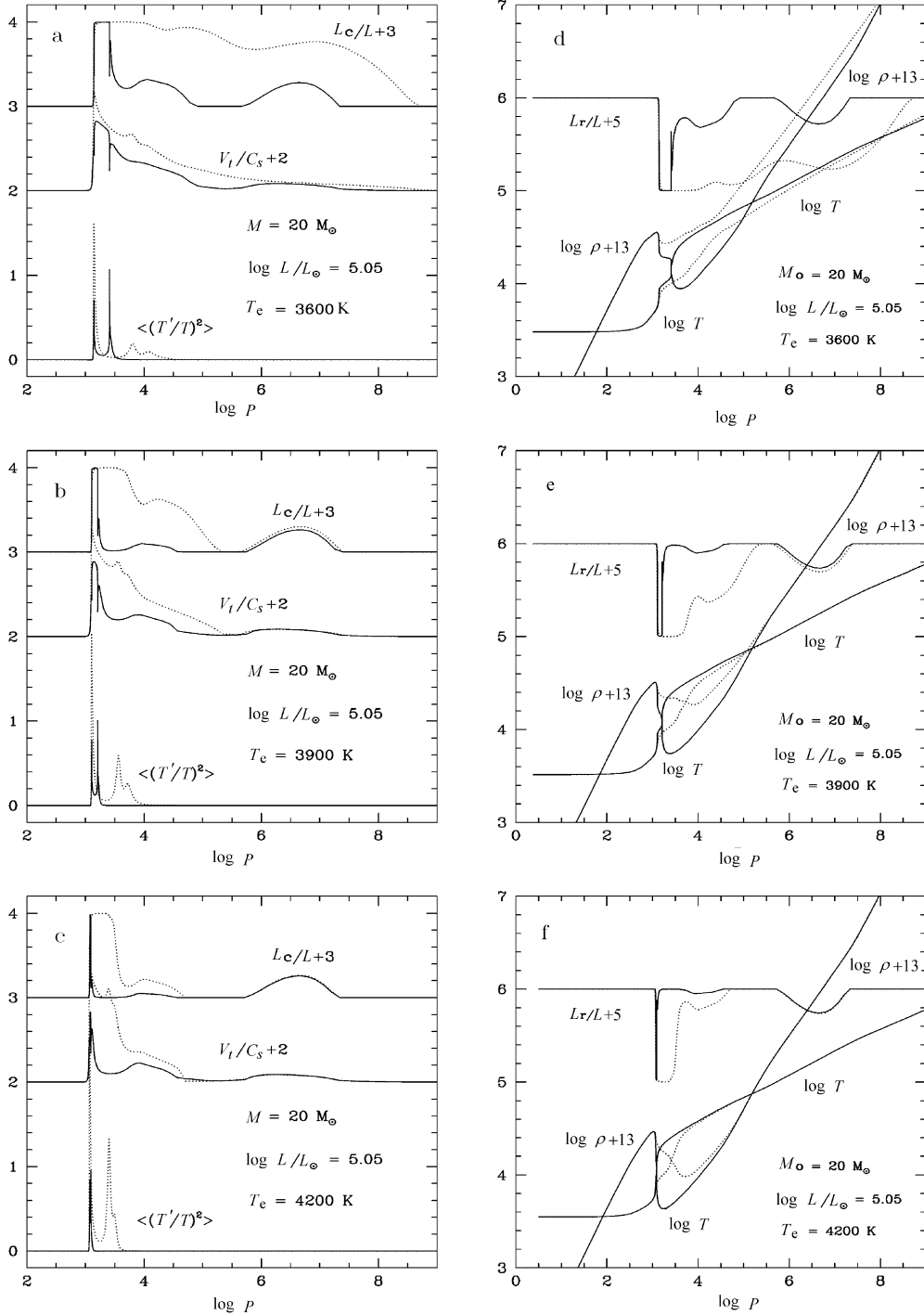


Fig. 1 Three models of yellow-red supergiants following the MLT. Plotted are the ratio between the turbulent velocity V_t and the local sound speed c_s , relative turbulent fluctuations in temperature $\log |\Delta T/T|$ and fractional convective energy flux L_c/L , all versus depth in $\log P$. The stellar parameters are: $M = 20 M_\odot$, $\log L/L_\odot = 5.05$, $X = 0.700$, $Z = 0.020$. The three models have the following effective temperatures: 3600 K, 3900 K and 4200 K respectively. The dotted lines are computed according to the Vitense (1958) formulas, while the solid line is given with our modified MLT formulas.

modeling of massive stars, that the theoretical location of the red supergiant phase does not match the observed red supergiants. The effective temperatures of observed stars are generally redder than predicted (See a comparison between the theoretical models, e.g. Bertelli et al 1994, Deng et al 1996, and the observations of Fitzpatrick & Garmany 1990, Blaha & Humphreys 1989). We also know that the location of the giant and supergiant branches in the theoretical frame is a sensitive function of the mixing length parameter used for the envelope (Weigert & Huang 1983). With a shorter mixing length (therefore, smaller α) in the MLT, one can reach the observed position. However, this parameter is one of the most critical parameters in the theoretical calculations of stellar structure and evolution. We cannot change this parameter at will. In general, this parameter is fixed by the calibration of the observed quantities of the Sun and the RGB of stellar clusters. Our new treatment has the advantage of effectively limiting the mixing length in supergiants while retaining the same for low luminosity stars, therefore not altering the results of the calibrations.

To study the temperature shift for the supergiant phase of massive stars, we have calculated an evolutionary sequence for a $20 M_{\odot}$ star, with $c = 0, 1.5, 3$. The input parameters for the models are: $X = 0.700$, $Z = 0.020$, the overshoot parameter is taken as 1.0 following Bressan et al (1981). The results are plotted in Figure 2, where the solid line is the normal MLT ($c = 0$), the dotted line is for $c = 1.5$, and the heavy line, $c = 3$. We can see that the red supergiant phase is shifted by about 200 K to lower effective temperature for $c = 3$).

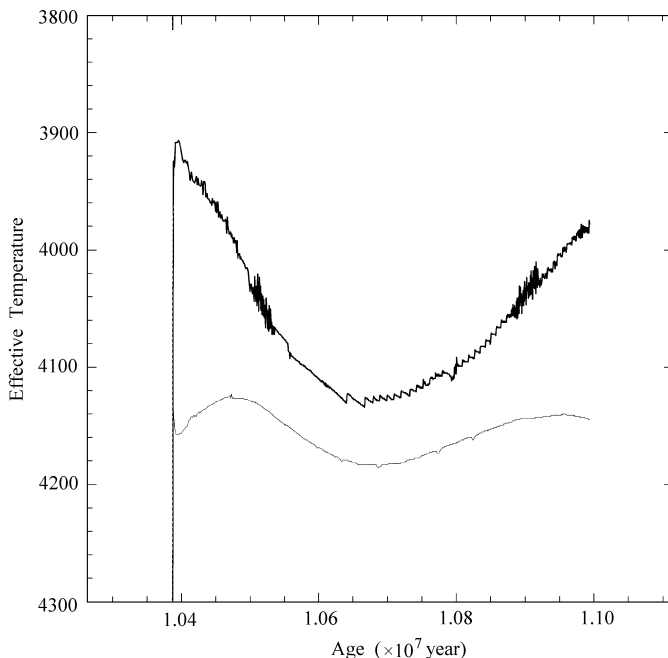


Fig. 2 Supergiant phase of a $20 M_{\odot}$ track. The thin solid line is given by the original MLT (which is exactly a coincidence of $c = 0$ in the new formalism, the dotted line and the heavy solid line are given by the modified MLT, take $c = 1.5$ and 3.0 respectively).

We have also computed a set of stellar evolutionary tracks for a $2 M_{\odot}$ stellar model, for $c = 0$ and 3 (solid and dotted). Figure 3 displays the evolutionary tracks for the $20 M_{\odot}$ and the $2 M_{\odot}$ stars, for $c = 0$ and 3. We find that, with our new formalism, the red supergiant phase is shifted towards the red side of the HR diagram for the massive star model while it does not affect at all the low mass star model. This confirms our conclusion above that the evolution of low mass stars remains unchanged, and it guarantees that all the calibrations for the MLT parameter α_0 , especially the solar calibration, are not altered.

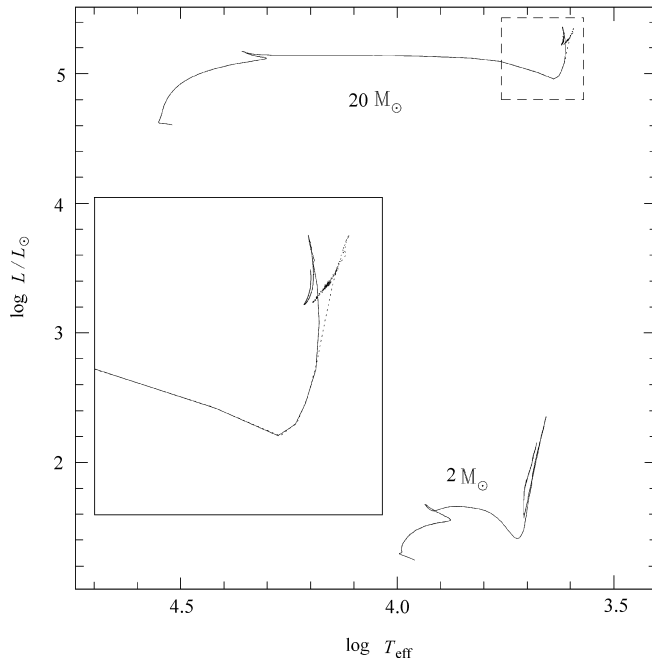


Fig. 3 The HR diagram for 20 and $2 M_{\odot}$ stars, using $c = 0$ (solid line) and 3 (dotted line). For the $2 M_{\odot}$ star, the tracks are completely identical. The red supergiant phase of the $20 M_{\odot}$ tracks are enlarged in the plot, which shows the difference between the original MLT (solid line) and our new treatment (dotted line).

4 DISCUSSION

Following the above discussion, we can conclude that when the convection is subsonic, i.e. $|T'/T| \ll 1$, the formulation of the original MLT is retained. We can further predict that the new expression of the MLT will not affect the evolution and structure patterns of late type stars of low luminosities. However, for luminous late type supergiants, the usual MLT predicts the existence of supersonic convection and $|T'/T| \gg 1$ in the hydrogen ionization zone. If the modified MLT (Eq. (5)) is applied, the mixing length will dramatically decrease in the superadiabatic regions where $|T'/T| \geq 1$. The turbulent fluctuations in temperature and velocity, and efficiency of convective energy transport are consequently greatly reduced in

the formulae of the new theory. In fact the depth of convective region is determined by the structure of the superadiabatic region. Lower convective energy transport efficiency can raise the temperature gradient in the superadiabatic gradient region, and therefore make the convective region shallower. Three models of envelope structure of yellow-red supergiants are plotted in Figure 1 the solid lines according to the new formula of the MLT Eq. (5), the dotted lines, the corresponding old models. A comparison of the two sets of lines shows that the supersonic convection vanishes already when $c = 3$, and the relative fluctuations in temperature becomes $|T'/T| \leq 1$ everywhere in the convective zone.

As the consequence of the modification, we predict redder positions than does the original MLT for red supergiants. This improves our modelling of massive star evolution.

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