Stability of LRS Bianchi type-I cosmological models in $f(R, T)$-gravity

Umesh Kumar Sharma$^1$, Rashid Zia$^1$, Anirudh Pradhan$^1$, A. Beesham$^2$

$^1$ Department of Mathematics, Institute of Applied Sciences & Humanities, GLA University, Mathura-281 406, Uttar Pradesh, India; pradhan.anirudh@gmail.com
$^2$ Department of Mathematical Sciences, University of Zululand, Kwa-Dlangezwa 3886, South Africa

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Abstract This paper looks at the stability of the transition from the early decelerating stage of the universe to the recent accelerating stage for the perfect fluid cosmological LRS Bianchi-I model in $f(R, T)$ theory. To determine the solution of field equations (FEs), the idea of a time-varying deceleration parameter (DP) which yields a scale factor for which the universe attains a phase transition scenario and consistent with recent cosmological observations is used. The time-dependent DP yields a scale factor $a = \exp\left[\beta \sqrt{2/\beta t + k}\right]$, where $\beta$ and $k$ are respectively arbitrary and integrating constants. By using the recent constraints ($H_0 = 73.8$, and $q_0 = -0.54$) from SN Ia data in combination with Baryonic Acoustic Oscillations (BAO) and Cosmic Microwave Background (CMB) observations (Giostri et al., arXiv:1203.3213v2[astro-ph.CO]), we obtain the values of $\beta = 0.0062$ and $k = 0.000016$ for which we have derived a cosmological model from early decelerated phase to the present accelerating phase. By using another recent constraints ($H_0 = 73.8$, $q_0 = -0.73$) from supernovae type Ia union data (Cunha, arXiv:0811.2379v1[astro-ph]), we obtain the values of $\beta = 0.0036$ and $k = 0.000084$ for which we have derived a cosmological model in accelerating phase only. We have compared the models in both experimental data. The stability of background solution has been examined also for the metric perturbations alongside the properties of future singularities in the Universe ruled by dark energy with the phantom type fluid. We demonstrate the presence of a stable fix point with a condition of state $\omega < -1$ and numerically affirm this is really a late-time attractor in the ghost overwhelmed universe. Some physical and geometric properties of the model are found and examined.

Key words: LRS Bianchi-I universe: Time-dependent deceleration parameter: $f(R, T)$ gravity theory: Transit universe

1 INTRODUCTION

The hypothetical contentions for the cosmic acceleration (late-time ) are being a noteworthy matter between the cosmologists of the twentieth century. In recent years, the importance of observational cosmology cannot be ignored. Recently, observed astronomical phenomena have revolutionized the understanding of cosmology. About twenty years ago the idea of a universe which is in an accelerated expansion phase was discovered by various observations (Riess et al. 1998; Perlmutter et al. 1999; Garnavich et al. 1998; Spergel et al. 2007a,b; Huang et al. 2006; Tegmark et al. 2004). Later on, numerous test confirmations additionally demonstrate the expansion of the universe is accelerated in nature.
(Ade et al. 2016; Naess et al. 2014). These perceptions demonstrate that our Universe is overwhelmed by bizarre cosmic fluid with huge pressure (negative), named, Dark Energy (DE), which constitutes $\simeq 3/4$ (Padmanabhan 2003; Sahni 2004) of the critical density. It is believed that the DE might be a dependable contender for the present cosmic acceleration. Yet, the nature of DE is secretive and its astounding inquiries, for example, where, how, why and when about the DE, are fascinating. The reason for the sudden change from the prior deceleration stage to the ongoing speeding upstage and the wellspring of accelerated expansion is as yet obscure. Be that as it may, the nature of dark energy is baffling and its bewildering questions, such as where, how, why and when about the DE, are intriguing.

In an essentially indistinguishable way, another modified theory has been started like Gauss-Bonnet gravity investigated by (Nojiri et al. 2005, 2006; Bamba et al. 2010, 2017) where the component of the invariant term (Gauss-Bonnet) $G$ is $f(G)$. $f(T)$ gravity (Bamba et al. 2013; Odintsov et al. 2015; Cai et al. 2016; Paliathanasis et al. 2016). Various candidates have been suggested for this strange DE, such as the constant (cosmological) (originally introduced by Einstein), quintessence, phantom, quantum, and so on. A more thorough survey is given in (Copeland et al. 2006). Soon after the development of General Relativity by Einstein, many alternatives to GR had been proposed. Most of them, however, were lacking simplicity as well as observational fitting. Nevertheless, modification proposals have been published by researchers ever-since. Numerous models have been proposed to clarify this present accelerated expansion. Basically, there are two approaches that allow one to modify gravity. The first is to adjust the right-hand side of the Einstein’s field equations (EFEs) by considering particular structures for the energy-momentum tensor ($T_{\mu\nu}$) having a negative pressure, which finishes in the proposition of an “exotic cosmic fluid” however the experimental information don’t clarify the expansion totally (Sharif & Zubair 2010a,b). Secondly, from a mathematical perspective, there is no apriori demand that forces one to consider the Einstein-Hilbert action (linear in $R$) as the fundamental action of gravity. At a show, the modified hypothesis of gravity has turned into a well-known contender to clarify the inception of DE. The main outline is supplanting Hilbert-Einstein term by a random function of scalar curvature $R$, named $f(R)$ gravity (Capozziello & Vignolo 2009, 2011, 2012; Nojiri & Odintsov 2017; Sepehri et al. 2016, 2017).

Sharma et al. (2018), the torsion scalar $T$ is beside a case of modified gravity. Utilizing diverse mixes of scalars these altered hypotheses of gravity summed up to $f(R, P, Q)$ and $f(\tilde{R}, G)$ (Nojiri & Odinstsov 2005; Cru-Dombritz & Sáez-Gómez 2012) where $P = R^{\mu\nu}R_{\mu\nu}$, what’s more, $Q = R^{\mu\nu\sigma\tau}R_{\mu\nu\sigma\tau}$ (here $R_{\mu\nu\sigma\tau}$ addresses the Riemann tensor and $R_{\mu\nu}$ addresses the Ricci tensor). Another kind of alternative hypothesis of gravity is $f(R, T, T_{\mu\nu})$ (Sharif & Zubair 2013; Yousaf et al. 2017). In this research, we concentrated on our regard for $f(R, T)$ gravity which was first developed by Harko et al. (2011) and this theory models, focusing of energy-momentum tensor $T_{\mu\nu}$ variation concerning the metric, relies on a source term. The clarification of picking this source term, in general, is a part of the Lagrangian $L_{\text{m}}$. Thulsy for every decision of $L_{\text{m}}$ there ought to be the time of a particular solution of equations. The field equations of this theory by Hilbert- Einstein variational principle was obtained by Harko et al. (2011) furthermore find the covariance divergence of the $T_{\mu\nu}$. In this way, alternative gravity speculations give distinctly a strategy for understanding the issue of DE and the likelihood of reproducing the gravitational field hypothesis that would have the ability to explain that the universe is in accelerated expansion phase presently. There are many researchers who have thought about cosmology in this theory of gravity. It isn’t conceivable to say every one of them yet, we give some pertinent and most recent references which are specifically identified with our present work (Adhav 2012; Shamir 2014; Chaubey & Shukla 2013; Chandel & Ram 2013; Velten & Carmes 2017; Chakraborty 2013; Moraes 2015; Singh & Bishi 2016; Sahu et al. 2017; Sahoo et al. 2017; Ahmed & Pradhan 2014; Pradhan et al. 2015; Mishra et al. 2016; Ahmed et al. 2016; Tiwari et al. 2017; Sharma & Pradhan 2018; Pradhan & Jaitswal 2018; Sahoo et al. 2018; Nagpal et al. 2018; Pullen & Kamionkowski 2007; Samal et al. 2008 and references therein).

It is fascinating to observe that in the dynamical history of our universe the acceleration expansion of the universe becomes an important part. Evaluating the EoS for dark energy in observational cosmology at present is one of the best undertakings. The dark energy show has been depicted routinely by the (EoS) parameter $\omega = \frac{p}{\rho}$ which isn’t generally constant (Caroll and Hoffman 2003). The display data
appear to imperceptibly bolster a propelling dark energy with EoS $\omega < -1$ at the present stage and $\omega > -1$ in the nearby past. Plainly, $\omega$ can’t cross $-1$ for phantom or quintessence alone. A couple of undertakings have been made to develop a dark energy show whose EoS may cross the phantom segment. The minimum complex DE part is the vacuum energy ($\omega = -1$), which is numerically similar to (Λ). The other normal decisions, which can be depicted by inconsequential coupled scalar fields, are the quintessence ($\omega > -1$) (Steinhardt and Wesley 1999), phantom $\omega < -1$ (Caldwell 2002), quintom and what’s more, have time subordinate EoS parameter. The EoS parameter $\omega$ may be taken as constant yet it is a mapping of time or redshift $z$ or scale factor $a$ additionally (Ratra and Peebles 1988; Jimenez 2003; Das et al. 2005).

The bulk of democratic cosmological models use the cosmological principle; that is, they assume that the universe is homogeneous and isotropic. On the other hand, the CMB temperature and polarization anisotropy fundamentals (Hu, 2003) and CMB polarization of complimentary information to anisotropies (Kaplan et al. 2003; Souradeep 2011; Buzzelli et al. 2016) suggest that the assumption of statistical isotropy is broken on the largest angular scales, leading to some intriguing anomalies. To provide predictions for the CMB anisotropies, one may consider the homogeneous but anisotropic cosmologies known as Bianchi type space times, which include the isotropic and homogeneous FRW models. The Bianchi type I, II, .....IX cosmological models are spatially homogeneous space times of dimension $1 + 3$ admitting a group of motions $G_3$ acting on space-like hyper-surfaces (Stephani et al. 2003; Wald 1984). More precisely, they are manifolds of the form $M = I \times G$ where $I \subset R$ is an interval and $G$ is a Lie group endowed with a Lorentzian metric of the form $-dt^2 + g_{i\ell}$ where $(g_{i\ell})_{\ell \in I}$ is a family of left-invariant Riemannian metrics on $G$. The physical substance of a space time $M$ is param-phased in terms of non-linear partial differential equations (PDEs) on its Lorentz metric: the so called Einstein’s equations. For Bianchi cosmological models, this PDEs reduce to a set of second order ordinary differential equations on the family of metric $(g_{i\ell})_{\ell \in I}$. However, it is not the case in LRS Bianchi type cosmological models where Einstein’s field equations lead to a set of non-linear differential equations. The study of Bianchi type-I cosmological models becomes more interest as these models contain isotropic special cases and permit arbitrarily small anisotropic levels at certain stages. When the Bianchi type-I space time flourishes equally in two spatial directions it is called locally rotationally symmetric. For simplification and description of the large scale behaviour of the actual universe, locally rotationally symmetric (LRS) Bianchi type-I space time has extensively well-read. This model is characterized by three metric functions $R_1(t), R_2(t)$ and $R_3(t)$ such that $R_1 \neq R_2 = R_3$. LRS Bianchi type-I space time is a generalization of flat FRW metric.

Inspired by the above discussion, in this paper, we have revisited in solutions as of late acquired by Pradhan et al. (2018). The paper is dedicated to contemplating cosmological LRS Bianchi I model in $f(R,T)$ altered hypothesis of gravity in presence of $\Lambda(T)$ which have the property that they show a progress from decelerating at early time to accelerating at late time. The paper is outlined as takes after the development of altered $f(R,T)$ hypothesis with $\Lambda(T)$ is given in segment 2. The metric and field conditions are given in segment 3. Solutions are exhibited in segment 4 and furthermore portrays the physical and geometric characteristics of the model with a brief talk of the outcomes. In section 5 the stability of corresponding solutions has been talked about. The conclusion is given in the last Section 6.

2 CONSTRUCTION OF MODIFIED $f(R,T)$ THEORY WITH $\Lambda(T)$ GRAVITY

For $f(R,T)$ gravity, the action integral is characterized by:

\[
S = \frac{c^4}{16\pi G} \int \sqrt{-g} f(R,T) d^4x + \int \sqrt{-g} L_m d^4x, \tag{1}
\]

Here

- $f(R,T) \rightarrow$ a random mapping of $R$ and $T$.
- $R \rightarrow$ Ricci scalar.
- $T = g^{\mu\nu} T_{\mu\nu}$ follow $T_{\mu\nu}$.
- $L_m \rightarrow$ ( matter) the Lagrangian density.
$T_{\mu\nu}$ is defined as:

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta \sqrt{-g} L_m}{\delta g^{\mu\nu}}.$$  \hspace{1cm} (2)

Let $L_m$ relies upon $g_{\mu\nu}$ as it were. $T_{\mu\nu}$ is characterized as (Harko & Lobo 2010):

$$T_{\mu\nu} = -2\frac{\partial L_m}{\partial g^{\mu\nu}} + g_{\mu\nu}L_m.$$  \hspace{1cm} (3)

Here we use the unit for $c = G = 1$. $F(R, L_M)$ gravity has been explored by Harko & Lobo (2010).

Likewise, a speculation in which the cosmological constant is made by a limit of $f(R)$ gravity has been explored by Poplawski (2006a).

Variation in $S$ with respect to $g^{\mu\nu}$ gives the field condition as:

$$(g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})f_R(R, T) - \frac{1}{2} f(R, T)g_{\mu\nu} + f_R(R, T)R_{\mu\nu}$$

$$= -f_T(R, T)\Theta_{\mu\nu} - f_T(R, T)T_{\mu\nu} + 8\pi T_{\mu\nu}.$$  \hspace{1cm} (4)

By the connection $\delta \left(\frac{g^{\gamma\tau}T_{\tau\gamma}}{\delta g^{\mu\nu}}\right) = \Theta_{\mu\nu} + T_{\mu\nu}, g^{\gamma\tau} \left(\frac{\delta T_{\tau\gamma}}{\delta g^{\mu\nu}}\right) \equiv \Theta_{\mu\nu}$ furthermore, $\nabla^{\gamma}\nabla_{\gamma} = \Box$, $f_R(R, T) \equiv \frac{\partial f(R, T)}{\partial R}$, $f_T(R, T) \equiv \frac{\partial f(R, T)}{\partial T}$ and $\nabla_{\gamma}$ speaks to the covariant derivative.

Utilizing the connection $\delta g_{\mu\nu} = -g_{\gamma\beta}g_{\delta\mu}g^{\gamma\delta}$ with $\delta g^{\mu\nu} = \frac{\delta g^{\mu\nu}}{\delta g_{\mu\nu}}$, which takes after from $g_{\gamma\beta}g_{\delta\mu}g^{\gamma\delta} = \delta_{\mu}^{\gamma}$, we acquire $\Theta_{\mu\nu}$ as given by:

$$\Theta_{\mu\nu} = -2T_{\mu\nu} + g_{\mu\nu}L_m - 2g^{\gamma\delta}\frac{\partial^2 L_m}{\partial g^{\mu\nu}\partial g^{\delta\gamma}}.$$  \hspace{1cm} (5)

The energy momentum-tensor of perfect fluid for $L_m$ is characterized as:

$$T_{\mu\nu} = (p + \rho)u_{\mu}u_{\nu} - pg_{\mu\nu},$$  \hspace{1cm} (6)

where the four velocities is $u^{\mu} = (0, 0, 0, 1)$ in the moving coordinates such that $u^\mu u_\mu = 1$ and $u^\mu \nabla_{\mu} u_\mu = 0$. Here $p$ is the pressure and $\rho$ is the energy density of the fluid. By utilizing the condition (5), we acquire:

$$\Theta_{\mu\nu} = -2T_{\mu\nu} - pg_{\mu\nu}.$$  \hspace{1cm} (7)

Since $f(R, T)$ field equations besides relying on the physical behaviour of the matter field (through the tensor $\Theta_{\mu\nu}$), many hypothetical models may be found for every choice of $f$. Harko et al. (2011) suggested three affirmation of the function $f$ as given below:

$$- 2f(T) + R$$

$$- f_2(T) + f_1(R)$$

$$- f_2(R)f_3(T) + f_1(R)$$

The results, cosmologically for the first case have been talked about by different scientists (Adhav 2012; Samanta 2013; Samanta & Dhal 2013 and references therein) and for the second case by (Shamir et al. 2012); Chaubey & Shukla (2013)). We are mulling over the cosmological results for $f(R, T) = \lambda_1 R + \lambda_2 T$. Our interpreted model is extraordinary and new from that of different researchers described above. So far $\Lambda$ which is responsible for dark energy stays less went to.

Now the equation (4) may be composed as:

$$f'_1(R)R_{\mu\nu} - \frac{1}{2} f_1(R)g_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_{\mu}\nabla_{\nu})f'_1(R) =$$

$$8\pi T_{\mu\nu} + f'_2(T)T_{\mu\nu} + \left(f'_2(T)p + \frac{1}{2} f_2(T)\right)g_{\mu\nu}.$$  \hspace{1cm} (8)
here differentiation is represented by prime indices with respect to argument. By choosing \( \lambda = \lambda_1 = \lambda_2 \) in this article with the objective that \( f(R, T) = \lambda R + \lambda T \). Here \( \lambda_1 \) and here \( \lambda_2 \) are random parameters. Condition (8) would now have the capacity to changed as:

\[
\lambda R_{\mu\nu} - \frac{1}{2} \lambda (R + T) g_{\mu\nu} + (g_{\mu\nu} \Box - \nabla_\mu \nabla_\nu) \lambda = 8\pi T_{\mu\nu} - \lambda T_{\mu\nu} + \lambda (2T_{\mu\nu} + pg_{\mu\nu}).
\]

(9)

Setting \((-\nabla_\mu \nabla_\nu + g_{\mu\nu} \Box) \lambda = 0\), we get:

\[
\lambda G_{\mu\nu} = 8\pi T_{\mu\nu} + \lambda T_{\mu\nu} + \left( \lambda p + \frac{1}{2} \lambda T \right) g_{\mu\nu},
\]

(10)

where \( G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \) is simply the Einstein tensor. This could be revised as:

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \left( \frac{8\pi + \lambda}{\lambda} \right) T_{\mu\nu} + \left( p + \frac{1}{2} T \right) g_{\mu\nu}
\]

(11)

Original Einstein reasoning with cosmological constant is:

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -8\pi T_{\mu\nu} + \Lambda g_{\mu\nu}.
\]

(12)

Picking a small negative value for the random \( \lambda \) with the objective having a comparable sign of the \( \text{R.H.S. of (10) and (11)} \), keeping this decision of \( \lambda \) all through. The \( (p + \frac{1}{2} T) \) term would now have the capacity to be seen as a \( \Lambda \). In this manner we form :

\[
\Lambda(T) \equiv \Lambda = p + \frac{1}{2} T.
\]

(13)

The \( T \) is dependent on the cosmological constant \( \Lambda \), that has been examined before by Poplawski (2006a) where \( \Lambda \) in the gravitational Lagrangian is a function of the \( T \), furthermore, in this way the demonstrator was shown “\( \Lambda(T) \) gravity”. By rejecting the matter pressure, \( \Lambda(T) \) gravity which more wide in comparison to Palatini \( f(R) \) can be obtained (Magnano 1995;, Poplawski 2006b, 2006c). \( T = \rho - 3p \), for our model, considering perfect fluid. In this manner Eq. (13) reduces to :

\[
\Lambda = \frac{1}{2} \rho - \frac{1}{2} p
\]

(14)

3 METRIC AND BASIC EQUATIONS

Considering the anisotropic LRS Bianchi type-I (spatially homogeneous ) metric of the frame as:

\[
ds^2 = dt^2 - A^2(t)dx^2 - A^2(t)dy^2 - B^2(t)dz^2.
\]

(15)

The LRS Bianchi type-I (the locally rotationally symmetric) has symmetric plane comparing to xy-plane and \( e = \sqrt{1 - \frac{B^2}{A^2}} \) is its eccentricity.

Other physical parameters like spatial volume, average scale factor and summed up mean Hubble parameter \( H \) for the Eq. (15) is defined as :

\[
V = A^2 B.
\]

(16)

\[
a = (A^2 B)^\frac{1}{3} = V^\frac{1}{3}.
\]

(17)

\[
H = \frac{1}{3}(2H_x + H_z),
\]

(18)
Here the Hubble parameters (directional) are \( H_x = H_y = \frac{\dot{A}}{A} \), \( H_z = \frac{\dot{B}}{B} \) in course of \( x \) and \( z \) individually and an overhead prime from now on, speaks to ordinary differentiation with respect to cosmic time “\( t \)” as it were.

By Eqs. (17) and (18), we get an essential connection:

\[
H = \frac{1}{3} \left( \frac{2\dot{A}}{A} + \frac{\dot{B}}{B} \right) = \frac{\ddot{a}}{a},
\]

(19)

Enunciations for the dynamic scalars, for instance, expansion scalar \((\theta)\), anisotropy parameter \((A_m)\) what’s more, the shear scalar \((\sigma)\) are described clearly:

\[
\theta = u^i_i = \left( 2\frac{\dot{A}}{A} + \frac{\dot{B}}{B} \right),
\]

(20)

\[
A_m = \frac{1}{3} \sum_{i=1}^{3} \left( \frac{H_i - H}{H} \right)^2,
\]

(21)

\[
\sigma^2 = \frac{1}{2} \sigma_{ij} \sigma^{ij} = \frac{1}{2} \left[ 2 \left( \frac{\dot{A}}{A} \right)^2 + \left( \frac{\dot{B}}{B} \right)^2 \right] - \frac{\theta^2}{6}.
\]

(22)

We characterize deceleration parameter (DP) \( q \) as:

\[
q = -\frac{a\ddot{a}}{a^2} = -\left( \frac{\dot{H} + H^2}{H^2} \right).
\]

(23)

Assuming the coordinate system as co-moving, the field Eqs. (11) for (15) and \( T_{\mu\nu} \) by Eq. (6) may be obtained in terms of directional Hubble parameters as (Pradhan et al. 2018):

\[
\dot{H}_x + \dot{H}_z + H_x^2 + H_z^2 + H_x H_z = -\left( \frac{8\pi + \lambda}{\lambda} \right) p - \Lambda,
\]

(24)

\[
2\dot{H}_x + 3H_x^2 = -\left( \frac{8\pi + \lambda}{\lambda} \right) p - \Lambda,
\]

(25)

\[
H_z^2 + 2H_x H_z = \left( \frac{8\pi + \lambda}{\lambda} \right) \rho - \Lambda.
\]

(26)

4 COSMOLOGICAL SOLUTIONS OF THE FIELD EQUATIONS AND THEIR PROPERTIES

Now we have a system of three equations given by (24)-(26) with four unknown, viz. \( p \), \( \rho \), \( A \) and \( B \). Therefore, we need one more physical condition for finding the deterministic solution of these equations.

By doing some calculations with Eqs. (24), (25) and (17) we get (Pradhan et a. 2018):

\[
A = k_1^\frac{1}{3} a \exp \left[ \frac{k_1}{3} \int \frac{dt}{a^3} \right],
\]

(27)

and

\[
B = k_2^\frac{1}{3} a \exp \left[ -\frac{2k_1}{3} \int \frac{dt}{a^3} \right],
\]

(28)

where \( k_1 \) and \( k_2 \) are integrating constants.
The expressions for pressure ($p$), energy density ($\rho$), and cosmological constant ($\Lambda$) for the universe can be obtained in terms of Hubble parameters as:

$$p = \frac{2(2\kappa + 1)\dot{H}_x + 2(3\kappa + 1)H_x^2 - H_x H_z}{2\kappa(\kappa + 1)},$$  (29)

$$\rho = \frac{2\dot{H}_x + 2(1 - \kappa)H_x^2 - (2\kappa + 1)H_x H_z}{2\kappa(\kappa + 1)},$$  (30)

$$\Lambda = -\frac{2\dot{H}_x + 4H_x^2 + H_x H_z}{2(\kappa + 1)},$$  (31)

where $\kappa = \frac{8\pi + \lambda}{\lambda}$.

The EoS parameter is obtained as:

$$\omega = \frac{2(2\kappa + 1)\dot{H}_x + 2(3\kappa + 1)H_x^2 - H_x H_z}{2\dot{H}_x + 2(1 - \kappa)H_x^2 - (2\kappa + 1)H_x H_z}.$$  (32)

The Ricci scalar $R$ is given by

$$R = -2\left[2\dot{H}_x + \dot{H}_z + 3H_x^2 + H_z^2 + 2H_x H_z\right].$$  (33)

From Eqs. (27) and (28), we observe that by knowing the $a$ (scale factor) value, one can assess the estimations of $A$ and $B$ and thus the field equations can be solved. So we consider a period subordinate deceleration parameter ($q$) which is supported by observations of SN Ia (Riess et al. 2004; Clocchiatti et al. 2006; Tonry et al. 2003) also CMB anisotropies (Hanany et al. 2000; de Bernardis et al. 2007). The inspiration to pick such a period subordinate DP is that the universe is at present experiencing accelerating expansion and decelerated expansion in the past as appeared by ongoing observations. Furthermore, ($z < 0.5$), the current acceleration and ($z > 0.5$), the past deceleration is supported by the SNe information. Additionally corrected redshift $z_t = 0.43 \pm 0.07$ by (1 $\sigma$) c.1. (Riess et al. 2007) from $z_t = 0.46 \pm 0.13$ at (1 $\sigma$) c.1. (Riess et al. 2004) as of late found be High-Z Supernova Search (HZSNS) group. Supernova Legacy Survey (SNLS) (Astier et al. 2006), and additionally the one as of late incorporated by Knop et al. (2003), yields $z_t \sim 0.6(1 \sigma)$ in better concurrence with flat $\Lambda$CDM demonstrate ($z_t = (2\Omega_\Lambda/\Omega_m)\frac{1}{2} - 1 \sim 0.66$). In this way, the DP by theory is the rate with which the universe decelerates, must show signature flipping (Riess 2001; Padmanabhan 2003; Amendola 2003).

We consider the deceleration parameter (23) as:

$$q = -\frac{\ddot{a}}{a^2} = \beta H + \alpha = \beta \frac{\dot{a}}{a} + \alpha,$$  (34)

here $\alpha$, $\beta$ arbitrary constants.

From above equation, we have $\frac{2\dot{a}}{a} + \beta \frac{\dot{a}}{a} + \alpha = 0$, which on solving, yields

$$a = \exp\left[-\frac{(1 + \alpha)}{\beta}t - \frac{1}{(1 + \alpha)} + \frac{l}{\beta}\right], \text{ provided } \alpha \neq -1$$  (35)

here constant of integration is $l$.

From Eq. (35), we calculate

$$\dot{a} = -\left(\frac{1 + \alpha}{\beta}\right)\exp\left[-\left(\frac{1 + \alpha}{\beta}\right)t - \frac{1}{(1 + \alpha)} + \frac{l}{\beta}\right],$$

$$\ddot{a} = \left(\frac{1 + \alpha}{\beta}\right)^2\exp\left[-\left(\frac{1 + \alpha}{\beta}\right)t - \frac{1}{(1 + \alpha)} + \frac{l}{\beta}\right]$$  (36)
Putting above values in Eq. (34), we obtain the DP value as $q = -1$. Similarly we also observed that $q = -1$ for $\alpha = 0$.

For $\alpha = -1$, we have to find another solution. In this case Eq. (34) reduces to

$$q = -\frac{a\ddot{a}}{a^2} = -1 + \beta H,$$

which yields the following differential equation:

$$\frac{a\ddot{a}}{a^2} + \beta \frac{\dot{a}}{a} - 1 = 0.$$  \hspace{1cm} (38)

The solution of above equation is found to be

$$a = \exp \left[ \frac{1}{\beta} \sqrt{2} \sqrt{2} t + k \right],$$  \hspace{1cm} (39)

here constant of integration is $k$. Eq. (39) is recently used by Tiwari et al. (2015)

Since we are interested to study the cosmic decelerated-accelerated transit universe, so we only consider the later case for which $\alpha = -1$.

Using Eq. (39) in Eqs. (27) and (28), we obtain

$$A = k_2 \frac{2}{\beta} \sqrt{2} \sqrt{2} \frac{t}{k} \exp \left[ \frac{k_1}{3} \int \exp \left\{ -\frac{3}{\beta} \sqrt{2} \sqrt{2} t + k \right\} dt \right],$$  \hspace{1cm} (40)

and

$$B = k_2 \frac{2}{\beta} \sqrt{2} \sqrt{2} \frac{t}{k} \exp \left[ -\frac{2k_1}{3} \int \exp \left\{ -\frac{3}{\beta} \sqrt{2} \sqrt{2} t + k \right\} dt \right].$$  \hspace{1cm} (41)

Using Eqs. (40) and (41) in (15), the model of the universe takes the form

$$ds^2 = dt^2 - k_2 \frac{2}{\beta} \sqrt{2} \sqrt{2} \frac{t}{k} \exp \left[ \frac{2k_1}{3} F(t) \right] (dx^2 + dy^2) - k_2 \frac{2}{\beta} \sqrt{2} \sqrt{2} \frac{t}{k} \exp \left[ -\frac{4k_1}{3} F(t) \right] dz^2,$$

where

$$F(t) = \int \exp \left\{ -\frac{3}{\beta} \sqrt{2} \sqrt{2} t + k \right\} dt =
\frac{e^{-\frac{2}{k} n}}{2k} \left[ e^{\frac{2}{k} n} \exp \left\{ \int E_1 \left\{ 8 \frac{-1 + \coth \left( \frac{1 - k t}{n} \right) \}}{n} \right\} \right] - \exp \left\{ \int E_1 \left\{ 8 \frac{1 + \coth \left( 1 - \frac{k t}{n} \right) \}}{n} \right\} \right] \right].$$  \hspace{1cm} (43)

Articulations for physical parameters, for example, the spatial volume $(V)$, directional Hubble parameters $(H_x)$, mean Hubble’s parameter $(H)$, extension scalar $(\theta)$, shear scalar $(\sigma)$ and anisotropy parameter $(A_m)$ for universe (42) are given by:

$$V = \exp \left[ \frac{3}{\beta} \sqrt{2} \sqrt{2} t + k \right],$$  \hspace{1cm} (44)

$$H_x = H_y = \frac{k_1}{3} \exp \left[ -\frac{3}{\beta} \sqrt{2} \sqrt{2} t + k \right] + \frac{1}{\sqrt{2} \sqrt{2} t + k},$$  \hspace{1cm} (45)

$$H_z = -\frac{2k_1}{3} \exp \left[ -\frac{3}{\beta} \sqrt{2} \sqrt{2} t + k \right] + \frac{1}{\sqrt{2} \sqrt{2} t + k},$$  \hspace{1cm} (46)

$$H = \frac{1}{\sqrt{2} \sqrt{2} t + k},$$  \hspace{1cm} (47)
The deceleration parameter is calculated from Eq. (37) as:

\[ q = -1 + \frac{\beta}{\sqrt{2\beta t + k}}. \]

From Eq. (51), we see that \( q > 0 \) for \( t < \frac{2\beta + k}{2\beta} \), while \( q < 0 \) for \( t > \frac{2\beta + k}{2\beta} \). So, in our determined model, one can choose the values of constants \( \beta \) and \( k \) in such a manner that we obtain the value of \( q \) consistent with observation range \(-1 < q < 0\).

From Eq. (51), deceleration parameter present value can be calculated as:

\[ q_0 = -1 + \frac{\beta}{\sqrt{2\beta t_0 + k}} = -1 + \beta H_0, \]

where \( H_0 \) is Hubble’s parameter present value and \( t_0 \) is the universe age at present.

We consider the following two cases based on two different data:

**Case 1: Model Based on SN Ia data in combination with BAO and CMB observations**

Putting \( H_0 = 73.8 \) and \( q_0 = -0.54 \), (Giofristi et al., 2012), we get \( \beta = 0.0062 \).

Also from Eq. (45) we have \( H_0 = \frac{1}{\sqrt{2\beta t_0 + k}} = \sqrt{\frac{1}{2\beta t_0 + k}} \), which on solving for \( k \) results into \( k = \frac{1}{\frac{H_0}{\sqrt{\beta t_0}} - 2\beta} \) and using the values of \( H_0 \) and \( \beta \), we obtain \( k = 0.000016 \). These values of \( \beta \) and \( k \) are used in plotting all the figures 1-9.

**Case 2: Model Based on supernovae type Ia union data**

Putting \( H_0 = 73.8 \) and \( q_0 = -0.73 \), (Cunha, 2009), we get \( \beta = 0.0036 \) and \( k = 0.000084 \).

Figure 1 corresponding to Eq. (51) describes the variation of DP \( q \) with time \( t \) for both cases. In case 1, for \( \beta = 0.0062 \) and \( k = 0.000016 \), we observe that the model exhibits the expansion from decelerating to accelerating era whereas, for \( \beta = 0.0036 \) and \( k = 0.000084 \), the model is in accelerating phase only. In case 1, at the early phase of the evolution of the universe, \( q \) was positive, which indicates that in the early universe the model was expanding but expansion rate was slowing down with time. It passed through a transition phase from positive to negative and currently, it is negative, indicating that the universe is expanding with an accelerated rate of expansion. Recent observations have also established that the current universe is undergoing in a cosmic acceleration. Hence our models in both cases are consistent with observations of SN Ia described in the introduction.

Putting the values from Eqs. (45) and (46) in Eqs. (24)–(26), the pressure, energy density and cosmological constant for model (42) are obtained as:

\[ p = \frac{1}{2\kappa(\kappa + 1)} \left[ \frac{2k_0^2}{g_c \sqrt{2\beta t + k}} + \frac{k_1}{3\sqrt{2\beta t + k}} \frac{(6\kappa + 1)}{(2\beta t + k)} - \frac{2\beta (2\kappa + 1)}{(2\beta t + k)^2} \right], \]

\[ \rho = \frac{1}{2\kappa(\kappa + 1)} \left[ \frac{2k_0^2}{g_c \sqrt{2\beta t + k}} + \frac{k_1}{3\sqrt{2\beta t + k}} \frac{(2\kappa + 1)}{(2\beta t + k)} - \frac{2\beta (2\kappa + 1)}{(2\beta t + k)^2} \right], \]

\[ \Lambda = -\frac{1}{2\kappa(\kappa + 1)} \left[ \frac{2k_0^2}{g_c \sqrt{2\beta t + k}} + \frac{5k_1}{3\sqrt{2\beta t + k}} \frac{5}{(2\beta t + k)} - \frac{2\beta (2\kappa + 1)}{(2\beta t + k)^2} \right]. \]
Fig. 1 Graph of DP $q$ with $t$ for two sets of $(\beta, k)$

$$
\omega = \frac{2k^2(3\kappa+2)}{9c^2\sqrt{2\beta+t}} - \frac{k_1}{3\sqrt{2\beta+t}c_0^2}\sqrt{2\beta+t} + \frac{(6\kappa+1)}{(2\beta+t+k)} + \frac{2\beta(2\kappa+1)}{(2\beta+t+k)^2}
$$

Figure 2 corresponding to the Eq. (53), depicts the variation of isotropic pressure $p$ with time $t$. We find that $p$ is negative and it approaches near zero as $t \to \infty$ for both cases. This negative pressure actually causes the accelerated expansion of the universe.

Figure 3 corresponding to the Eq. (54), describes the energy density $\rho$ variation with time $t$. It is observed that when $t \to 0$, $\rho \to \infty$, indicating the Big-bang scenario i.e. the density was very high in the early universe. As the time progressed, the concentrated matter and radiation dispersed and so the density decreased. At the current time, it is also decreasing but with a moderate rate, indicating the expansion is still going on. This result of our model is also consistent with recent observations. The $\rho$ is a positive decreasing function of time and it approaches near zero as $t \to \infty$. It is worth mentioned here that $\rho$ in case 1 is fast decreasing in comparison to $\rho$ in case 2.

The behaviour of the cosmological term $\Lambda$ with time in both cases is shown in Figure 4. From the observations, it is clear that in the early time the cosmological constant $\Lambda$ is negative and it increases quickly in brief day and age drawing closer to little negative an incentive almost zero. The $\Lambda$ of the case 1 is speedy increasing in contrast with case 2. It has been found that we get a more negative pressure than the previous one in every next time. The reason for this is the Physics. The cosmological models have been explored by Yadav (2011), Saha and Boyadjiev (2004), Pedram et al. (2008), Biswas and Mazumdar (2009) and Jotania et al. (2011) in which the cosmological constant is negative. Really at present the estimation of $\Lambda$ isn’t simply trapped in any case, it is uncertain and circumlocutory too. In any case, the Einstein-Maxwell speculation shows to the other approach which looks not so much troublesome but rather more immense, since a likelihood has appeared for $\Lambda \leq 0$ i.e. for the circumstance when the presence of $\Lambda$ decelerates the expansion of the universe. Late cosmological recognitions (Perlmutter et al. 1998; Riess et al. 1999, 2004) propose the presence of a positive cosmological predictable $\Lambda$ with the magnitude $\Lambda(G\hbar/c^3) \approx 10^{-123}$. These recognitions on size and red-shift of supernova Ia suggest that our universe may be an accelerating one with prompted cosmological den-
Fig. 2 Graph of pressure $p$ with $t$ for two sets of $(\beta, k)$, $\kappa = -1.2, k_1 = 1$

Fig. 3 Graph of energy density $\rho$ versus $t$ for two sets of $(\beta, k)$, $\kappa = -1.2, k_1 = 1$
Figure 4 corresponds to the Eq. (56), giving the variation of EoS parameter $\omega$ with $t$ and it is observed that $\omega$ is a negative decreasing function of time and approaches to a small negative value near zero as $t \to \infty$. Here, it is observed that $\omega$ vanishes for $t = t_c$, where $t_c$ is a critical time given by the
relation
\[
\frac{2k_1^2(3\kappa + 2)}{9e^{\kappa \sqrt{2\beta t + k}}} + \frac{(6\kappa + 1)}{(2\beta t + k)} = \frac{k_1}{3\sqrt{2\beta t + k}e^{\kappa \sqrt{2\beta t + k}}} + \frac{2\beta(2\kappa + 1)}{(2\beta t + k)^2}
\]

We know that \(\omega = 0\), \(\omega > 0\) and \(\omega < 0\) represents the dusty universe, real matter dominated universe and dark energy dominated universe respectively. Earlier to that (i.e. at \(t < t_c\)) real matter dominates, where and when \(t > t_c\), \(\omega < 0\), indicating dark energy dominated phase of the universe. In our derived model, at the early universe (when \(t < t_c\)) \(\omega > 0\) indicating that the early universe was a matter-dominated universe. It passes through a dusty universe phase at \(t = t_c\) and at present \(\omega < 0\) shows that at present the universe is dominated by dark energy. The phantom dominated universe ends up with a finite-time future singularity called Big Rip or Cosmic Doomsday (Caldwell et al. 2003; McInnes 2002; Gonzalez-Diaz 2004; Sami and Toporensky 2004; Nojiri and Odintsov 2004). From Figure 5 we observe that in case 1, the model starts their evolution from matter dominated era to dark energy phase. In other words, we say that this model starts evolution from matter dominated phase, then reaches to the quintessence dark energy phase and lastly ends in phantom dark energy scenario. In case 2 the model is dominated by phantom dark energy as it is shown clearly in figure 5.

Figure 6 corresponds to the Eq. (49) and plots the anisotropic parameter \(A_m\) versus time \(t\). It is observed from the figure that in the early phase of the evolution, it is very high which means the early universe was highly anisotropic. It decreases rapidly and approaches to zero as \(t \to \infty\) in both cases 1 & 2, indicating that the universe will become isotropic in the very long run.

**Energy Conditions**

We examine the conceivable outcomes of energy conditions to be fulfilled or not in our model. We realize that:

- \(\rho \geq 0\) and \(\rho + p \geq 0\) are the energy conditions (weak).
- \(\rho \geq |p|\) i.e. \(\rho + p \geq 0\) and \(\rho - p \geq 0\) are the energy conditions (dominant)
- \(\rho + 3p \geq 0\) are the energy conditions (strong)
Utilizing articulations for $\rho$ and $p$, we have plotted the diagrams for energy conditions. The left-hand side of the energy conditions has been plotted concerning cosmic time $t$ in Figures 7 and 8 for the case 1 and 2 separately. From Figures 7 and 8, plainly all the three sorts of energy conditions are not fulfilled in both cases 1 & 2. The presence of the locale with $\omega < 1$ (if such a stage in the universe advancement without a doubt happens) opens up various major inquiries. For example, the entropy of such a universe is negative (or the trademark temperatures ought to be negative). The DEC for ghost matter is violated, as a rule (Nojiri et al. 2005). It is worth mentioned here that our derived model is dominated by phantom dark energy fluid. So, the violation of SEC is consistent with well-established law.

### The Ricci Scalar and Trace

The Ricci scalar $R$ and the trace $T$ for this model are obtained as

$$ R = -6 \left[ \frac{k_1^2}{9 e^{\frac{3}{2} \sqrt{2 \beta t + k}}} + \frac{2}{(2 \beta t + k)} - \frac{\beta}{(2 \beta t + k)^{\frac{3}{2}}} \right], \quad (58) $$

$$ T = \frac{1}{2\kappa(\kappa + 1)} \left[ -\frac{8(2\kappa + 1)k_1^2}{9 e^{\frac{3}{2} \sqrt{2 \beta t + k}}} + \frac{2(1 - \kappa)}{3 \sqrt{2 \beta t + k} e^{\frac{3}{2} \sqrt{2 \beta t + k}}} - \frac{2(11\kappa + 1)}{(2 \beta t + k)} + \frac{4(3\kappa + 1)}{(2 \beta t + k)^{\frac{3}{2}}} \right]. \quad (59) $$

The function $f(R, T) = f_1(R) + f_2(T)$ for this model is obtained as

$$ f(R, T) = \lambda \left[ -\frac{2k_1^2(\kappa + 2)(3\kappa + 1)}{3\kappa(\kappa + 1)e^{\frac{3}{2} \sqrt{2 \beta t + k}}} + \frac{2\beta(3\kappa^2 + 6\kappa + 1)}{\kappa(\kappa + 1)(2 \beta t + k)^{\frac{3}{2}}} - \frac{(12\kappa^2 + 23\kappa + 1)}{\kappa(\kappa + 1)(2 \beta t + k)} + \frac{(1 - \kappa)e^{-\frac{3}{2} \sqrt{2 \beta t + k}}}{3\kappa(\kappa + 1)\sqrt{2 \beta t + k}} \right]. \quad (60) $$

**Fig. 7** Graph of energy conditions with $t$ for case 1 $\beta = 0.0062, k = 0.000016, \kappa = -1.2, k_1 = 1$
Fig. 8 Graph of energy conditions with $t$ for case $2 \beta = 0.0036, k = 0.000084, \kappa = -1.2, k_1 = 1$

5 IS THE CORRESPONDING SOLUTIONS STABLE?

A thorough investigation by conjuring a perturbation approach should be possible on the stability of the relating solutions. Perturbations of the fields of a gravitational framework against the background evolutionary solutions ought to be investigated to guarantee the stability of the correct or approximated background solution are talked about by Chen and Kai (2001) and Kai (2001). Presently we shall examine the background solution stability regarding metric perturbations. Considering perturbations for every one of the three expansion factors $a_i$ by means of:

$$a_i \to a_{B_i} + \delta a_i = a_{B_i}(1 + \delta b_i)$$ (61)

Focusing $\delta b_i$ variables rather than $\delta a_i$ starting now and into the foreseeable future for comfort. In this way, the perturbations of the volume scale factor $V_B = \Pi_{i=1}^{3} a_i$, directional Hubble factors $\theta_i = \frac{\dot{a_i}}{a_i}$, and the average Hubble factor $\theta = \sum_{i=3}^{3} \frac{\theta_i}{3} = \frac{\dot{V}}{V}$ can be appeared to be:

$$V \to V_B + V_B \sum_i \delta b_i, \quad \theta_i \to \theta_{B_i} + \sum_i \delta b_i, \quad \theta \to \theta_B + \frac{1}{3} \sum_i \delta b_i$$ (62)

where $V_B$ represents the scale volume factor (background). Now one may demonstrate that the metric linear order perturbations $\delta b_i$, comply with the accompanying conditions:

$$\sum_i \delta \dot{b}_i + 2 \sum_i \theta_{B_i} \delta b_i = 0$$ (63)

$$\delta \ddot{b}_i + \frac{V_B}{V_B} \delta \dot{b}_i + \sum_j \delta \dot{b}_j \theta_{B_j} = 0$$ (64)

$$\sum \delta \dot{b_i} = 0.$$ (65)
Fig. 9 Graph of $\delta a_i$ with $t$ for two sets of $(\beta, k)$, $\kappa = -1.2, k_1 = 1$

We can without much of a stretch find by three conditions given above:

$$\delta \ddot{b}_i + \frac{\dot{V}_B}{V_B} \delta \dot{b}_i = 0,$$

(66)

$V_B$ for our case is taken as:

$$V_B = \exp \left[ \frac{3}{\beta} \sqrt{2/3t + k} \right],$$

(67)

Utilizing above condition in Eq. (66) and after integration we have:

$$\delta b_i = -c_i \left[ \frac{\beta + 3\sqrt{2/3t + k}}{9 \exp \left( \frac{\sqrt{2/3t + k}}{\beta} \right)} \right],$$

(68)

here integration constant is $c_i$. So, for each expansion factor $\delta a_i = a_B \delta b_i$ the actual fluctuations are given by:

$$\delta a_i \rightarrow -c_i \left[ \frac{\beta + 3\sqrt{2/3t + k}}{9 \exp \left( \frac{\sqrt{2/3t + k}}{\beta} \right)} \right],$$

(69)

From the above equation we see that $\delta a_i$ approaches zero when $t \rightarrow \infty$. The variation of $\delta a_i$ versus $t$ is shown in Fig. 9. Thus, against the perturbations of the graviton field, the background solution is stable.

6 CONCLUDING REMARKS

In the present paper, we have examined two kinds of LRS Bianchi I cosmological models inside the outline work of $f(R, T)$ theory with $\Lambda(T)$. The two models are depending on the two different data: SN Ia data in combination with BAO and CMB observations (Giostrì et al. 2012) and from supernovae type Ia union data (Cunha 2009) respectively. We have thought about the aftereffects of two cosmological
models of the universe, determined in the present work. The field conditions have been fathomed precisely with reasonable physical assumption. The relating correct solutions are found for the particular model \( f(R, T) = \lambda (R + T) \), for which the universe expands as \( a = \exp \left[ \frac{1}{\beta} \sqrt{2\beta t + k} \right] \). This creates a progress of the universe from the early decelerating stage to the ongoing accelerating stage. The physical parameters \( \rho, p, \Lambda \) evolves with an initial singularity and tends to a small value in late time which is predictable with the ongoing observations (expanding universe). The models are anisotropic in early universe however at late time \( t \to \infty \) the model behaves as an isotropic FRW model.

The primary highlights of the models are as per the following:
- In outline, two expansion phases of the universe which are naturally unified by modified gravity have been considered: early time inflation and astronomical speeding up at current age. Our inferred models depend on two cases as said above, in case 1, for \( \beta = 0.0062 \) and \( k = 0.000016 \) we acquired a cosmological model from early decelerated stage to the present accelerating stage. In case2, for \( \beta = 0.0036 \) and \( k = 0.000084 \), we have obtained cosmological model in accelerating stage only. In case 1, the models start evolution from a matter dominated era, then reach to quintessence dark energy phase and lastly ends in phantom dark energy era (Fig. 5). In case 2, we obtain the phantom dark energy era only (Fig. 5).
- The models depend on correct solution of the \( f(R, T) \) gravity field conditions for the anisotropic LRS Bianchi space-time with perfect fluid with time subordinate deceleration parameter.
- The model speaks to a shearing, non-rotating, expanding and transit (from decelerating to accelerating) universe.
- In both cases1 & 2 the anisotropic parameter \( A_m \) tends to zero at \( t \to \infty \) (Fig. 6) indicating that the universe will become isotropic in late time. In an early phase of the evolution, anisotropy is very high which means that the early universe was highly anisotropic. But as the present time, the model exhibits isotropic in nature.
- It is a standard practice in literature to think about constant deceleration parameter. The nature of the deceleration parameter for a Universe which is accelerating at present must show flipping signature as of now talked about. Subsequently, our thought of variable DP is physically supported. Our determined display is decelerating in an early period of the universe while it is accelerating at present stage.
- In our derived models we observe that energy conditions are violated.
- The solution of our model is steady against the perturbation of the gravitational field as talked about in Sect. 5.

Consequently, the results found in this article may be useful for better comprehension of the behavior for cosmological LRS Bianchi-I dark energy models in the evolution of the universe inside the \( f(R, T) \) gravity speculation.

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