Self-similar orbit-averaged Fokker-Planck equation for isotropic spherical dense clusters (iii) Application to Galactic globular clusters

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Abstract Fitting parametric models to globular clusters’ structural profiles has been essential for the stellar-dynamics study. It provides their important structural parameters, such as the concentrations and core radii of the clusters. However, existing parametric models can apply only to non-collapsing-core clusters at the early relaxation-evolution stage. Hence, a single parametric model could not provide globular clusters’ structural parameters in both the early and late evolution stages. We have recently found an accurate spectral solution of the self-similar orbit-averaged Fokker-Planck (OAFP) equation to model collapsing-core clusters at the late evolution stage. The present work establishes a new parametric model by combining the self-similar OAFP- and polytropic- models. Although it is a single-mass and isotropic model, the new model applies to at least fifty-five Galactic globular clusters with resolved cores in all the evolution stages. As a main result, we show the characteristics of the relaxation times against the concentrations of the clusters. We also show that the structures of low-concentration clusters are polytropic in the Milky Way.

Key words: (Galaxy:) globular clusters: general — Galaxy: kinematics and dynamics — methods: numerical

1 INTRODUCTION

In the late relaxation-evolution stage, the standard stellar dynamics expects globular clusters to experience a self-similar evolution and core-collapse (e.g., Spitzer 1988; Heggie & Hut 2003; Binney & Tremaine 2011). The corresponding mathematical model is called the self-similar orbit-averaged Fokker-Planck (OAFP) equation (Heggie & Stevenson 1988). We recently found an accurate Gauss-Chebyshev spectral solution of the equation for isotropic, spherical clusters in the pre-collapse phase (Ito 2021). Based on the pre-collapse solution, the present paper proposes an energy-truncated self-similar OAFP model. It reasonably fits the projected structural profiles of Galactic globular clusters, including collapsing-core and collapsed-core clusters, with resolved cores. In the rest of the present section, we review the applicability of fitting models to Galactic globular clusters. We primarily focus on the most fundamental parametric fitting model (the King model (King 1966)) (Section 1.1) and time-dependent OAFP models (Section
1.2. We also explain why we may apply the pre-collapse solution of the ss-OAFP (self-similar-OAFP) equation to even the post-collapse clusters (Section 1.3).

1.1 The applicability of the King model to Galactic globular clusters

The King model (King 1966) is the most fundamental parametric fitting model for globular clusters’ structures. It is a single-component, isotropic, spherical cluster model. The model can reasonably apply to clusters in the early relaxation-evolution stage. The fitting of the King model depends on the three numerical parameters; the central projected density \( \Sigma_c \), core radius \( r_c \), and dimensionless central potential \( K = \varphi(r = 0)/\sigma_c \), where \( \varphi(r = 0) \) is the central cluster potential and \( \sigma_c \) the central velocity dispersion. Only with the three degrees of freedom, the King model adequately fits the structural profiles of approximately 80% of globular clusters in the Milky Way. The rest of the clusters are considered to be undergoing or have undergone a core-collapse at least once (Djorgovski & King 1986). If the King model well fits a globular cluster’s structural profile, then one conventionally calls the cluster a ‘normal’ or ‘King-model’ (KM) cluster, otherwise a ‘post-collapsed-core’ or ‘post-core-collapse’ (PCC) cluster. There are three differences between the KM- and PCC- clusters. (i) The projected structural profile of a typical KM cluster flattens in the core while a typical PCC cluster has a cusp (a power-law projected density profile) approximately \( r^{-1} \), where \( r \) is the distance from the center of the cluster. (ii) The concentrations \( c \) of PCC clusters are high (\( c \gtrsim 2 \)) while those of KM clusters are low (\( 0.7 \lesssim c \lesssim 1.8 \)) (See, e.g., Meylan & Heggie 1997). The concentration \( c \) of a cluster is a possible measure to describe the dynamical state of the cluster and defined by \( \log \left[ r_{\text{tid}}/r_c \right] \). (iii) The outer halos of PCC clusters are more anisotropic than those of KM clusters because of the cores’ relaxation processes. Realistic outer halos’ structures depend on not only the effect of the relaxation process but also the tidal effects from the Galaxy and other factors. On the one hand, numerical simulations (Giersz & Spurzem 1994; Takahashi 1995; Drukier et al. 1999) showed that the anisotropy could be less significant in the core and inner halo due to the shorter relaxation time. Hence, one may expect that even isotropic models, like the self-similar OAPF model, may reasonably apply to the central parts of globular clusters.

1.2 The applicabilities of the OAFP- and other models to PCC clusters

Time-dependent spherical OAFP models may fit PCC clusters’ structural profiles. However, it is not an easy task to self-consistently solve a time-dependent OAFP equation coupled with Poisson’s equation. Generally, one applies a time-dependent OAFP model to a certain globular cluster as a case study to examine the detailed structure. The examples of the clusters are NGC 6838 (Drukier et al. 1992), NGC 6397 (Drukier 1995), and NGC 7088 (Murphy et al. 2011). On the one hand, the King model builds only on the Poisson equation, easy to be solved for the potential of a spherical cluster. It has been preferably used in homogeneous surveys to capture the common properties (the characteristic sizes of clusters and dynamical states) of as many globular clusters as possible. The surveys typically neglect the detailed information of each cluster. The concentration and core- and tidal radii obtained from the King model have been the fundamental structural parameters in the compilation works for globular-cluster studies (e.g., Peterson & King 1975; Trager et al. 1995; Noyola & Gebhardt 2006; Miocchi et al. 2013; Merafina 2017) and in Harris catalog (Harris 1996, (2010 edition)). To the best of our knowledge, there does not exist a single-mass isotropic model only based on the Poisson

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1 One may refer to some models that focus on the outer halo, including realistic effects, such as anisotropy (Michie 1963; Meylan 1987), collisionless relaxation based on \( f_c \) model (de Vita et al. 2016), and the effect of escapers to discuss the elongated outer halos of some clusters (Claydon et al. 2019). Those models build on the modifications of isotropic models. Hence, similar modifications can readily apply to our new isotropic model (introduced in Section 2) as well.
equation that applies to both PCC- and KM- clusters due to their different core structures. For
PCC clusters, one has employed a modified power-law profile (e.g., Lugger et al. 1995; Ferraro
et al. 2003) or non-parametric model (e.g., Noyola & Gebhardt 2006). For KM clusters, one
has used the single-component King model, its variants (e.g., Woolley 1961; Wilson 1975), and
generalized models (e.g., Gomez-Leyton & Velazquez 2014; Gieles & Zocchi 2015). Although
the present work focuses on a single-mass model for simplicity, a multi-mass King model is
known to fit some PCC cluster (King et al. 1995). There is no strict argument that rules out
the multi-mass King model from a proper PCC cluster model (Meylan & Heggie 1997).

1.3 The relationship of the self-similar OAFP model with PCC clusters

We expect that the ss-OAFP model in the pre-collapse phase can fit PCC clusters with
resolved cores. In principle, the ss-OAFP model applies only to globular clusters at the moment
of complete core-collapse and approximately collapsing-core clusters in the late relaxation-
evolution stage (Heggie & Stevenson 1988; Ito 2021). Also, the ss-OAFP model itself is unreal-
istic. In actual clusters, the cores’ densities can reach high enough to form binaries from single
stars. The binaries release gravitational energy and halt the collapse before the cores develop an
infinite central density. After the core-collapse holds, time-dependent- and self-similar-conduc-
tive gaseous models predict that the clusters successively repeat a core expansion (due to the
energy released from binaries) and core-collapse (due to the relaxation process and self-gravity)
(Sugimoto & Bettwieser 1983; Bettwieser & Sugimoto 1984; Goodman 1984, 1987). This process
is called the ‘gravothermal oscillation’ in the post-core-collapse phase since it shows a nonlinear
oscillation of the core density with time. Time-dependent OAFP models (Cohn et al. 1989;
Murphy et al. 1990; Takahashi 1996) and N-body simulations (Makino 1996; Breen & Heggie
2012) also predict the same oscillation.

PCC clusters may form different structures in the post-collapse phase depending on binary
stars and total stellar numbers $N$. A time-dependent conductive gaseous model (Sugimoto &
Bettwieser 1983; Bettwieser & Sugimoto 1984) found that the core structure is similar to the
non-singular isothermal sphere (except at the moment of the core-collapse) in the post-collapse
phase. This feature contradicts the result of a self-similar gaseous model (Inagaki & Lynden-
Bell 1983) that revealed a central-cusp structure after a core-collapse. The latter result appears
proper to model PCC clusters. However, the formation of a cusp in the core is conditional.
Another self-similar gaseous model (Goodman 1984), including the mass-ejection effect from
the core, examined the evolution on longer time scales compared to (Inagaki & Lynden-Bell
1983). It also showed a cusp in the core. Moreover, the model clarified that the core radius gets
smaller with an increasing stellar number $N$. On the one hand, the same model as (Goodman
1984) showed that the gravothermal oscillation could occur if the cluster had enough stars in
it ($N \geq 7 \times 10^3$), and more efficient binary heating was employed (Goodman 1987). In other
words, efficient binary heating is necessary to produce a core like a non-singular isothermal
sphere. To avoid unrealistically small cores, efficient binary heating with primordial binaries
must also occur (Goodman & Hut 1989). These discussions infer that PCC clusters may have
various core structures, such as a non-singular isothermal core, resolved core with a cusp, and
unsolved core. Especially, if binary heating is efficient, the structural profiles of both the
gaseous model (Goodman 1987) and the OAFP model (Takahashi 1996) are similar in between
the post-collapse and pre-collapse phases. There is no way to differentiate the structural profiles
in the two phases only from observational data (Meylan & Heggie 1997) unless one acquires
accurate kinematic data to see the temperature inversion. Hence, the ss-OAFP model can

\[ \text{A distinct difference appears in the radial velocity dispersion profile between the OAFP models in the pre-}
\text{collapse and post-collapse phases (Sugimoto & Bettwieser 1983). When the core expands in the pose-collapse}
\text{phase, the temperature (velocity dispersion) increases with an increasing cluster radius around the cluster center.}
\text{This temperature gradient causes a heat flow inward toward the center. It cools down the center due to the}
\text{‘negative heat capacity’ of the cluster (See (Lynden-Bell 1999) for a review on negative heat capacity.) However,}
\]
model some PCC clusters with resolved cores. This idea motivated us to apply the (pre-collapse) ss-OAFP model to PCC clusters.

The present paper’s purpose is to establish a parametric model comparable with the King model. We propose an energy-truncated ss-OAFP model. It can apply to Galactic KM- and PCC- clusters with resolved cores reported in (Kron et al. 1984; Djorgovski & King 1986; Trager et al. 1995; Lugger et al. 1995; Drukier et al. 1993; Ferraro et al. 2003; Miocchi et al. 2013). Unfortunately, we did not have access to the data of (Djorgovski & King 1986; Lugger et al. 1995; Miocchi et al. 2013). Hence, we employed WebPlotDigitizer (Rohatgi, Ankit 2019) to extract the data points and uncertainties of the projected structural profiles from their figures. The present paper is organized as follows. Section 2 introduces the energy-truncated ss-OAFP model that we applied to Galactic globular clusters’ projected structural profiles. Section 3 explains the result of fitting the new model to PCC clusters. Section 4 shows the relationship between the completion rate of the core-collapse and concentration based on fitting our model to KM- and PCC- clusters. It also suggests that a polytropic model could fit low-concentration globular clusters in the Milky Way. Hence, Section 5 discusses whether the low-concentration clusters can have structures described by polytropic spheres of index $m$. Section 6 concludes this paper. For the sake of brevity, in Appendixes B and C, we show the majority of the projected structural profiles fitted by the energy-truncated ss-OAFP model.

2 ENERGY-TRUNCATED SS-OAFP MODEL

The present section introduces a new model, i.e., an energy-truncated ss-OAFP model. Section 2.1 shows the relationship between the ss-OAFP model and the isothermal sphere first, and then explains the motivation for truncating the energy domain of the ss-OAFP model. Section 2.2 details the new model. The new model does not depend on dimensionless central potential $K$, unlike the King model. Hence, Section 2.3 explains how to regularize the new model’s concentration and core radius so that the structural parameters are comparable with those of the King model. Also, our model is composed of a polytrope of $m$ and the ss-OAFP model. Section 2.4 explains how we found an optimal value of $m$.

2.1 The relationship between the ss-OAFP model and the isothermal sphere

The ss-OAFP model could model KM clusters since it has a flat core like the isothermal sphere. Figure 1 depicts that the ss-OAFP model’s core has almost the same density profile as that of the isothermal sphere. For the figure, we rescaled the ss-OAFP model’s radius by multiplying by 3.739. This scaling makes the two models’ cores approximately the same in size. This core structure infers that one may obtain a model similar to the King model at small radii by adequately truncating the energy domain of the ss-OAFP model. We discuss how to truncate the energy in Section 2.2.

kinematic surveys generally provide more significant uncertainty in velocity dispersion than structural profile data (Meylan & Heggie 1997). Hence, one can not easily determine whether a well-relaxed (or high-concentration) cluster is currently in the pre-collapse or post-collapse phases.

3 During preparing the present manuscript, we were informed by Mario Pasquato that the numeric data for (Miocchi et al. 2013) were available at http://www.cosmic-lab.eu/catalog/index.php.
2.2 Energy-truncated ss-OAFP model

We energy-truncate the ss-OAFP model so that the new model’s outer halo behaves like a polytrope of $m$. Hence, the model is phenomenological, unlike the King model (King 1966). The King model counts the effects of the escaping stars or imitates the tidal effect from Galaxy’s potential by truncating energies available to stars. Simple arguments have explained the model’s physical origin based on the ‘test particle’ method of kinetic theories. The method assumes that particles (stars) follow Maxwellian-Boltzmann DF (hereafter Maxwellian DF), while the test particle (star) does not have to. It explains that the King model is compatible with the isothermal sphere enclosed in a square well. Their relationship was studied for a stationary Fokker-Planck model (Spitzer & Harm 1958; Michie 1962; King 1965) and OAFP model (Spitzer & Shapiro 1972). Also, the stellar DF for the King model is proportional to $E$ as $E \to 0$. This feature may be related to the asymptotic stationary solution of the OAFP model with a constant stellar flux at the fringe (Spitzer & Shapiro 1972). To obtain the King model (or lowered-Maxwellian DF), one must subtract the DF for polytrope of $m = 2.5$ from the Maxwellian DF

\[
F(E) = \begin{cases} 
\frac{\rho_e}{(2\pi\sigma_e^2)^{3/2}I(K)} \left( \exp \left[ -\frac{E}{\sigma_e^2} \right] - 1 \right), & (E < 0) \\
0, & (E > 0)
\end{cases}
\]

\[
I(K) = \exp(K) \text{erf} \left( \sqrt{K} \right) - \left( 1 + \frac{2K}{3} \right) \sqrt{\frac{4K}{\pi}}
\]

(2.1a)

(2.1b)

where the parameter $K$ and the error function $\text{erf}(x)$ read

\[
K = -\frac{\varphi(r = 0)}{\sigma_e^2},
\]

\[
\text{erf}(x) = \frac{2}{\sqrt{\pi}} \exp(x^2) \int_0^x \exp(-y^2) \, dy.
\]

(2.2a)

(2.2b)

A prescribed DF $F(E)$ provides the mean-field (MF) potential $\varphi(r)$ through the Poisson’s equation

\[
d^2 \varphi \over dr^2 + \frac{2 d\varphi}{r \, dr} = 4\pi GD(\varphi) \equiv 16\pi^2 G \int_{E(r)}^{0} F(E) \sqrt{2[E - \varphi(r)]} \, dE',
\]

(2.3)
where $G$ is the gravitational constant.

Our new model incorporates the escaping stars’ effect in the same way as the King model by truncating high energies available to stars. However, the mathematical operation for combining DFs differs from that of the King model. Our model adds the DFs up for a polytrope of $m$ and the ss-OAFP model as follows

$$F(E) \equiv \begin{cases} 
\rho_c & (E > 0) \\
\frac{F_o(E) + \delta (-E)^{m-3/2}}{4\sqrt{2\pi}\sigma_0^3 D_o(\varphi = -1) + \delta B(m - 1/2, 3/2)} & (E < 0)
\end{cases}$$

(2.4)

where $\delta$ and $m$ are positive real numbers, $D_o(\varphi)$ is the density of the ss-OAFP model, and $B(a, b)$ is the beta function defined as $B(a, b) = 2 \int_0^1 t^{2a-1} (1 - t^2)^{b-1} \, dt$ with $a > 1/2$ and $b > 1$. The factor $(D_o(\varphi = -1) + \delta B(m - 1/2, 3/2))^{-1}$ is inserted in the DF so that the density profile for the DF $F(E)$ has a certain central density $\rho_c$ as $R \to 0$. In equation (2.4), $F_o(E)$ is the DF for the ss-OAFP model and proportional to $(-E)^{8.178}$ as $E \to 0$ (Ito 2021). On the one hand, we limit $m$ to be less than 5, which results in $F(E) \propto (-E)^{m-3/2}$ as $E \to 0$. Our model can be finite in size by this limit, following the discussion of (Chandrasekhar 1939). Hence, a polytrope of $m$ controls the model’s outer halo. Also, the new DF (equation (2.4)) behaves like the ss-OAFP model’s DF beyond the order of $\delta$, while it is like approximately a polytropic sphere’s DF below $\delta$.

We employ a polytrope of $m$ to truncate the energy in equation (2.4) since a simple argument can not generally provide an explicit form of energy truncation. The tidal effect on globular clusters significantly depends on where they orbit in the Milky Way. Also, more realistic effects (mass spectrum, collisionless relaxation, and dark matter) could have affected the current outer halos’ structures. Hence, one can not universally determine their structures. For example, the King model does not reasonably fit all the KM clusters’ outer halos. The Wilson model (Wilson 1975) and Woolley model (Woolley 1961) can fit some clusters’ structural profiles better than the King model. The Wilson model’s outer halo behaves like the polytrope of $m = 3.5$, while that of the Woolley model is approximately the polytrope of $m = 1.5$ in the limit of $K \to 0$. Using more different values of $m$ provides a more flexible fitting to outer halos. This generalization was carried out in the truncated $\gamma$ exponential (fractional-power) model proposed in (Gomez-Leyton & Velazquez 2014). Our model has the same parameter-dependence as the model. Of course, our purpose is to build a parametric model comparable with the King model. Hence, we must first specify the value of $m$ for our model based on physical arguments (with numerical experiments) and observational data. In the present work, we carried out only the latter process (Section 2.4). In this sense, we consider our model to be phenomenological.

The rest of the present section shows the numerically calculated density profile, MF potential, and projected density profile of the energy-truncated ss-OAFP model. One can analytically derive the explicit form of the density profile of polytropes. Hence the density profile of our new truncated model reads

$$D(\varphi) = \rho_c \frac{D_o(\varphi) + \delta B(m - 1/2, 3/2) (-\varphi)^m}{D_o(\varphi = -1) + \delta B(m - 1/2, 3/2)}.$$  

(2.5)

The density of the ss-OAFP model in a Chebyshev-expansion form was obtained from (Ito 2021) as follows

$$D_o(\varphi) = (-\varphi)^{\beta+3/2} \sum_{n=1}^{63} D_n T_{n-1}(\varphi),$$  

(2.6a)

$$\beta = 8.1783711596581004.$$  

(2.6b)
where $T_n$ is the Chebyshev polynomials of the first kind. The numerical values of the Chebyshev coefficients $\{D_n\}$ are given in Appendix A. The dimensionless form of the Poisson’s equation (2.3) reads

$$
\frac{\partial^2 \phi}{\partial r^2} + \frac{2}{r} \frac{\partial \phi}{\partial r} = \tilde{D}(\phi) = \frac{D_o(\phi) + \delta B(m - 1/2, 3/2)(-\phi)^m}{D_o(\phi = -1) + \delta B(m - 1/2, 3/2)},
$$

(2.7)

where the potential $\phi(r)$, radius $r$, and density $\rho(r)$ are made in dimensionless form using

$$
\tilde{\phi} = \frac{\phi(r)}{\sigma_c^2},
$$

(2.8a)

$$
\tilde{r} = r \sqrt{\frac{4\pi G \rho_c}{\sigma_c^2}},
$$

(2.8b)

$$
\tilde{D}(r) = \frac{D(r)}{\rho_c},
$$

(2.8c)

where the variables with subscript $c$ are corresponding to the time-dependent variables in the self-similar analysis (Heggie & Stevenson 1988; Ito 2021). The boundary conditions for the Poisson’s equation (2.7) are

$$
\tilde{\phi}(\tilde{r} = 0) = 1, \quad \frac{d \tilde{\phi}}{d \tilde{r}}(\tilde{r} = 0) = 0.
$$

(2.9)

The dimensionless potential $\tilde{\phi}$ is an independent variable in the ss-OAFP model (Heggie & Stevenson 1988; Ito 2021). Hence, following the method of inverse mapping (Ito et al. 2018; Ito 2021), we must solve the following Poisson’s equation in its inverse form for $R(\tilde{\phi})$

$$
R \frac{d^2 R}{d \tilde{\phi}^2} - 2 \left( \frac{dR}{d\tilde{\phi}} \right)^2 = \left( \frac{dR}{d\tilde{\phi}} \right)^3 \frac{D_o(\tilde{\phi}) + \delta B(m - 1/2, 3/2)(-\tilde{\phi})^m}{D_o(\tilde{\phi} = -1) + \delta B(m - 1/2, 3/2)}.
$$

(2.10)

The numerical integration of the Poisson’s equation provided the density profile (Figure 2) and MF potential (Figure 3) for an optimal index $m = 3.9$. (We explain the reason why $m = 3.9$ is an optimal choice in Section 2.4.) In the figures, the value of $\delta$ spans $10^{-5}$ through $10^3$. For large $\delta (\gg 1)$, the profiles appear to be the same regardless of the value of $\delta$. They behave like the polytrope of $m = 3.9$. On the one hand, the profile is similar to the ss-OAFP model for small $\delta (\lesssim 10^{-2})$. 


In applying the density $D(\phi)$ to globular clusters, one needs to convert $D(\phi)$ to the projected density profile

$$\Sigma(r) = 2 \int_0^\infty \frac{D(\phi)}{\sqrt{1 - (r/r')^2}} \, dr'. \quad (2.11)$$

The corresponding inverse form with dimensionless variables is

$$\Sigma(\phi) = -2 \int_0^2 \sqrt{\frac{1 - \mu_R(\phi, \phi')}{1 + \mu_R(\phi, \phi')}} \left[ -2 \frac{dD}{d\phi'} R(\phi') + D(\phi') S(\phi') \frac{1}{1 + \mu_R(\phi, \phi')} \right] d\phi', \quad (2.12)$$
where $\mu_R(\varphi, \varphi') \equiv R(\varphi)/R(\varphi')$ and $S \equiv -dR/d\varphi$. Figure 4 depicts the projected density profiles for different $\delta$. As $\delta$ decreases, the power-law profile $R^{-1.23}$ develops more clearly in the inner halo (as expected from the asymptotic density profile of the ss-OAFP model, i.e., $D \propto R^{-2.23}$ as $R \to \infty$). This power-law profile appears at radii between $R \sim 10$ and $R \sim 100$ for $\delta = 10^{-4}$. In addition, one can find a similar power-law profile for larger $\delta$. For $\delta = 10^{-2}$ and $10^{-3}$, $\Sigma$ shows power-law-like structures $R^{-1.0} \sim R^{-1.1}$ at radii between $R \sim 1$ and $R \sim 10$. This property is desirable to fit our model to PCC clusters’ structural profiles. PCC clusters with resolved cores have similar power-law profiles near the core (e.g., Djorgovski & King 1986; Lugger et al. 1995).

Fig. 4: Projected density $\Sigma$ of the energy-truncated ss-OAFP model for different $\delta$. The power-law profile $R^{-1.23}$ corresponds to the asymptotic approximation of the ss-OAFP model as $R \to \infty$.

2.3 Regularization for the concentration and King radius

The energy-truncated ss-OAFP model differs from the King model in the sense of how concentrations and core radii depend on the dimensionless central potential $K$. We must properly regularize the structural parameters for the sake of comparison. By plugging the lowered-Maxwellian DF (equation (2.1)) in the Poisson’s equation (2.3) and employing the dimensionless variables (equations (2.8a) - (2.8c)) with new variable $\hat{\varphi} \equiv \varphi/K$, one can reduce the King model to

$$\frac{d^2\hat{\varphi}}{d\hat{r}^2} + \frac{2}{\hat{r}} \frac{d\hat{\varphi}}{d\hat{r}} - \frac{1}{K} \frac{I(\hat{\varphi})}{I(1)} = 0,$$

with the boundary conditions $\hat{\varphi}(\hat{r} = 0) = 1$ and $\frac{d\hat{\varphi}}{d\hat{r}}(\hat{r} = 0) = 0$. Due to the $K$-dependence of the equation, $c \to 0$ as $K \to 0$. Of course, one can make equation (2.13) independent of $K$ by further regularizing the radius $\hat{r}$ as

$$\hat{r} = \hat{r}/\sqrt{K}.$$ 

Then, the King model’s tidal radius is equal to that of polytrope of $m = 2.5$, that is, 5.355275459 (e.g., Boyd 2011) regardless of the value of $K$. On the one hand, the energy-truncated ss-OAFP model does not depend on $K$ since the dimensionless central potential $\hat{\varphi}(\hat{r} = 0)$ is unity.
(equation (2.9)), unlike the King model. One may like to find the same order of the structural parameters as the King model. To do so, one can regularize the structural parameters as follows

\[
\begin{align}
\tilde{r}_{\text{Kin}} & = \frac{r_{\text{Kin}}}{\sqrt{K^{(m)}}}, \\
\tilde{c} & = \log \left[ \frac{r_{\text{tid}}}{r_{\text{Kin}} \sqrt{K^{(m)}}} \right],
\end{align}
\]

where \( K^{(m)} \) is the central potential, \( r_{\text{Kin}} \) the King (core) radius, and \( r_{\text{tid}} \) the tidal radius of the energy-truncated ss-OAFP model. Equation (2.15) reduces equation (2.7) to

\[\frac{d^2 \varphi}{d\tilde{r}^2} + \frac{2}{\tilde{r}} \frac{d \varphi}{d\tilde{r}} - \frac{D(\varphi)}{K^{(m)}} = 0, \quad (2.16)\]

which has the same \( K \)-dependence as equation (2.13).

Using equation (2.15), one can obtain the concentration of the energy-truncated ss-OAFP model. Strictly speaking, the value of \( K^{(m)} \) depends on both \( m \) and \( \delta \) since we would like to determine the value of \( K^{(m)} \) so that the ss-OAFP model’s structural parameters are the same order as those of the King model. To reduce the complexity of determining \( K^{(m)} \), we resort to the property that our model’s outer halo builds on a polytrope. We consider our model’s size to be approximately equal to that of the polytrope of \( m \). For a specific value of \( m \), we can calculate the tidal radius \( r_{\text{tid}}^{(\text{poly})} \) of the polytrope through the Lane-Emden equation of the first kind (Chandrasekhar 1939)

\[\frac{d^2 \varphi}{d\tilde{r}^2} + \frac{2}{\tilde{r}} \frac{d \varphi}{d\tilde{r}} - \varphi^m = 0, \quad (2.17)\]

with the boundary conditions (equation (2.9)). Then, we must solve the King model (equation (2.13)) and find the value of \( K \) so that the tidal radius of the King model, \( r_{\text{tid}}^{(\text{King})} \), is approximately equal to \( r_{\text{tid}}^{(\text{poly})} \). For example, if one chooses \( m = 3.9 \), then the tidal radius of the polytrope of \( m = 3.9 \) is \( r_{\text{tid}}^{(\text{poly})} = 13.4731 \). In this case, one must determine \( K \) of the King model so that \( r_{\text{tid}}^{(\text{King})}/\sqrt{K} \) is close to 13.4731. This value can be achieved when \( K = 4.82 \) with \( r_{\text{tid}}^{(\text{King})}/\sqrt{K} = 13.444 \). Hence, \( K^{(m)} = 4.82 \) for \( m = 3.9 \). We show in Section 2.4 that the concentrations calculated by this scaling are reasonably close to those of the King model.

Figure 5 depicts the concentration \( \tilde{c} \) for \( m = 3.9 \). As \( \delta \) increases, the concentration approaches that of the polytrope of \( m = 3.9 \). Our primary focus is \( \delta \lesssim c_4^*(\equiv 0.3032) \) with which the energy-truncated ss-OAFP model approaches the ss-OAFP model and differentiates itself from the isothermal sphere and King model. The corresponding concentration is \( \tilde{c} \lesssim 1.45 \). On the one hand, our model would behave like the King model for \( 1 < \tilde{c} \lesssim 1.45 \) and like the polytrope of \( m = 3.9 \) in the limit of \( \tilde{c} \to 1 \).

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4 One should not regularize the tidal radius of the energy-truncated ss-OAFP model since it is still the radius at which the projected density reaches zero numerically or on a graph. After adequately fitting the model to a globular cluster’s projected structural profile, one can find the tidal radius.

5 Following the mathematical form of equation (2.14), one must regularize equation (2.17) so that it has the same \( K \)-dependence as equation (2.16).

6 One can obtain the value of \( c_4^* \) in the DF \( F_0(E) \) for the ss-OAFP model (Ito 2021). In the limit of \( E \to 0 \), \( F_0(E) = c_4^*(-E)^{1/2} \). Hence, one can approximately characterize the structure of the energy-truncated ss-OAFP model by comparing \( c_4^* \) with \( \delta \) in equation (2.4).
Fig. 5: Concentration of the energy-truncated ss-OAFP model. The horizontally dashed line represents the concentration of the polytropic sphere of \( m = 3.9 \).

2.4 An optimal value of the polytropic index: \( m = 3.9 \)

We determined the index \( m \) to be 3.9 in the energy-truncated ss-OAFP model after having preliminarily applied the model to the projected surface densities of six KM clusters and a PCC cluster that we chose. Initially, we expected that \( m = 2.5 \) could be an optimal choice for the energy-truncated ss-OAFP model following (Spitzer & Shapiro 1972), though it was not the case. The values in \( 3.5 \leq m \leq 4.4 \) were useful. Among the values of \( m \), we chose 3.9 as the optimal value in the present work. It provided the same order of the structural parameters as that of the existing works based on the King model. Table 1 shows the results obtained by fitting our model with \( m = 3.9 \) to the projected surface densities of six Milky-Way globular clusters reported in (Kron et al. 1984). Only the six clusters were reported in all the compilation works based on the King model (Peterson & King 1975; Kron et al. 1984; Chernoff & Djorgovski 1989; Trager et al. 1993; Miocchi et al. 2013) that we chose to compare this time. Many of the compilation works are inhomogeneous surveys in the sense that they depend on different instruments, photometry methods, and statistical analyses. However, the structural parameters obtained from our model are reasonably close to their results. On the one hand, when we chose \( m \leq 3.8 \) or \( 4.3 \leq m \) for our model’s fitting, the structural parameters’ orders are approximately ten times less or greater than those of the compilation works. Interestingly, our tidal radii are close to those obtained from the King model rather than the Wilson model (Table 1). The Wilson model’s index \( m \) is 3.5 in the limit of \( K \to 0 \) and greater than that of the King model. A higher index \( m \) provides a larger tidal radius of the polytropic sphere (e.g., Chandrasekhar 1939). The reason why our model with the high \( m (= 3.9) \) does not overestimate the tidal radius would be that the density profile of the ss-OAFP model more rapidly decays compared to the isothermal sphere in the inner halo (Figure 1). Refer to Appendix B, to find the energy-truncated ss-OAFP model with \( m = 3.9 \) reasonably fitted to the projected structural profiles of thirty-nine KM clusters reported in (Kron et al. 1984; Miocchi et al. 2013).

Another reason why we chose \( m = 3.9 \) is that the energy-truncated ss-OAFP model with \( m = 3.9 \) agreeably fits the relatively new data for NGC 6752 reported in (Ferraro et al. 2003). The result of (Ferraro et al. 2003) provided data and errors of the projected surface density for NGC 6752. Their data were convenient for us to test our model since we did not need to artificially extract data from their graph. The King model does not well fit the central part of the projected density profile in their work since the cluster is one of the (possible) PCC clusters with a power-law profile near the core. Hence, following (Lugger et al. 1995), they employed modified power-law profile \( \Sigma \propto (1 + (R/3.1)^2)^{-0.525} \), where \( R \) is measured in log [arcsec]. This profile well fits the central part, as shown in Figure 6 (left top). On the one hand, our model
Table 1: Concentrations and core- and tidal- radii obtained by fitting the energy-truncated ss-OAFP model to six KM clusters. The structural parameters are compared to the previous compilation works based on the King- and Wilson- models.

<table>
<thead>
<tr>
<th>Cluster</th>
<th>NGC 1904</th>
<th>NGC 2419</th>
<th>NGC 6205</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>c</td>
<td>r_c</td>
<td>r_tid</td>
</tr>
<tr>
<td>Energy-truncate ss-OAFP</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The present work based on data of (Kron et al. 1984)</td>
<td>1.86</td>
<td>0.191</td>
<td>13.9</td>
</tr>
<tr>
<td>King model</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Miocchi et al. 2013)</td>
<td>1.76</td>
<td>0.15</td>
<td>9.32</td>
</tr>
<tr>
<td>(Harris 1996)</td>
<td>1.70</td>
<td>0.16</td>
<td>8.0</td>
</tr>
<tr>
<td>(Trager et al. 1993)</td>
<td>1.72</td>
<td>0.159</td>
<td>8.35</td>
</tr>
<tr>
<td>(Chernoff &amp; Djorgovski 1989)</td>
<td>1.90</td>
<td>0.132</td>
<td>10.5</td>
</tr>
<tr>
<td>(Kron et al. 1984)</td>
<td>1.75</td>
<td>0.178</td>
<td>10.0</td>
</tr>
<tr>
<td>(Peterson &amp; King 1975)</td>
<td>1.60</td>
<td>0.27</td>
<td>10.7</td>
</tr>
<tr>
<td>Wilson model</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Miocchi et al. 2013)</td>
<td>2.14</td>
<td>0.18</td>
<td>28</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Cluster</th>
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<td></td>
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<td>Energy-truncate ss-OAFP</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>The present work based on data of (Kron et al. 1984)</td>
<td>1.45</td>
<td>0.178</td>
<td>5.00</td>
</tr>
<tr>
<td>King model</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Miocchi et al. 2013)</td>
<td>1.65</td>
<td>0.13</td>
<td>6.12</td>
</tr>
<tr>
<td>(Harris 1996)</td>
<td>1.50</td>
<td>0.12</td>
<td>3.79</td>
</tr>
<tr>
<td>(Trager et al. 1993)</td>
<td>1.61</td>
<td>0.13</td>
<td>5.39</td>
</tr>
<tr>
<td>(Chernoff &amp; Djorgovski 1989)</td>
<td>1.40</td>
<td>0.167</td>
<td>4.19</td>
</tr>
<tr>
<td>(Kron et al. 1984)</td>
<td>1.25</td>
<td>0.173</td>
<td>3.08</td>
</tr>
<tr>
<td>(Peterson &amp; King 1975)</td>
<td>1.41</td>
<td>0.22</td>
<td>5.62</td>
</tr>
<tr>
<td>Wilson model</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(Miocchi et al. 2013)</td>
<td>1.82</td>
<td>0.16</td>
<td>12.0</td>
</tr>
</tbody>
</table>

with $m = 3.0$ does not match the cluster’s profile at all on the figure. However, our model more reasonably fit the same data with a greater $m$. The model with $m = 4.2$ best fits the data except in the tail of the cluster. Even with $m = 3.9$, one can find a good fit to the data. Hence, the present work chose $m = 3.9$ as the optimal value to consistently accumulate the data for both KM clusters and PCC clusters.
Fig. 6: Fitting of the energy-truncated ss-OAFP model to the projected density profile of NGC 6752 (Ferraro et al. 2003) for different $m$. The unit of $\Sigma$ is stellar number per arcminute squared, and $\Sigma$ is normalized so that the density is unity at the smallest radius of data points. In the legends, (c) means a PCC cluster as judged so in (Ferraro et al. 2003). In the left top panel, the modified power-law profile is depicted as done in (Ferraro et al. 2003). In the two bottom panels, $\Delta \log[\Sigma]$ for $m = 3.9$ and $m = 4.2$ depicts the corresponding deviation of log[\Sigma] from our model.

3 FITTING OF THE SS-OAFP MODEL TO PCC CLUSTERS

The energy-truncated ss-OAFP model with $m = 3.9$ reasonably fits the projected structural profiles of PCC clusters with resolved cores at $R \leq 1$ arcminute. Other than NGC 6752, we also had access to the numerical data of the projected density profile for NGC 6397 from (Drukier 1995). Our model reasonably fits the density profile of NGC 6397 with $\chi^2_p = 1.52$ (Figure 7)
where the reduced chi-square is defined as follows

$$
\chi^2 = \sum \frac{\chi^2}{n_{d.f.}},
$$

(3.1)

where $\chi^2$ is the chi-square value between the observed data and our model, and $n_{d.f.}$ is the degree of freedom. We chose $n_{d.f.} = 3$ in the same way as the King model since we fixed index $m$ of our model to 3.90. The fitting parameters of our model are $\Sigma_c$, $r_c$, and $\delta$ only. Unfortunately, we did not have access to numeric values for the rest of the projected structural profiles of PCC clusters reported in (Djorgovski & King 1986; Lugger et al. 1995). Hence, our error analysis becomes less trustworthy hereafter. However, it appears enough to capture the applicability of our model to the PCC clusters. For example, a review work (Meylan & Heggie 1997) introduced NGC 6388 and Terzan 2 as examples of a KM cluster and PCC cluster by citing the clusters’ surface brightness profiles from (Djorgovski & King 1986). The energy-truncated ss-OAFP model reasonably fits both the density profiles at radii $R \lesssim 1$ arcminute (Figure 8).

In addition to NGC 6752 and NGC 6397, we applied our model to thirteen PCC clusters with resolved and unresolved cores reported in (Djorgovski & King 1986; Lugger et al. 1995) (See Appendix C for the results).

Table 2 shows the values of $\chi^2$ for both the KM- and PCC- clusters for which we could obtain the uncertainties in the observed densities from the numeric values or graphs. The result shows that the ss-OAFP model adequately fits the KM clusters’ profiles at all the data points given. On the one hand, the model fits only the PCC clusters with resolved cores reported in (Lugger et al. 1995). For example, our model fits PCC clusters with partially-resolved cores (NGC 6453, NGC 6522, and NGC 7099) and resolved cores (NGC 6397 and NGC 6752) at $R \lesssim 1$ arcminute with $\chi^2 \lesssim 2$. It reasonably applies to even a PCC cluster with unresolved core (NGC 6342) similarly, though the present work does not count the ‘seeing-effect’ that comes from the seeing-disk’s finiteness. However, the model does not fit the rest of PCC clusters’ structural profiles with unresolved cores (e.g., NGC 5946 and NGC 6624). It does not fit even only the observed cores since the cores have steeper power-law profiles than our model. This failure was expected since our model does not correctly count the binaries’ effect, as explained in Section 1.3. Also, we do not assess that our (single-component) model is the best model for PCC clusters. PCC clusters are supposed to have experienced the mass segregation. Hence, we need to incorporate the mass spectrum’s effect in our model and would be able to improve the fitting, as done for the King model (Costa & Freeman 1976).

---

For the fitting of our model, we first determined the value of $\delta$. We then numerically integrated equation (2.16) with the fixed $\delta$. We fitted the computed projected-density profile to the observed data by adjusting the values of $\Sigma_c$ and $r_c$ so that $\chi^2$-value reached its minimum for the fixed $\delta$. We repeated these steps for different $\delta$ until we found the minimum of $\chi^2$ for any $\delta$. 
Energy-truncated self-similar OAFP model

Fig. 7: Fitting of the energy-truncated ss-OAFP model ($m = 3.9$) to the projected surface density of NGC 6397 reported in (Drukier et al. 1993). The unit of $\Sigma$ is stellar number per arcminute squared, and $\Sigma$ is normalized so that the density is unity at the smallest radius for data. In the legends, (c) means a PCC cluster as judged so in (Djorgovski & King 1986). $\Delta \log \Sigma$ is the corresponding deviation of $\Sigma$ from the model on the log scale.

Fig. 8: Fitting of the energy-truncated ss-OAFP model ($m = 3.9$) to the surface brightness profiles of Terzan 2 and NGC 6388 reported in (Djorgovski & King 1986). The unit of the surface brightness (SB) is B magnitude per arcsecond squared. The brightness is normalized by the magnitude $SB_o$ observed at the smallest radius point. In the legends, (n) means a normal or KM cluster and (c) means a PCC cluster as judged so in (Djorgovski & King 1986). $\Delta (SB_o - SB)$ is the corresponding deviation of $(SB_o - SB)$ from the model.
Table 2: Values of $\chi^2_{2}$ between the energy-truncated ss-OAFP model and observed structural data for Galactic globular clusters. The data we used for the KM clusters are from (Miocchi et al. 2013), for NGC 6397 from (Drukier et al. 1993), for Terzan 2 from (Djorgovski & King 1986), for NGC 6752 from (Ferraro et al. 2003), and for the rest of the PCC clusters from (Lugger et al. 1995). $N_b$ is the number of data points at large radii excluded from the calculation.

<table>
<thead>
<tr>
<th>KM cluster</th>
<th>$\chi^2_{2}$</th>
<th>$N_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NGC 288</td>
<td>0.45</td>
<td>0</td>
</tr>
<tr>
<td>NGC 1851</td>
<td>0.56</td>
<td>0</td>
</tr>
<tr>
<td>NGC 5466</td>
<td>2.07</td>
<td>0</td>
</tr>
<tr>
<td>NGC 6121</td>
<td>0.72</td>
<td>0</td>
</tr>
<tr>
<td>NGC 6205</td>
<td>1.05</td>
<td>0</td>
</tr>
<tr>
<td>NGC 6254</td>
<td>0.57</td>
<td>0</td>
</tr>
<tr>
<td>NGC 6626</td>
<td>0.47</td>
<td>0</td>
</tr>
<tr>
<td>NGC 6809</td>
<td>0.44</td>
<td>0</td>
</tr>
<tr>
<td>Pal 3</td>
<td>0.06</td>
<td>0</td>
</tr>
<tr>
<td>Pal 4</td>
<td>0.34</td>
<td>0</td>
</tr>
<tr>
<td>Pal 14</td>
<td>0.31</td>
<td>0</td>
</tr>
<tr>
<td>Trz 5</td>
<td>2.23</td>
<td>0</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>PCC cluster</th>
<th>$\chi^2_{2}$</th>
<th>$N_b$</th>
</tr>
</thead>
<tbody>
<tr>
<td>NGC 6342</td>
<td>1.73</td>
<td>3</td>
</tr>
<tr>
<td>NGC 6397</td>
<td>1.52</td>
<td>0</td>
</tr>
<tr>
<td>NGC 6453</td>
<td>1.89</td>
<td>5</td>
</tr>
<tr>
<td>NGC 6522</td>
<td>2.52</td>
<td>5</td>
</tr>
<tr>
<td>NGC 6558</td>
<td>2.17</td>
<td>5</td>
</tr>
<tr>
<td>NGC 6752</td>
<td>2.00</td>
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<tr>
<td>NGC 7099</td>
<td>2.12</td>
<td>2</td>
</tr>
<tr>
<td>Trz 1</td>
<td>2.41</td>
<td>5</td>
</tr>
<tr>
<td>Trz 2</td>
<td>1.94</td>
<td>0</td>
</tr>
<tr>
<td>NGC 5946</td>
<td>6.75</td>
<td>5</td>
</tr>
<tr>
<td>NGC 6624</td>
<td>7.18</td>
<td>5</td>
</tr>
</tbody>
</table>

4 MAIN RESULT: RELAXATION TIME AND COMPLETION RATE OF CORE COLLAPSE AGAINST CONCENTRATION

Concentration $\bar{c}$ is a possible measure to characterize the globular clusters' states in the relaxation evolution, especially for the cores. The present section compares $\bar{c}$ to the core relaxation time and completion rate of core-collapse. The energy-truncated ss-OAFP model can reasonably apply not only to KM clusters (Appendix B) but also to PCC clusters (Appendix C). One may systematically discuss their relationship. We first discuss how the core relaxation time depends on the concentration. Figure 9(a) depicts the characteristics of the core relaxation time $t_{c,r}$ against the concentration $\bar{c}$. Figure 9(b) depicts the corresponding characteristics based on the King model reported in the Harris catalog (Harris 1996, (2010 edition)). All the relaxation times on both the figures are the values reported for PCC- and KM- clusters in the catalog (Harris 1996, (2010 edition)). The catalog depicts some concentrations as ‘2.50c’ for clusters whose projected structural profiles are not well fitted by the King model. Hence, we assumed that the concentrations of the clusters are 2.50 in Figure 9 (b). In the figure, the relaxation time decreases with increasing concentration $c$ for KM clusters. However, it is not clear if the PCC clusters have the same tendency. On the one hand, Figure 9(a) shows not only that the relaxation time decreases with concentration $\bar{c}$ for KM clusters but also that the time drops down almost vertically for PCC clusters when the relaxation time is long. This tendency well captures the feature of PCC clusters. Their projected profiles can be close to the ss-OAFP model near the moment of a core-collapse. They are still similar to the King model (KM clusters) in the expansion phase after a core-collapse.
Fig. 9: Core relaxation time against (a) concentration $\tilde{c}$ obtained from the energy-truncated ss-OAFP model ($m = 3.9$) and (b) concentration $c$ based on the King model reported in (Harris 1996, (2010 edition)).

We discuss compare our and the King model using the completion rate $\eta_c$ of core-collapse. To find the value of $\eta_c$, we use the formula employed in (Lightman 1982)

$$\eta_c \equiv \frac{t_{\text{age}}}{t_{\text{c.r.o}}} = \frac{-(1 + Aq_o) + \sqrt{(1 + Aq_o)^2 + 4ABq_o^2}}{2B},$$

where $A = 35$, $B = 4.8$, and $q_o = t_{\text{age}}/t_{\text{c.r.o}}$. The time $t_{\text{c.r.o}}$ is the estimated relaxation time at the beginning of each cluster’s relaxation evolution based on an $N$-body simulation. Figure 10 (a) shows the completion rate against concentration $\tilde{c}$ obtained from the energy-truncated ss-OAFP model. The majority of data plots lie within the region between two lines $\eta_c = 0.75(\tilde{c} - 2.0) + 1.05$ and $\eta_c = 0.75(\tilde{c} - 2.0) + 0.40$ that are empirical lines of equation, not based on physical arguments. Figure 10 (b) shows that the corresponding characteristics of $\eta_c$ against concentration $c$ based on the King model, and the same two lines reasonably include most data plots between them. The horizontal difference between the two lines is approximately unity in both Figures 10 (a) and (b). Recalling the logarithmic dependence of the concentration, we assess that our model can provide the same order of structural parameters as the King model.

From Figure 10 (a), we can find several results. (i) Clusters with $c > 2.0$ are PCC clusters, as explained in (Meylan & Heggie 1997), if the completion rate is above 0.8. (ii) Clusters with

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8 The ages of most Galactic globular clusters are available from recent surveys (e.g., Santos & Piatti 2004; Forbes & Bridges 2010; VandenBerg et al. 2013). However, the ages of some clusters (e.g., Terzan 5) are not available in the Literature. Also, our focus is to figure out the difference and similarity between our and the King models. Hence, using the ages’ order $t_{\text{age}}$ for both models does not ruin our purpose.
over a completion rate of 0.8 are PCC- or possible PCC- clusters except for a cluster NGC 6517. (iii) KM clusters with high concentrations \( c \geq 2.0 \) reported in (Harris 1996, (2010 edition)) are reasonably close to the other KM clusters on the figure. Their concentrations \( \tilde{c} \) are lowered to values smaller than 2.0 based on our model. Our model suggests that two KM clusters (NGC 1851 and NGC 6626) have high concentrations \( \tilde{c} \geq 2.0 \), and their profiles are close to the ss-OAFP model. (iv) A PCC cluster (NGC 6544) differentiates itself from the KM clusters and the rest of the PCC clusters. NGC 6544 has a high completion rate (0.989) compared to the rest of the KM clusters and a low concentration \( \tilde{c} = 1.61 \) compared to the rest of PCC clusters. Hence, the cluster may be a good candidate for a PCC cluster that may have one of the most expanded cores, though it was judged only as a ‘possible PCC’ in (Djorgovski & King 1986). (v) Our model with \( \tilde{c} \approx 1 \) fits the projected structural profiles of low-concentration clusters. This result infers that the clusters may have structures similar to the polytrope of \( m \approx 3.9 \).

The result (i) just confirmed an expected property of the core-collapse process, and the results (ii) through (iv) require a detailed case study for each cluster to deepen our understanding of the results. The discussion of the latter results are out of the scope of the present work. Hence, we further discuss only the result (v) in Section 5.

Fig. 10: Completion rate of core-collapse against (a) concentration \( \tilde{c} \) based on the energy-truncated ss-OAFP model with \( m = 3.9 \) and (b) concentration \( c \) based on the King model reported in (Harris 1996, (2010 edition)).
5 DISCUSSION: ARE LOW-CONCENTRATION GLOBULAR CLUSTERS LIKE SPHERICAL POLYTROPES?

The present section discusses the relationship between the polytropic sphere and low-concentration ($c \approx 1$) globular clusters. The result of Section 4 shows that some clusters (e.g., Palomar 3 and Palomar 4) have concentrations $c$ close to one. This result indicates that the polytropic spheres of $m \approx 3.9$ can model the low-concentration clusters’ structural profiles. Our interest in the present section is to see whether the projected structural profiles of low-concentration globular clusters are polytropic. We computed the density profile of the polytropic sphere of $m$ using equation (2.17). Appendix D shows the fitting of polytropic-sphere models to the globular clusters’ structural profiles reported in (Kron et al. 1984; Trager et al. 1995; Miocchi et al. 2013). We found that the model could well fit the projected structural profiles of eighteen low-concentration globular clusters. In the present section, we show the results for NGC 288 and NGC 6254 as examples (Figure 11). Their concentrations are 1.30 and 1.64 based on the energy-truncated ss-OAFP model, while those of the King model are 1.0 and 1.41. NGC 288 is a good example of polytropic globular clusters, while NGC 6254 is an example of non-polytropic clusters. Figure 12 also depicts another example of a polytropic globular cluster (NGC 5139) whose surface brightness profile was reported in (Meylan 1987). The central part of the cluster data deviates from the polytrope model due to the weak cusp, though the polytropic sphere well fits the inner- and outer- halos. In the rest, we first discuss that many-body relaxation may bring the low-concentration globular clusters into structures like polytropic spheres (Section 5.1). We then show that the many-body relaxation effect could well characterize polytropic globular clusters (Section 5.2). Lastly, we criticize the concept of polytropic globular clusters (Section 5.3).

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9 We obtained the concentrations in Figure 11 by applying the same regularization explained in Section 2.3.

10 We consider that the concentration of NGC 288 is 1.0 based on our fitting of the King model. We confirmed the concentrations $c$ and values of $\chi_v$ reported in Table 2 of (Miocchi et al. 2013) based on our calculation, but not for NGC 288. We found the same result ($\chi_v = 1.7$ with $W_0 = 5.8$) for NGC 288 as their result. However, we found that the concentration for NGC 288 was $c = 1.0$ for $W_0 = 5.0$ that provided $\chi_v = 0.48$. This value is smaller than their value $\chi_v = 1.7$ with $W_0 = 5.8$ and close to unity.
Fig. 11: Fitting of the polytropic sphere of index $m$ to the projected densities $\Sigma$ of NGC 288 and NGC 6254 reported in (Miočchi et al. 2013). The unit of $\Sigma$ is number per arcminute squared, and $\Sigma$ is normalized so that the density is unity at the smallest radius data point. In the legends, (n) means a ‘normal’ or KM cluster as judged so in (Djorgovski & King 1986). $\Delta \log[\Sigma]$ is the corresponding deviation of $\log[\Sigma]$ from the model.

Fig. 12: Fitting of the polytropic sphere of index $m$ to the surface brightness profile of NGC 5139 reported in (Meylan 1987). The unit of the surface brightness (SB) is V magnitude per arcminute squared. The brightness is normalized by the magnitude $SB_0$ observed at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster as judged so in (Djorgovski & King 1986). $\Delta (SB_0 - SB)$ is the corresponding deviation of ($SB_0 - SB$) from the model.
5.1 Discussion of why low-concentration clusters’ structures are like polytropes

The low-concentration clusters’ cores are in non-equilibrium states, possibly modeled by polytropic spheres rather than a state of (local) thermodynamic equilibrium, if the mass loss from the clusters is less significant. The results of Section 4 and Appendix B show that both the King- and energy-truncated-ss-OAFP- models reasonably fit the projected structural profiles of low-concentration globular clusters in the Milky Way. These results imply that the low-concentration clusters’ cores are well relaxed. However, it is not evident whether their DFs have already reached a local Maxwellian DF since their core-relaxation times $t_{c.r.}$ are long ($\gtrsim 1$ Gyr (See, e.g., Harris 1996, (2010 edition))). Also, even the cores of star clusters undergoing a complete core-collapse can not reach a local Maxwellian DF in principle (Ito 2020).

In the initial relaxation-evolution stage, globular clusters’ cores could be described by a non-Maxwellian DF because of the non-dominant (many-body-relaxation) effect. According to the standard stellar dynamics (e.g., Spitzer 1988; Heggie & Hut 2003; Binney & Tremaine 2011), the two-body relaxation drives globular clusters’ evolution. The two-body relaxation was originally introduced as the ‘dominant effect’ of the evolution in (Chandrasekhar 1943). During the two-body relaxation evolution, a star can ‘encounter’ another star when their distance is as large as the mean stellar distance $(n^{-1/3} \sim N^{-1/3})$, where $n$ is the mean stellar density) and as small as the order of $r_{\text{tid}}/N(\sim 1/N)$,\(^{12}\) Hence, the classical stellar-dynamics theory expects the two-body relaxation to not occur between two distant stars. However, direct $N$-body simulations (e.g., Aarseth & Heggie 1998) have assessed that the two-body relaxation could not be enough to quantify the evolution. One must include the non-dominant (many-body-relaxation) effect. However, the mathematical expression of the relaxation time, including the many-body effect, is complicated. It mitigates in both the multiple collision integrals and the summation of Fourier-series expansion in an orbit-averaged kinetic equation (Polyachenko & Shukhman 1982; Ito 2018). One has conventionally assumed that the two-body relaxation is dominant even at the apocenter $r_{\text{tid}} \sim 1$ (Cohen et al. 1950). The standard stellar dynamics assumes the half-mass relaxation time to be $t_r/t_{\text{cross}} \sim N/\ln[0.1N]$, where $t_{\text{cross}}$ is the crossing time (e.g., Heggie & Hut 2003). The time $t_r$ is proper to characterize the two-body relaxation process in homogeneous clusters. On the one hand, actual clusters are inhomogenous systems. We also need to consider the many-body relaxation effect on the secular-evolution time scales $t_{\text{sec}} \sim Nt_{\text{cross}}$. Loosely speaking, during the late relaxation-evolution stage, the two-body relaxation is approximately $\ln[0.1N]$ times more significant than the many-body one. However, it may not be significant in the early stage, and the non-dominant effect can be more effective. The pericenter of stars may be much larger on average. Ideally, many stars separate from the other stars by at least the order of $1/n^{1/3}$. Extreme cases were discussed and mathematically formulated in (Kandrup 1981, 1988). Kandrup (1985) discussed some simple examples of this matter by neglecting the effect of evaporation and gravothermal instability so that the two-body relaxation processes are not dominant. The work confined a self-gravitating system in a box and examined its secular evolution. This analysis concluded that the many-body relaxation could cause a deviation of the stellar DF from the Maxwellian DF on the secular-evolution time scales in the beginning of the evolution, given that the evaporation did not occur.

Our concern is whether little evaporation could bring globular clusters’ DFs into those of polytropes. Taruya & Sakagami (2003) discussed the realization of polytropic clusters based on an $N$-body simulation. The study examined a self-gravitating system of equal-masses enclosed in an adiabatic container. It found that DFs for polytropes can well approximate the simulated DFs even on time scales much longer than the half-mass relaxation time. This result was also confirmed using a time-dependent Fokker-Planck model (Taruya & Sakagami 2004). Taruya & Sakagami (2003) also tested the system without an adiabatic wall. Of course, due to the

\(^{11}\) The encounter causes the deflection of a star’s path from its original path determined by the MF potential. The deflection is due to the pair-wise Newtonian interaction between the two stars.

\(^{12}\) See (Chavanis 2013) for the scaling of the physical quantities in terms of total stellar number $N$. 
evaporation, the simulated stellar DF largely deviates from the stellar polytrope as time elapses. On the one hand, in the early relaxation-evolution stage, the simulated DF seems reasonably fitted by the DF for a polytrope (See \( m = 5.7 \) at \( T = 50 \) in their work). Also, the inner parts of their model clusters and stellar DF at low energies are well modeled by the DFs for polytropes, regardless of the effect of escaping stars. Their results imply that the stellar DF and structural profile of a star cluster can be like a polytrope unless the effect of evaporation is dominant.

5.2 The secular evolution and polytropic globular clusters in Milky Way

We found that the secular-evolution time \( t_{\text{sec}} \) could well characterize the physical states of polytrope globular clusters in the Milky Way. Table 3 shows the time scales and parameters for the globular clusters. Such as current and estimated-initial relaxation times \( t_{c.r.} \) and \( t_{c.r.o.} \), cluster’s age \( t_{\text{age}} \), and the total stellar mass \( M \). We estimated the values of \( t_{c.r.o.} \) using the analysis of (Lightman 1982) that we employed in Section 4. Fortunately, all the polytropic clusters’ ages are available in the Literature. Hence, we employed the actual ages rather than the order of the ages in the present section, unlike Section 4. The present section examines how many initial secular-evolution times have already passed during the cluster ages. We measure this property by defining the ‘secular-evolution’ parameter

\[
\eta_M \equiv \frac{t_{\text{age}}}{t_{\text{sec.o}}},
\]

where the initial secular-evolution time is defined as

\[
t_{\text{sec.o}} \equiv \ln \left( 0.11 \frac{M}{M_\odot} \right) t_{c.r.o},
\]

where \( M_\odot \) is the solar mass. We introduced the natural log and factor 0.11, presuming that the mathematical expression for \( t_{c.r.o.} \) obeys that of the core relaxation time of (Spitzer 1988) and quantitatively the result of an \( N \)-body simulation (Aarseth & Heggie 1998). Following the discussion of Section 5.1, we employed the parameter \( \eta_M \) to examine the dynamical states of clusters’ cores. We can presume that if \( \eta_M \lesssim 1 \), then the cluster cores may be in a state of the local thermodynamic equilibrium described by a local Maxwellian DF at present. Otherwise, the cores may be in non-equilibrium states. This threshold would provide us with some insight to judge whether polytrope spheres can model low-concentration clusters. Table 3 shows that globular clusters have low \( \eta_M \) (0.20 < \( \eta_M \lesssim 1 \)), given that polytrope models adequately fit their structural profiles. On the one hand, the clusters with high \( \eta_M \) (1 \( \lesssim \eta_M \leq 3.77 \)) could not be fitted by polytropes. The maximum value (3.77) was achieved by NGC 7099 (one of the PCC clusters). We classified NGC 3201 and NGC 4590 into the intermediate class in which polytrope models fitted the projected structural profiles at only part of cluster radii. Figure 13 shows the secular-evolution rate against concentration \( c \). For the concentration, we employed the values reported in (Harris 1996, (2010 edition)). It appears that \( \eta_M = 1 \) is a good threshold to separate polytropic clusters from non-polytropic ones. When \( c \approx 1.5 \) and \( \eta_M \approx 1 \), both the polytropic- and non-polytropic clusters coexist.
Table 3: Secular-evolution parameters calculated from the core relaxation times and ages of polytropic- and non-polytropic- clusters. The current relaxation time $t_{c.r.}$ and concentration $c$ are values reported in (Harris 1996, 2010 edition). The total mass $M$ is the dynamical mass for each cluster reported in (Mandushev et al. 1991). We adapted the ages of clusters from (Forbes & Bridges 2010) and resorted to other sources when we found more recent data or could not find cluster ages in (Forbes & Bridges 2010).

Fig. 13: Secular-relaxation parameter $\eta_M$ against concentration $c$. The values of concentration are adapted from (Harris 1996, 2010 edition)).
5.3 Criticism of the concept of polytropic globular clusters

The discussion of polytropic globular clusters in Section 5.1 is oversimplified in the sense that actual globular clusters are subject to mass spectrum (segregation) with stellar evolution and tidal effects (shock). In an isolated $N$-body system of equal masses, the cluster loses a small fraction ($\sim 0.1\%$) of the total stellar mass in the first five initial-relaxation time scale (Baumgardt et al. 2002). However, more realistically, mass segregation and tidal effect make faster the process that leads to a core-collapse, as discussed for both multi-mass OAFP models (e.g., Chernoff & Weinberg 1990; Takahashi & Lee 2000) and $N$-body simulations (e.g., Fukushige & Heggie 1996; Portegies Zwart et al. 1998; Baumgardt & Makino 2003) in tidal fields. Also, a relatively new observation (Marchi et al. 2007) showed an unexpected feature of low-concentration clusters. Low-mass stars are more depleted in low-concentration clusters’ mass functions than high-concentration ones. This result implies that the lower-concentration clusters have lost more stars due to evaporation or tidal stripping. However, the excessive loss of low-mass stars from low-concentration clusters contradicts the standard stellar dynamics. Conventionally, higher-concentration clusters are supposed to have lost more low-mass stars due to more frequent two-body relaxation processes and mass stratification (segregation) (Spitzer 1988). An $N$-body simulation (Baumgardt et al. 2008) explained one possible interpretation for this issue. It showed that the low-concentration clusters had already undergone primordial mass segregation in the early relaxation-evolution stage due to the stellar evolution. This idea was extended to a sophisticated case study for one of the low-concentration clusters, Palomar 4 (Zonoozi et al. 2017). The study reported that the total mass of Palomar 4 rapidly decreased only in the first 0.1 Gyr, and the mass of the cluster calmly kept decreasing with time. The decrease in the mass depends on the orbit of Palomar 4 in the Milky Way, though the total stellar number decreases by approximately 60 % in 10 Gyr. Based on the results of the Literature, it appears that the reason why the low-concentration clusters have polytropic structures is not directly because of little mass loss from the clusters.

However, we can not jump to conclusions without more careful consideration. The direct relationship is currently unknown between the DFs for polytropes and globular clusters that have experienced mass segregation. Also, the present work does not discuss the projected line-of-sight velocity dispersion profiles of the energy-truncated ss-OAFP model. Many of the polytropic clusters are low-concentration clusters, which implies that accurate observational data are hard to be obtained compared to high-concentration clusters (e.g., Meylan & Heggie 1997). Perhaps, more recent data from Gaia 2 (e.g., Baumgardt et al. 2018), the ESO Multi-instrument Kinematic Survey (MIKiS) (Ferraro et al. 2018), and more accurate kinematic data may differentiate polytropic model from other models, including the King- and energy-truncated ss-OAFP models.

6 CONCLUSION

The present work introduced a phenomenological model, i.e., the energy-truncated ss-OAFP model. It can apply to at least fifty-five Galactic globular clusters, including PCC clusters with resolved cores. Our new model is a linear algebraic combination of the DFs for the ss-OAFP model and a polytropic sphere of $m$. We weighed the latter by a factor $\delta$. We determined the optimal value of $m$ to be 3.9 by comparing the structural parameters obtained from the King- and our models. After this procedure, the new model has only three degrees of freedom that are the same as those of the King model.

Our new model can reasonably fit the projected structural profiles of both the PCC- and KM- clusters with resolved cores. The fitting provided the completion rates of core-collapse against the concentrations of the clusters. The characteristics of the completion rates are consistent with the standard stellar dynamics. Also, our model provided the same order of structural parameters as those of the King model. This feature infers that our model can apply to glob-
ular clusters as adequately as the King model. On the other hand, our model is more useful compared to the King model to single out KM clusters whose structure is similar to the complete core-collapse cluster, i.e., high concentration ($\bar{c} \geq 2.0$) clusters. The examples are NGC 1851, NGC 6626, and NGC 6517. We also found that low-concentration ($\bar{c} \approx 1$) clusters may have structures close to the polytropes of $m = 3.9$. This result motivated us to discuss the relationship between low-concentration clusters and polytropic-sphere models.

We found that eighteen low-concentration globular clusters could be polytropic in the Milky Way. Polytrope models well fitted their projected structural profiles. We also showed that the secular-evolution time $t_{\text{sec}}$ could well characterize the low-concentration clusters' dynamical states in the cores. This feature implies that many-body relaxation processes are more significant than two-body ones in the early relaxation-evolution stage. However, one has not carried out detailed physical arguments and numerical and observational studies regarding the polytropic globular clusters, including the effects of mass spectrum and segregation. Hence, we consider that the polytropic clusters are a tentative idea, which intrigues us to work on two topics in the future. We will examine (i) the relationship between the mass spectrum (segregation) in star clusters and the stellar DF for polytropes and (ii) the applicability of the King-, polytrope-, and the ss-OAFP- models to kinematic profiles of Galactic globular clusters.

We will determine whether $m = 3.9$ is the best value in our model using more recent data, such as Gaia 2 survey, in the future. The present paper primarily focused on whether our model can fit globular clusters' structural profiles as reasonably as the King model. Strictly speaking, we do not need to choose the optimal value by comparing our and the King models, unlike Section 2.4. We also do not need to employ the regularization made in Section 2.3. We introduced those tedious processes in the present work only for the sake of comparison to the King model. We will examine our model’s parametric-space dependence more carefully using data based on a recent homogeneous survey. Part of the work has been done in (Ito 2020).

We will also extend our model by incorporating extra effects. As explained in Section 1, the King model has been extended to anisotropic and multi-component models including the effects of black holes, dark matter, and others. We will follow the same path as the King model to ‘fine-tune’ our model for specific clusters. We especially consider the multi-mass effect to be important to our model. As we examined in Section 3, our model does not fit PCC clusters as adequately as KM clusters near the cores. We believe that the multi-mass effect resolves the issue. Hence, we are planning to resort to the method of (Costa & Freeman 1976) with a realistic mass spectrum.

Acknowledgements The present work is partial fulfillment of the degree of Philosophy at The Graduate Center of CUNY. This research has made use of the products of the Cosmic-Lab project funded by the European Research Council, by digitizing the graphs of (Miocchi et al. 2013) using the WebPlotDigitizer.

Appendix A: CHEBYSHEV COEFFICIENTS FOR THE DENSITY PROFILE OF THE SS-OAFP MODEL

Table A.1 provides the Chebyshev coefficients for the density profile obtained from the Gauss-Chebyshev spectral solution of the ss-OAFP equation (Ito 2021).
Table A.1: Chebyshev coefficients for $D(\varphi)$ obtained from the solution of the ss-OAFP equation in (Ito 2021)

Appendix B: FITTING THE ENERGY-TRUNCATED SS-OAFP MODEL TO KM CLUSTERS

The present appendix shows the fitting of the energy-truncated ss-OAFP model to Galactic KM clusters’ projected structural profiles reported in (Kron et al. 1984; Miocchi et al. 2013). For fitting the model to data of (Miocchi et al. 2013), we determined the fitting parameters’ values when $\chi^2_\nu$ reached its minimum. To fit the data of (Kron et al. 1984), we minimized the model’s absolute deviation from the observed data. Sections B.1 and B.2 show that our model fits the KM stars reported in (Miocchi et al. 2013) and (Kron et al. 1984).

B.1. KM cluster (Miocchi 2013)

Figures B.1, B.2, and B.3 depict the energy-truncated ss-OAFP model fitted to KM clusters’ projected density profiles reported in (Miocchi et al. 2013). Table B.1 compares the structural parameters obtained from our, the King-, and the Wilson- models. The majority of the param-

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Parameters obtained from our model are slightly greater than those of the King model. However, they are less than those of the Wilson model. Our model does not well fit the structures of NGC 5466 and Terzan 5, while the King- and Wilson- models fit them. This result implies that the clusters are less close to the core-collapse cluster and the polytropic sphere of $m = 3.9$.

Fig. B.1: Fitting of the energy-truncated ss-OAFP model ($m = 3.9$) to the projected density $\Sigma$ of NGC 288, NGC 1851, NGC 5466, and NGC 6121 reported in (Miocchi et al. 2013). The unit of $\Sigma$ is stellar number per arcminute squared, and $\Sigma$ is normalized so that the density is unity at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster, as judged so in (Djorgovski & King 1986). $\Delta \log[\Sigma]$ is the corresponding deviation of $\Sigma$ from the model on the log scale.
Fig. B.2: Fitting of the energy-truncated ss-OAFP model ($m = 3.9$) to the projected density $\Sigma$ of NGC 6254, NGC 6626, Palomar 3, and Palomar 4 reported in (Miocchi et al. 2013). The unit of $\Sigma$ is stellar number per arcminute squared, and $\Sigma$ is normalized so that the density is unity at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster, as judged so in (Djorgovski & King 1986). $\Delta \log[\Sigma]$ is the corresponding deviation of $\Sigma$ from the model on the log scale.
Fig. B.3: Fitting of the energy-truncated ss-OAFP model ($m = 3.9$) to the projected density $\Sigma$ of Palomar 14, Terzan 5, and NGC 6809 reported in (Miocchi et al. 2013). The unit of $\Sigma$ is stellar number per arcminute squared, and $\Sigma$ is normalized so that the density is unity at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster, as judged so in (Djorgovski & King 1986). $\Delta \log[\Sigma]$ is the corresponding deviation of $\Sigma$ from the model on the log scale.
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Table B.1: Core- and tidal- radii obtained by fitting the energy-truncated ss-OAFP model to projected density profiles of KM clusters reported in (Miocchi et al. 2013).

B.2. KM clusters (Kron 1984)

Figures B.4- B.7 show the projected density profiles reported in (Kron et al. 1984), fitted by the energy-truncated ss-OAFP model. In (Kron et al. 1984), the King model’s fitting does not include several data points near the clusters’ centers because of uncertainty in the data originating from too high brightness in the cores. However, the present work included them since our model adequately fits almost all the data plots (except for NGC 5053 and NGC 5897). Also, the data near the center did not affect the fitting parameters significantly.
Fig. B.4: Fitting of the energy-truncated ss-OAFP model ($m = 3.9$) to the projected density profiles of NGC 2419, NGC 4590, NGC 5272, NGC 5634, NGC 5694, and NGC 5824 reported in (Kron et al. 1984). The unit of the projected density $\Sigma$ is stellar number per arcminute squared. In the legends, (n) means a ‘normal’ cluster or KM, as judged so in (Djorgovski & King 1986).
Fig. B.5: Fitting of the energy-truncated ss-OAFP model \((m = 3.9)\) to the projected density profiles of NGC 6093, NGC 6205, NGC 5229, NGC 6273, NGC 6304, and NGC 6333 reported in (Kron et al. 1984). The unit of the projected density \(\Sigma\) is stellar number per arcminute squared. In the legends, \((n)\) means a ‘normal’ or KM cluster and \((n?)\) a ‘probable normal’ cluster, as judged so in (Djorgovski & King 1986).
Energy-truncated self-similar OAFP model

Fig. B.6: Fitting of the energy-truncated ss-OAFP model \((m = 3.9)\) to the projected density profiles NGC 6341, NGC 6356, NGC 6401, NGC 6440, NGC 6517, and NGC 6553 reported in (Kron et al. 1984). The unit of the projected density \(\Sigma\) is stellar number per arcminute squared. In the legends, \((n)\) means a ‘normal’ or KM cluster, as judged so in (Djorgovski & King 1986).
Fig. B.7: Fitting of the energy-truncated ss-OAFP model ($m = 3.9$) to the projected density profiles of NGC 6569, NGC 6638, NGC 6715, NGC 6864, NGC 6934, and NGC 7006 reported in (Kron et al. 1984). The unit of the projected density $\Sigma$ is stellar number per arcminute squared. In the legends, (n) means a 'normal' or KM cluster, as judged so in (Djorgovski & King 1986).
Appendix C: FITTING THE ENERGY-TRUNCATED SS-OAFP MODEL TO PCC CLUSTERS

The present appendix shows the results of applying the energy-truncated ss-OAFP model to PCC clusters reported in (Kron et al. 1984; Djorgovski & King 1986; Lugger et al. 1995). For fitting the model to data of (Kron et al. 1984), we minimized the infinite norm of the deviations of the model from the data to find the structural parameters, while $\chi^2$-value was minimized for fitting to data of (Lugger et al. 1995; Djorgovski & King 1986). Sections C.1 and C.2 show the fitting to PCC clusters reported in (Kron et al. 1984) and (Lugger et al. 1995).

C.1. PCC? clusters (Kron 1984)

Figure C.1 depicts the energy-truncated ss-OAFP model with $m = 3.9$ fitted to the projected density profiles of possible PCC clusters reported in (Kron et al. 1984). The work of (Kron et al. 1984) considered NGC 1904, NGC 4147, NGC 6544, and NGC 6652 as possibly PCC clusters.
Fig. C.1: Fitting of the energy-truncated ss-OAFP model to the projected densities of NGC 1904, NGC 4147, NGC 6544, and NGC 6652 reported in (Kron et al. 1984). The unit of the projected density $\Sigma$ is stellar number per arcminute squared. In the legends, (c) means a ‘core-collapse’ or PCC cluster, (c?) a probable/possible PCC cluster, and (n?c?) weak indications of a PCC cluster as judged so in (Djorgovski & King 1986).

Figures C.2 and C.3 show the results of fitting the energy-truncated ss-OAFP model with $m = 3.9$ to the surface brightness profiles of PCC clusters reported in (Lugger et al. 1995) and Terzan 1 in (Djorgovski & King 1986). For clusters with resolved cores, our model well fits the core and halo at up to 1 arcminute. As expected, the fitting to the clusters with unresolved cores (NGC 5946 and NGC 6624) is not satisfactory. On the one hand, NGC 6342 is one of them, but our model reasonably fits the profile.
Fig. C.2: Fitting of the energy-truncated ss-OAFP model to the surface brightness profiles of NGC 5946, NGC 6342, NGC 6624, and NGC 6453 reported in (Lugger et al. 1995). The unit of the surface brightness (SB) is U magnitude per arcsecond squared. The brightness is normalized by the magnitude SB$_0$ at the smallest radius point. In the legend, (c) means a PCC cluster as judged so in (Djorgovski & King 1986). ∆(SB$_0$ − SB) is the corresponding deviation of (SB$_0$ − SB) from the model.
Fig. C.3: Fitting of the energy-truncated ss-OAFP model to the surface brightness profiles of NGC 6522, NGC 6558, and NGC 7099 reported in (Lugger et al. 1995) and Terzan 1 reported in (Djorgovski & King 1986). The unit of the surface brightness (SB) is U magnitude per arcsecond squared, except for Terzan 1 for which B band is used. The brightness is normalized by the magnitude $SB_0$ at the smallest radius point. In the legend, (c) means a PCC cluster as judged so in (Djorgovski & King 1986). $\Delta(SB_\circ - SB)$ is the corresponding deviation of $(SB_\circ - SB)$ from the model.

Appendix D: FITTING POLYTROPIC SPHERE MODEL TO LOW-CONCENTRATION CLUSTERS

The present appendix shows the result of fitting polytropic spheres of index $m$ to the projected structural profiles of low-concentration globular clusters reported in (Miocchi et al. 2013; Trager et al. 1995; Kron et al. 1984). When we fitted the polytropes to the data of (Miocchi
et al. 2013), we minimized $\chi^2_\nu$ and determined the structural parameters. We minimized the infinite norm of the differences between the fitted curve and the data of (Kron et al. 1984; Trager et al. 1995).

D.1. Polytropic cluster (Miocchi 2013) and (Kron 1984)

Figures D.1 and D.2 show successful applications of the polytrope models to the projected density profiles of NGC 5466, NGC 6809, Palomar 3, Palomar 4, and Palomar 14 reported in (Miocchi et al. 2013). In the figures, the polytropes of $3.0 < m < 5$ reasonably fit to the projected surface densities of the globular clusters whose concentrations range $1 < c < 1.4$. Figure D.3 shows the projected density of NGC 4590 fitted with a polytrope. NGC 4590 is one of the examples that could fall in either of polytropic and non-polytropic clusters.
Fig. D.1: Fitting of the polytropic spheres of index $m$ to the projected density profiles $\Sigma$ of NGC 5466, NGC 6809, Palomar 3, and Palomar 4 reported in (Miocchi et al. 2013). The unit of $\Sigma$ is stellar number per arcminute squared, and $\Sigma$ is normalized so that the density is unity at smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster, as judged so in (Djorgovski & King 1986). $\Delta \log[\Sigma]$ is the corresponding deviation of $\Sigma$ from the model on the log scale.
Fig. D.2: Fitting of the polytropic sphere of index $m$ to the projected density of Palomar 14 reported in (Miocchi et al. 2013). The unit of the projected density $\Sigma$ is stellar number per arcminute squared. In the legends, (n) means a 'normal' or KM cluster, as judged so in (Djorgovski & King 1986).

Fig. D.3: Partial success of fitting the polytropic sphere of index $m$ to the projected density of NGC 4590 reported in (Kron et al. 1984). The unit of the projected density $\Sigma$ is stellar number per arcminute squared. In the legends, (n) means a 'normal' or KM cluster, as judged so in (Djorgovski & King 1986). Following (Kron et al. 1984), data at small radii are ignored due to the depletion of projected density profile in the core.

D.2. Polytropic cluster (Trager 1995)

Figures D.4 and D.5 depict the fitting of the polytropic sphere model to the Chebyshev approximation to the surface brightness profiles reported in (Trager et al. 1995). The polytropic indices $m = 3.3 \sim 4.99$ are useful to fit the polytrope models to the low-concentration clusters whose core-relaxation times are the order of 1 Gyr (from (Harris 1996, (2010 edition))’s catalog). The polytropes themselves do not rapidly decay near their tidal radii. Hence, their concentrations are relatively high, such as $c = 3.34$ for $m = 4.99$. On the one hand, Figure D.6 shows the clusters whose surface brightness profiles are not close to the polytropes. Such non-polytropic clusters have shorter core-relaxation times (< 0.5 Gyr) and relatively high concentrations ($c > 1.5$ based on the King model), as shown in Table 3.
Fig. D.4: Fitting of the polytropic spheres of index $m$ to the surface brightness profiles of NGC 1261, NGC 5053, NGC 5897, NGC 5986, NGC 6101, and NGC 6205 reported in (Trager et al. 1995). The unit of the surface brightness (SB) is $V$ magnitude per arcsecond squared. The brightness is normalized by the magnitude SB$_0$ at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster, as judged so in (Djorgovski & King 1986).
Fig. D.5: Fitting of the polytropic sphere of index $m$ to the surface brightness profiles of NGC 6402, NGC 6496, NGC 6712, NGC 6723, and NGC 6981 reported in (Trager et al. 1995). The unit of the surface brightness (SB) is $V$ magnitude per arcsecond squared. The brightness is normalized by the magnitude $SB_o$ at the smallest radius point. In the legends, (n) means a 'normal' or KM cluster, as judged so in (Djorgovski & King 1986).
Fig. D.6: Failure of fitting the polytrope models of index $m$ to the surface brightness profiles of NGC 3201, NGC 6144, NGC 6273, NGC 6352, NGC 6388, and NGC 6656 reported in (Trager et al. 1995). The unit of the surface brightness (SB) is $V$ magnitude per arcsecond squared. The brightness is normalized by the magnitude $S_{B_0}$ at the smallest radius point. In the legends, (n) means a ‘normal’ or KM cluster, as judged so in (Djorgovski & King 1986).

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